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ON SUBMANIFOLDS WITH CONSTANT MEAN CURVATURE IN A REAL SPACE FORM

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Abstract: The purpose of this paper is to classify submanifolds with constant mean curvature in a real space form. We put S the squared norm of the second fundamental form and $|\phi|^2 = S - nH^2$. Denote by A_H and B_H the squares of the positive roots of the equations

$$x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2+c) = 0 \text{ and } \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2+c) = 0,$$

respectively. We prove the following: First, le M^n be a complete, connected and orientable submanifold with nonzero constant mean curvature H in $S^{n+p}(c)(p \geq 3)$. If $|\phi|$ satisfies $|\phi^2 \leq B_H$ for all $x \in M^n$, then M^n lies in a totally geodesic submanifold $S^{n+1}(c)$ of $S^{n+p}(c)$, and $|\phi|^2 \equiv 0$ and M^n is totally umbilic. Next, let M^n be a complete, connected and orientable hypersurface with constant mean curvature $H > 1$ in $H^{n+1}(-1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M^n$. Then (i) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic or $|\phi|^2 \equiv A_H$. (ii) $|\phi|^2 \equiv A_H$ if and only if M^n is isometric to $S^{n-r} \times H^1(-\frac{1}{r^2+1})$ for some $r > 0$. Moreover, we prove a gereralization of this result of the hypersurface in a hyperbolic space.

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1. Introduction

Let $\tilde{M}^{n+p}(c)$ be an $(n+p)$-dimensional complete, connected and simply connected Riemannian manifold of constant sectional curvature c. We call it a space form. A space form $\tilde{M}^{n+p}(c)$ is one of the following:

(i) If $c > 0$, then $\tilde{M}^{n+p}(c)$ is an $(n+p)$-dimensional Euclidean sphere $S^{n+p}(c)$,

(ii) If $c = 0$, then $\tilde{M}^{n+p}(c)$ is an $(n+p)$-dimensional Euclidean space R^{n+p},

(iii) If $c < 0$, then $\tilde{M}^{n+p}(c)$ is an $(n+p)$-dimensional hyperbolic sphere $H^{n+p}(c)$.

Let M^n be an $n-$dimensional, connected and orientable submanifold isometrically immersed in $\tilde{M}^{n+p}(c)$. Denote by h^α_{ij} the local component of the second fundamental form for each $i, j, \alpha (1 \leq i, j \leq n, n+1 \leq \alpha \leq n+p)$. We set

$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h^\alpha_{ij})^2 \text{ and } H := \frac{1}{n}\sqrt{\sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h^\alpha_{ii})^2}$$

be the squared norm of the second fundamental form and the mean curvature of M^n in $\tilde{M}^{n+p}(c)$, respectively.

Now, we denote by A_α the $n \times n$ matrix of h^α_{ij} with respect to indices i, j. Define linear maps $\phi_\alpha : T_xM \to T_xM$ by

$$< \phi_\alpha X, Y >:= \frac{1}{n}\text{trace}A_\alpha < X, Y > - < A_\alpha X, Y > \text{ for } n+1 \leq \alpha \leq n+p,$$

where $<,>$ is the Riemannian metric of M^n. Moreover, we define the bilinear map $\phi : T_xM \times T_xM \to T_x^\perp$ by

$$\phi(X, Y) = \sum_{\alpha=n+1}^{n+p} < \phi_\alpha X, Y > e_\alpha,$$

where $\{e_{n+1}, \ldots, e_{n+p}\}$ denotes an orthonormal basis. It is easy to check that trace $\phi = 0$ and that

$$|\phi|^2 := \sum_{\alpha=n+1}^{n+p} \text{trace } \phi_\alpha^2 = S - nH^2.$$

Let

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c)$$

and

$$Q_H(x) = \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c)$$

be the polynomials for each real number $H \in \mathbb{R}$. We put A_H the square of the positive root of $P_H(x) = 0$ and B_H one of $Q_H(x) = 0$.

Besides, in the case of $p = 1$, we denote by h_{ij} the local component of the second fundamental form for each $i, j (1 \leq i, j \leq n)$ and by A the $n \times n$ matrix of h_{ij} with respect to indices i, j. We choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. Then we have $H = \frac{1}{n}|\sum_{i=1}^{n} \lambda_i|$ and $S = \sum_{i=1}^{n} \lambda_i^2$. In the hypersurface we may put $\phi = \phi_{n+1}$. Then $\phi : T_x M \to T_x M$ satisfies

$$< \phi X, Y >:= \frac{1}{n} \text{ trace } A < X, Y > - < AX, Y > .$$

It easily check that trace $\phi = 0$ and that

$$|\phi|^2 := \text{trace } \phi^2 = \frac{1}{2n} \sum_{i,j} (\lambda_i - \lambda_j)^2.$$

Hence we get that $|\phi|^2 = 0$ if and only if M^n is totally umbilic.

We study generalizations of the results of the following theorems. Moreover, we also study in the case of $c = -1$.

Theorem 1. (see Alencar and do Carmo [1]) *Let M^n be a compact and orientable hypersurface with constant mean curvature H in $S^{n+1}(1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M$. Then:*

(i) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if:

(A) $H = 0$ and M^n is a Clifford torus in $S^{n+1}(1)$, i.e., M^n is a product of spheres $S^{n_1}(r_1) \times S^{n_2}(r_2), n_1 + n_2 = n$, of appropriate radii.

(B) $H \neq 0, n \geq 3$, and $M^n = S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, where $r^2 < \frac{n-1}{n}$.

(C) $H \neq 0, n = 2$, and $M^2 = S^1(r) \times S^1(\sqrt{1-r^2})$ in $S^3(1)$, where $r^2 \neq \frac{1}{2}$.

Theorem 2. (see Uchida and Matsuyama [13]) *Let M^n be a complete, connected and orientable submanifold with nonzero constant mean curvature H in $S^{n+2}(c)$. If $|\phi|$ satisfies $|\phi|^2 \leq A_H$ for all $x \in M^n$, then M^n lies in a totally geodesic hypersurface $S^{n+1}(c)$ of $S^{n+2}(c)$ and*

(i) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if:

(B) $n \geq 3$, and $M^n = S^{n-1}(r_1) \times S^1(r_2)$ in $S^{n+1}(c)$, where $r_1^2 + r_2^2 = \dfrac{1}{c}$ and $r_1^2 < \dfrac{n-1}{nc}$.

(C) $n = 2$ and $M^2 = S^1(r_1) \times S^1(r_2)$ in $S^3(c)$, where $r_1^2 + r_2^2 = \dfrac{1}{c}$ and $r_1^2 \neq \dfrac{1}{2c}$.

Theorem 3. *Let M^n be a complete, connected and orientable submanifold with nonzero constant mean curvature H in $S^{n+p}(c)(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then M^n lies in a totally geodesic submanifold $S^{n+1}(c)$ of $S^{n+p}(c)$, and $|\phi|^2 \equiv 0$ and M^n is totally umbilic.*

Theorem 4. *Let M^n be a complete, connected and orientable hypersurface with constant mean curvature $H > 1$ in $H^{n+1}(-1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M^n$. Then:*

(i) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if M^n is isometric to $S^{n-1}(r) \times H^1(-\dfrac{1}{r^2+1})$ for some $r > 0$.

Theorem 5. *Let M^n be a complete, connected and orientable submanifold with constant mean curvature $H > 1$ in $H^{n+2}(-1)$. If $|\phi|$ satisfies $|\phi|^2 \leq A_H$ for all $x \in M^n$, then M^n lies in a totally geodesic hypersurface $H^{n+1}(-1)$ of $H^{n+2}(-1)$ and:*

(i) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if M^n is isometric to $S^{n-1}(r) \times H^1(-\dfrac{1}{r^2+1})$ for some $r > 0$.

Theorem 6. *Let M^n be a complete, connected and orientable submanifold with nonzero constant mean curvature $H > 1$ in $H^{n+p}(c)(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then M^n lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$, and $|\phi|^2 \equiv 0$ and M^n is totally umbilic.*

2. Preliminaries

Let $\tilde{M}^{n+p}(c)$ be an $(n+p)$-dimensional space form of constant curvature c and M^n an $n-$dimensional, complete, connected and orientable submanifold in $\tilde{M}^{n+p}(c)$. We choose a local field of an orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ adapted to the Riemannian metric of $\tilde{M}^{n+p}(c)$ and the dual coframe $\{\omega_1, \ldots, \omega_{n+p}\}$ are tangent to M^n. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq n+p, \quad 1 \leq i, j, k, \cdots \leq n,$$

$$n+1 \leq \alpha\beta, \gamma, \cdots \leq n+p.$$

Then the structure equations of $\tilde{M}^{n+p}(c)$ are given by

$$d\omega_A = -\sum_{B=1}^{n+p} \omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_{C=1}^{n+p} \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,C=1}^{n+p} K_{ABCD}\omega_C \wedge \omega_D,$$

with $K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$, where K_{ABCD} denotes the component of the curvature tensor of $\tilde{M}^{n+p}(c)$. We restrict these forms to M^n. Then we get

$$\omega_\alpha = 0.$$

Since $0 = d\omega_\alpha = -\sum_{i=1}^{n} \omega_{\alpha i} \wedge \omega_i$, by Cartan's Lemma, we may write

$$\omega_{\alpha i} = \sum^{n} h_{ij}^\alpha \omega_j, h_{ij}^\alpha = h_{ji}^\alpha. \tag{1}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,\ell}^{n} R_{ijk\ell}\omega_k \wedge \omega_\ell.$$

with

$$R_{ijk\ell} = c(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p} (h^\alpha_{ik}h^\alpha_{j\ell} - h^\alpha_{i\ell}h^\alpha_{jk}), \tag{2}$$

where $R_{ijk\ell}$ is the component of the curvature tensor of M^n. Denoting by R_{jk} the component of the Ricci curvature of M^n, from (2), we have

$$R_{jk} = (n-1)c\delta_{jk} + \sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h^\alpha_{ii}h^\alpha_{jk} - \sum_{i=1}^{n} h^\alpha_{ik}h^\alpha_{ji}). \tag{3}$$

We also have

$$d\omega_{\alpha\beta} = -\sum_{\gamma=1}^{n+p} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2}\sum_{i,j=1}^{n} R_{\alpha\beta ij}\omega_i \wedge \omega_j,$$

where

$$R_{\alpha\beta ij} = \sum_{\ell=1}^{n}(h^\alpha_{i\ell}h^\beta_{\ell j} - h^\alpha_{j\ell}h^\beta_{\ell i}).$$

The Riemannian connection of M^n is defined by (ω_{ij}). The form $(\omega_{\alpha\beta})$ defines a connection in the normal bundle of M^n. The second fundamental form $\prod$ and the mean curvature vector h of M^n are defined by

$$\prod := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h^\alpha_{ij}\omega_i\omega_j e_\alpha \text{ and } h := \frac{1}{n}\sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h^\alpha_{ii})e_\alpha,$$

respectively. On the other hand, the mean curvature H of M^n is defined by

$$H := \frac{1}{n}\sqrt{\sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h^\alpha_{ii})^2}$$

(see Introduction). We take the exterior differentiation of (1) and define h^α_{ijk} by

$$\sum_{k=1}^{n} h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_{k=1}^{n} h^\alpha_{ik}\omega_{kj} - \sum_{k=1}^{n} h^\alpha_{jk}\omega_{ki} - \sum_{\beta=n+1}^{n+p} h^\beta_{ij}\omega_{\beta\alpha}. \tag{4}$$

By straightforward computations we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{5}$$

Similarly, we take the exterior differentiation of (4) and define $h^\alpha_{ijk\ell}$ by

$$\sum_{\ell=1}^{n} h^\alpha_{ijk\ell}\omega_\ell = dh^\alpha_{ijk} - \sum_{\ell=1}^{n} h^\alpha_{\ell jk}\omega_{\ell i} - \sum_{\ell=1}^{n} h^\alpha_{i\ell k}\omega_{\ell j} - \sum_{\ell=1}^{n} h^\alpha_{ij\ell}\omega_{\ell k} - \sum_{\beta=n+1}^{n+p} h^\beta_{ijk}\omega_{\beta\alpha}.$$

Then the Ricci formula for the second fundamental form is given by

$$h^\alpha_{ijk\ell} - h^\alpha_{ij\ell k} = \sum_{m=1}^{n} h^\alpha_{mj} R_{kol\ell} + \sum_{m=1}^{n} h^\alpha_{im} R_{mjk\ell} + \sum_{\beta=n+1}^{n+p} h^\beta_{ij} R_{\beta\alpha k\ell}. \tag{6}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by

$$\triangle h^\alpha_{ij} := \sum_{k=1}^{n} h^\alpha_{ijkk}.$$

From the Codazzi equation (5) we obtain

$$\sum_{k=1}^{n} h^\alpha_{ijkk} = \sum_{k=1}^{n} h^\alpha_{kijk}$$

for any $\alpha, n+1 \le \alpha \le n+p$. Moreover, using the Ricci formula (6), we have

$$\begin{aligned} \triangle h^\alpha_{ij} &= \sum_{k=1}^{n} h^\alpha_{kijk} \qquad (7)\\ &= \sum_{k=1}^{n} h^\alpha_{kikj} + \sum_{k,m=1}^{n} h^\alpha_{km} R_{mijk} + \sum_{k,m=1}^{n} h^\alpha_{mi} R_{mkjk} + \sum_{k=1}^{n}\sum_{\beta=n+1}^{n+p} h^\beta_{ki} R_{\beta\alpha jk}\\ &= \sum_{k=1}^{n} h^\alpha_{kkij} + \sum_{k,m=1}^{n} h^\alpha_{km} R_{mijk} + \sum_{k,m=1}^{n} h^\alpha_{mi} R_{mkjk} + \sum_{k=1}^{n}\sum_{\beta=n+1}^{n+p} h^\beta_{ki} R_{\beta\alpha jk}. \end{aligned}$$

Let

$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h^\alpha_{ij})^2$$

denote the squared norm of the second fundamental form of M^n (see Introduction). Assuming the mean curvature vector $h \neq 0$ on M^n, we know that $e_{n+1} = h/H$ is a normal vector field defined globally on M^n. We define S_1 and S_2 by

$$S_1 := \sum_{i,j=1}^{n} (h^{n+1}_{ij} - H\delta_{ij})^2 \text{ and } S_2 := \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h^\alpha_{ij})^2,$$

respectively. Then S_1 and S_2 are functions defined on globally and they are independent of the choice of the orthonormal frame $\{e_1, \ldots, e_n\}$. Also we have

$$|\phi|^2 = S - nH^2 = S_1 + S_2. \tag{8}$$

From the definition of the mean curvature vector h we know that

$$nH = \sum_{i=1}^{n} h_{ii}^{n+1} \text{ and } \sum_{i=1}^{n} h_{ii}^{\alpha} = 0$$

for $n+2 \leq \alpha \leq n+p$.

We establish the following lemmas for the proofs of theorems:

Lemma 1. (see Cheng [3]) *Let M^n be an $n-$dimensional submanifold with the mean curvature vector $h \neq 0$ in a space form $\tilde{M}^{n+p}(c)$. Then we have*

$$\begin{aligned}
\frac{1}{2}\triangle S_2 = & \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^{\alpha})^2 + nc \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2 \\
+ & nH \sum_{\alpha=n+2}^{n+p} trace(A_{n+1}A_{\alpha}^2) - \sum_{\alpha=n+2}^{n+p} [trace(A_{n+1}A_{\alpha})]^2 \\
+ & \sum_{\alpha,\beta=n+2}^{n+p} trace(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})^2 - \sum_{\alpha\beta=n+2}^{n+p} [trace(A_{\alpha}A_{\beta})]^2 \\
+ & \sum_{\alpha=n+2}^{n+p} trace(A_{n+1}A_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p} trace(A_{n+1}^2 A_{\alpha}^2).
\end{aligned}$$

Lemma 2. (see Cheng [3]) *Let a_i and b_{ij} be real numbers satisfying*

$$\sum_{i=1}^{n} a_i = 0, \quad \sum_{i=1}^{n} b_{ii} = 0, \sum_{i,j=1}^{n} b_{ij}^2 = b$$

and $b_{ij} = b_{ji}$ for $i, j = 1, \ldots, n$. Then we obtain

$$-(\sum_{i=1}^{n} b_{ij}a_i)^2 + \sum_{i,j=1}^{n} b_{ij}^2 a_i a_j - \sum_{i,j=1}^{n} b_{ij}^2 a_i^2 \leq -\sum_{i=1}^{n} a_i^2 b.$$

Lemma 3. (see Cheng [3]) *Let b_i be real numbers such that* $\sum_{i=1}^{n} b_i = 0$ *and* $\sum_{i=1}^{n} b_i^2 = B$ *for* $i = 1, \ldots, n$. *Then we have*

$$\sum_{i=1}^{n} b_i^4 - \frac{B^2}{n} \leq \frac{(n-2)^2}{n(n-1)} B^2.$$

Lemma 4. (see Cheng [3]) *Let a_i and b_i be real numbers satisfying* $\sum_{i=1}^{n} a_i = 0$ *and* $\sum_{i=1}^{n} a_i^2 = a$ *for* $i = 1, \ldots, n$. *Then we obtain*

$$\sum_{i=1}^{n} a_i b_i^2 \geq -\sqrt{\sum_{i=1}^{n} b_i^4 - \frac{1}{n}(\sum_{i=1}^{n} b_i^2)^2} \sqrt{a}.$$

Lemma 5. (see Li and Li [7]) *For symmetric matrices $A_1, \ldots, A_p (p \geq 2)$, we put*

$$S_{\alpha\beta} = trace(A_\alpha B_\beta), S = \sum_{\alpha=1}^{p} S_{\alpha\alpha} \text{ and } traceA_\alpha^2 = trace({}^t A_\alpha A^\alpha).$$

Then we have

$$-\sum_{\alpha,\beta=1}^{p} trace(A_\alpha A_\beta - A_\beta A_\alpha)^2 + \sum_{\alpha,\beta=1}^{p} S_{\alpha\beta}^2 \leq \frac{3}{2} S^2,$$

and the equality holds if and only if one of the following conditions holds:

(1) $A_1 = A_2 = \cdots = A_p = 0$.

(2) Only two of the matrices $A_1, \ldots, A_p$ are different from zeor. Moreover, assuming $A_1 \neq 0, A_2 \neq 0$ and $A_3 = \cdots = A_p = 0$, then $S_{11} = S_{22}$ and there exists an orthonormal matrix T such that

$${}^t T A_1 T = \sqrt{\frac{1}{2} S_{11}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$ {}^tTA_2T = \sqrt{\frac{1}{2}S_{11}} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. $$

Lemma 6. (see Alencar and do Carmo [1]) *Let* $\{\mu_i\}_{i=1}^n$ *be a set of real numbers satisfying* $\sum_{i=1}^n \mu_i = 0$ *and* $\sum_{i=1}^n \mu_i^2 = \beta^2$, *where* $\beta \geq 0$. *Then we obtain*

$$ -\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_{i=1}^n \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, $$

and the equality holds in the right-hand (resp. left-hand) side if and only if $(n-1)$ *of the* μ_i*'s are non-positive and equal (resp.* $(n-1)$ *of the* μ_i*'s are non-negative and equal).*

Remark 6. (see Alencar and do Carmo [1]) It is convenient to observe from the proof that the equality holds in the right-hand side if and only if $(n-1)\mu_i$'s are of the form $-\frac{1}{\sqrt{n(n-1)}}\beta$ and the remaining one is $\sqrt{\frac{n-1}{n}}\beta$.

In order to represent our thorems, we need some notations, for detail, see Lawson [6], Ryan [11] or Liu [8]. We give a description of the real hyperbolic space $H^{n+1}(c)$ of constant curvature $c(< 0)$. For any two vectors x and y in R^{n+2}, we set

$$ g(x, y) = \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2}. $$

(R^{n+2}, g) is the so-called Minkowski space. Denote $\rho = \sqrt{-1/c}$. We define

$$ H^{n+1}(c) = \{x \in R^{n+2} | g(x, x) = -\rho^2, x_{n+2} > 0\}. $$

Then $H^{n+1}(c)$ is a connected and simply connected hypersurface of R^{n+2}. Hence we obtain models of a real hyperbolic space. We define

$$ \begin{aligned} M_1 &= \{x \in H^{n+1}(c) | x_1 = 0\}, \\ M_2 &= \{x \in H^{n+1}(c) | x_1 = r > 0\}, \\ M_3 &= \{x \in H^{n+1}(c) | x_{n+2} = x_{n+1} + \rho\}, \end{aligned} $$

$$\begin{aligned} M_4 &= \{x \in H^{n+1}(c) | \sum_{i=1}^{n+1} x_i^2 = r^2 > 0\}, \\ M_5 &= \{x \in H^{n+1}(c) | \sum_{i=1}^{k+1} x_i^2 = r^2 > 0, \sum_{j=k+2}^{n+2} x_j^2 - x_{n+2}^2 = -\rho^2 - r^2\}. \end{aligned}$$

$M_1, \ldots, M_5$ are often called the standard examples of complete hypersurfaces in $H^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \ldots, M_4$ are totally umbilic. In the sense of Chen [2], they are called the hyperspheres of $H^{n+1}(c)$. M_3 is called the horosphere and M_4 the geodesic sphere of $H^{n+1}(c)$. Ryan [11] obtained the following:

Lemma 7. (see Ryan [11]) *Let M^n be a complete hypersurface in $H^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of TM^n, the shape operator over TM^n is expressed as a matrix A. If M^n has at most two distinct constant principal curvatures, then it is congruent to one of the following:*

(1) In the case of M_1, $A = 0$ and M_1 is totally geodesic. Hence M_1 is isometric to $H^n(c)$,

(2) In the case of M_2, $A = \dfrac{1/\rho^2}{\sqrt{1/\rho^2 + 1/r^2}} I_n$, where I_n denotes the identity matrix of degree n, and M_2 is isometric to $H^n(-1/(r^2 + \rho^2))$,

(3) In the case of M_3, $A = \dfrac{1}{\rho} I_n$ and M_3 is isometric to a Euclidean space R^n,

(4) In the case of M_4, $A = \sqrt{1/\rho^2 + 1/r^2} I_n$ and M_4 is isometric to a round sphere $S^n(r)$ of radius r,

(5) In the case of M_5, $A = \lambda I_k \oplus \mu I_{n-k}$, where

$$\lambda = \sqrt{1/\rho^2 + 1/r^2} \text{ and } \mu = \frac{1/\rho^2}{\sqrt{1/\rho^2 + 1/r^2}},$$

and M_5 is isometric to $S^k(r) \times H^{n-k}(-1/(\rho^2 + r^2))$.

The following generalized maximum principle due to Omori [10] and Yau [14] will be used in order to prove our theorems:

Generalized Maximum Principle. (see Omori [10] and Yau [14]) *Let M^n be a complete Riemannian manifolds whose Ricci curvature is bounded* *from the below and $f \in C^2(M)$ a funtion bounded from the above on M^n. Then, for any $\epsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \epsilon,\ |\mathrm{grad} f|(p) < \epsilon \text{ and } \triangle f(p) < \epsilon.$$

3. Proofs of Theorems

Proof of Theorem 3. We first compute the Laplacian $\triangle S_2$ and show that $\triangle S_2 \geq 0$. In the case of $p \geq 3$, from Lemma 1, we have

$$\begin{aligned} \frac{1}{2}\triangle S_2 &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h^\alpha_{ijk})^2 + nc\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}(h^\alpha_{ij})^2 \\ &+ nH\sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}A_\alpha^2) - \sum_{\alpha=n+2}^{n+p}[\mathrm{trace}(A_{n+1}A_\alpha)]^2 \\ &+ \sum_{\alpha\beta=n+2}^{n+p}\mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{trace}(A_\alpha A_\beta)]^2 \\ &+ \sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}A_\alpha)^2 - \sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}^2A_\alpha^2). \end{aligned} \tag{9}$$

According to Lemma 5 and the definition of S_2, we obtain

$$\sum_{\alpha,\beta=n+2}^{n+p}\mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{trace}(A_\alpha A_\beta)]^2 \geq -\frac{3}{2}S_2^2. \tag{10}$$

Since $e_{n+1} = h/H$, we get $\mathrm{trace}A_\alpha = 0$ for $\alpha = n+2, \ldots, n+p$ and $\mathrm{trace}A_{n+1} = nH$. Hence we have

$$\begin{aligned} &- \sum_{\alpha=n+2}^{n+p}[\mathrm{trace}(A_{n+1}A_\alpha)]^2 + \sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}A_\alpha)^2 - \sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}^2A_\alpha^2) \\ &= \sum_{\alpha=n+2}^{n+p}[-[\mathrm{trace}(A_{n+1}A_\alpha)]^2 + \mathrm{trace}(A_{n+1}A_\alpha)^2 - \mathrm{trace}(A_{n+1}^2A_\alpha^2)] \\ &= \sum_{\alpha=n+2}^{n+p}[-[\mathrm{trace}\{(A_{n+1} - HI)A_\alpha\}]^2 + H\mathrm{trace}A_\alpha \end{aligned}$$

$$
\begin{aligned}
& +\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}^2+2H\mathrm{trace}(A_{n+1}A_\alpha^2)-H^2\mathrm{trace}A_\alpha^2 \\
& -\mathrm{trace}\{(A_{n+1}-HI)^2A_\alpha^2\}-2H\mathrm{trace}(A_{n+1}A_\alpha^2)+H^2\mathrm{trace}A_\alpha^2] \\
= & \sum_{\alpha=n+2}^{n+p}[-[\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}]^2+\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}^2 \\
& -\mathrm{trace}\{(A_{n+1}-HI)^2A_\alpha^2\}],
\end{aligned}
$$

where I denotes the identity matrix.

For a fixed $\alpha, n+2 \leq \alpha \leq n+p$, we can take a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. Thus we obtain $\mathrm{trace}A_\alpha = \sum_{i=1}^n \lambda_i^\alpha = 0$ and $\mathrm{trace}A_\alpha^2 = \sum_{i=1}^n (\lambda_i^\alpha)^2$. Let $B := A_{n+1} - HI = (b_{ij})$. Then we get $\sum_{i=1}^n b_{ii} = 0$, $\sum_{i,j=1}^n b_{ij}^2 = S_1$ and $b_{ij} = b_{ji}$ for any $i, j = 1, \ldots, n$. Hence we have

$$
\begin{aligned}
- & \ [\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}]^2+\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}^2 \\
& -\mathrm{trace}\{(A_{n+1}-HI)^2A_\alpha^2\} \\
= & -[\mathrm{trace}(BA_\alpha)]^2+\mathrm{trace}(BA_\alpha)^2-\mathrm{trace}(B^2A_\alpha^2) \\
= & -(\sum_{i=1}^n b_{ii}\lambda_i^\alpha)^2+\sum_{i,j=1}^n b_{ij}^2\lambda_i^\alpha\lambda_j^\alpha-\sum_{i,j=1}^n b_{ij}^2(\lambda_i^\alpha)^2.
\end{aligned}
$$

Since λ_i^α and b_{ij} satisfy the conditions in Lemma 2 for $i, j = 1, \ldots, n$, we get

$$
\begin{aligned}
- & \ [\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}]^2+\mathrm{trace}\{(A_{n+1}-HI)A_\alpha\}^2 \\
& -\mathrm{trace}\{(A_{n+1}-HI)^2A_\alpha^2\} \\
\geq & -S_1\mathrm{trace}A_\alpha^2.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
- & \sum_{\alpha=n+2}^{n+p}[\mathrm{trace}(A_{n+1}A_\alpha)]^2+\sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}A_\alpha)^2 \\
& -\sum_{\alpha=n+2}^{n+p}\mathrm{trace}(A_{n+1}^2A_\alpha^2) \qquad (11) \\
= & \sum_{\alpha=n+2}^{n+p}[-[\mathrm{trace}(A_{n+1}A_\alpha)]^2+\mathrm{trace}(A_{n+1}A_\alpha)^2-\mathrm{trace}(A_{n+1}^2A_\alpha^2)]
\end{aligned}
$$

$$\begin{aligned} &\geq -S_1 \sum_{\alpha=n+2}^{n+p} \mathrm{trace} A_\alpha^2 \\ &= -S_1 S_2. \end{aligned}$$

By making use of the same assertion as above we have

$$\begin{aligned} nH \sum_{\alpha=n+2}^{n+p} \mathrm{trace}(A_{n+1}A_\alpha^2) &= nH \sum_{\alpha=n+2}^{n+p} \mathrm{trace}\{(A_{n+1} - HI)A_\alpha^2\} \\ &\quad + nH^2 \sum_{\alpha=n+2}^{n+p} \mathrm{trace} A_\alpha^2 \\ &= nH \sum_{\alpha=n+2}^{n+p} \mathrm{trace}\{(A_{n+1} - HI)A_\alpha^2\} + nH^2 S_2 \\ &= nH \sum_{\alpha=n+2}^{n+p} \sum_{i=1}^{n} b_{ii}(\lambda_i^\alpha)^2 + nH^2 S_2. \end{aligned}$$

From Lemma 3 and Lemma 4 we obtain

$$\begin{aligned} \sum_{i=1}^{n} b_{ii}(\lambda_i^\alpha)^2 &\geq -\sqrt{\sum_{i=1}^{n}(\lambda_i^\alpha)^4 - \frac{1}{n}(\sum_{i=1}^{n}(\lambda_i^\alpha)^2}\sqrt{\sum_{i=1}^{n} b_{ii}^2} \\ &\geq -\frac{n-2}{\sqrt{n(n-1)}}\sqrt{S_1}\mathrm{trace} A_\alpha^2. \end{aligned}$$

Thus we get

$$nH \sum_{\alpha=n+2}^{n+p} \mathrm{trace}(A_{n+1}A_\alpha^2) \geq -\frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S_1}S_2 + nH^2 S_2. \tag{12}$$

From (9)-(12) we have

$$\begin{aligned} \frac{1}{2}\triangle S_2 &\geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 + ncS_2 \\ &\quad + S_2(nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S_1}) - S_1 S_2 - \frac{3}{2}S_2^2 \\ &= \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 + S_2\{-\frac{3}{2}(S_1 + S_2) - \frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S_1} \end{aligned} \tag{13}$$

$$
\begin{aligned}
& +n(H^2+c)+\frac{1}{2}S_1\} \\
= & \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h^\alpha_{ijk})^2+S_2\{-\frac{3}{2}|\phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S-nH^2-S_2} \\
& +n(H^2+c)+\frac{1}{2}S_1\} \\
\geq & \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h^\alpha_{ijk})^2+S_2\{-\frac{3}{2}|\phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|+n(H^2+c)\} \\
= & \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h^\alpha_{ijk})^2+S_2\{-Q_H(|\phi|)\}
\end{aligned}
$$

Under our assumption $|\phi|^2 \leq B_H$, i.e., $-Q_H(|\phi|) \geq 0$ we get

$$
\frac{1}{2}\triangle S_2 \geq 0.
$$

Next, we show that the Ricci curvature R_{ii} of M^n is bounded from the below and S_2 is bounded from the above, when $|\phi|^2 \leq B_H$. From (3) and (8) we obtain

$$
\begin{aligned}
R_{ii} &= (n-1)c+\sum_{\alpha=n+1}^{n+p}\sum_{k=1}^{n}\{h^\alpha_{kk}h^\alpha_{ii}-(h^\alpha_{ik})^2\} \\
&= (n-1)c+\sum_{\alpha=n+1}^{n+p}(h^\alpha_{ii}\sum_{k=1}^{n}h^\alpha_{kk})-\sum_{\alpha=n+1}^{n+p}\sum_{k=1}^{n}(h^\alpha_{ik})^2 \\
&= (n-1)c+h^{n+1}_{ii}\sum_{k=1}^{n}h^{n+1}_{kk}-\sum_{\alpha=n+1}^{n+p}\sum_{k=1}^{n}(h^\alpha_{ik})^2 \\
&= (n-1)c+nHh^{n+1}_{ii}-\sum_{\alpha=n+1}^{n+p}\sum_{k=1}^{n}(h^\alpha_{ik})^2 \\
&\geq (n-1)c+nHh^{n+1}_{ii}-\sum_{\alpha=n+1}^{n+p}\sum_{i,k=1}^{n}(h^\alpha_{ik})^2 \\
&= (n-1)c+nHh^{n+1}_{ii}-(|\phi|^2+nH^2) \\
&\geq (n-1)c+nHh^{n+1}_{ii}-B_H-nH^2.
\end{aligned}
$$

and $S_2=|\phi|^2-S_1 \leq B_H-S_1 \leq B_H$, respectively. Since H is nonzero constant, there exist real numbers c_1, c_2 such that

$$R_{ii} \geq c_1 \text{ and } S_2 \leq c_2.$$

Since the Ricci curvature of M^n and S_2 satisfy the above conditons, we can apply the generalized maximum principle due to Omori [10] and Yau [14] to the function S_2. Then there exists a sequence $\{p_\ell\}$ in M^n such that

$$\lim_{\ell\to\infty} S_2(p_\ell) = \sup S_2 \text{ and } \lim_{\ell\to\infty} \sup \triangle S_2(p_\ell) \leq 0.$$

Hence we have

$$0 \geq \frac{1}{2}\triangle S_2 \geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 + S_2\{-Q_H(|\phi|)\} \geq 0.$$

Thus we get

$$\sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 = 0$$

on M^n. Since the equalities of (13) hold, we obtain

$$S_1 = S_2 = 0, \text{ i.e., } |\phi|^2 = 0$$

on M^n.

On the other hand, from (4), we have

$$\begin{aligned}
\sum_{i,k=1}^{n} h_{iik}^\alpha \omega_k &= \sum_{i=1}^{n} dh_{ii}^\alpha - \sum_{i,k=1}^{n} h_{ik}^\alpha \omega_{ki} - \sum_{i,k=1}^{n} h_{ik}^\alpha \omega_{ki} - \sum_{\beta=n+1}^{n+p} \sum_{i=1}^{n} h_{ii}^\beta \omega_{\beta\alpha} \\
&= -2\sum_{i,k=1}^{n} h_{ik}^\alpha \omega_{ki} - \sum_{i=1}^{n} h_{ii}^{n+1} \omega_{(n+1)\alpha} \\
&= -nH\omega_{(n+1)\alpha}
\end{aligned}$$

for any $\alpha, n+2 \leq \alpha \leq n+p$. From $H \neq 0$ and the above equality, we infer $\omega_{(n+1)\alpha} = 0$. Thus we know that e_{n+1} is parallel in the normal bundle $T^\perp M$ of M^n. Hence, if we denote by N_1 the normal subbundle spanned by e_α of the normal bundle of M^n, then M^n is totally geodesic with respect to N_1. Since the e_{n+1} is parallel in the normal bundle, we know that the normal subbundle N_1 is invariant under parallel translation with respect to the norma connection of M^n. Then, from Theorem 1 in [15], we get that M^n lies in a totally geodesic submanifold $S^{n+1}(c)$ of $S^{n+p}(c)$. This completes the proof of Theorem 3.

Proof of Theorem 4. From (7) we have

$$\begin{aligned}\frac{1}{2}S &= \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i,j=1}^{n} h_{ij}\triangle h_{ij}\\ &= \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i,j,k=1}^{n} h_{ij}h_{kkij} + \sum_{i,j,k,m=1}^{n} h_{ij}h_{km}R_{mijk}\\ &\quad + \sum_{i,j,k,m=1}^{n} h_{ij}h_{mi}R_{mkjk}.\end{aligned}$$

We choose a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $h_{ij}=\lambda_i\delta_{ij}$. Then, by a standar and direct computation, we get

$$\frac{1}{2}\triangle S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i=1}^{n}\lambda_i(nH)_{ii} + \frac{1}{2}\sum_{i,j=1}^{n} R_{ijij}(\lambda_i-\lambda_j)^2, \tag{14}$$

where $R_{ijij}=-1+\lambda_i\lambda_j(i\neq j)$ denotes the sectional curvature of the secton spanned by $\{e_i,e_j\}$.

We first compute the last term oon the right-hand side of (14). Since $R_{ijij}=-1+\lambda_i\lambda_j$, we have

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^{n}(-1+\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2 &= -n\sum_{i=1}^{n}\lambda_i^2 + (\sum_{i=1}^{n}\lambda_i)^2 - (\sum_{i=1}^{n}\lambda_i^2)^2\\ &\quad +(\sum_{j=1}^{n}\lambda_j)(\sum_{i=1}^{n}\lambda_i^3)\\ &= -nS + n^2H^2 - S^2 + nH\sum_{i=1}^{n}\lambda_i^3.\end{aligned}$$

Let $\{e_1,\dots,e_n\}$ be an orthonormal frame which diagonalizes ϕ at each point of M^n, that is, $\phi e_i=\mu_i e_i$. By the definiton of ϕ we get $\mu_i=H-\lambda_i$. Then we obtain

$$|\phi|^2 = \sum_{i=1}^{n}\mu_i^2 = \sum_{i=1}^{n}(H-\lambda_i)^2 = S - nH^2$$

and

$$\sum_{i=1}^{n}\lambda_i^3 = \sum_{i=1}^{n}(H-\mu_i)^3$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(H^3 + 3\mu_i^2 H - 3\mu_i H^2 - \mu_i^3) \\
&= nH^3 + 3H\sum_{i=1}^{n}\mu_i^2 - \sum_{i=1}^{n}\mu_i^3 \\
&= nH^3 + 3H|\phi|^2 - \sum_{i=1}^{n}\mu_i^3.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j=1}^{n} R_{ijij}(\lambda_i - \lambda_j)^2 =& - nS + n^2H^2 - S^2 \\
& + nH(nH^3 + 3H|\phi|^2 - \sum_{i=1}^{n}\mu_i^3) \qquad (15) \\
=& - n(S - nH^2) + nH^2(S - nH^2) - (S - nH^2)^2 \\
& - nH\sum_{i=1}^{n}\mu_i^3 \\
=& - n|\phi|^2 + nH^2|\phi|^2 - |\phi|^4 - nH\sum_{i=1}^{n}\mu_i^3 \\
=& n(H^2 - 1)|\phi|^2 - |\phi|^4 - nH\sum_{i=1}^{n}\mu_i^3.
\end{aligned}
$$

Putting (15) in (14), we obtain

$$
\frac{1}{2}\triangle S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i=1}^{n}\lambda_i(nH)_{ii} + n(H^2 - 1)|\phi|^2 - |\phi|^4 - nH\sum_{i=1}^{n}\mu_i^3.
$$

Since the assumption of Theorem 4 that the mean curvature H is constant, we get

$$
\frac{1}{2}\triangle S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + n(H^2 - 1)|\phi|^2 - |\phi|^4 - nH\sum_{i=1}^{n}\mu_i^3. \qquad (16)
$$

By using Lemma 6 in (16) we have

$$
\frac{1}{2}\triangle S \quad \geq \quad \sum_{i,j,k=1}^{n} h_{ijk}^2 + n(H^2 - 1)|\phi|^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3
$$

$$
\begin{aligned}
&= \sum_{i,j,k=1}^{n} h_{ijk}^2 + |\phi|^2\{-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(H^2-1)\} \\
&= \sum_{i,j,k=1}^{n} h_{ijk}^2 + |\phi|^2\{-P_H(|\phi|)\}.
\end{aligned}
$$

Since $|\phi|^2 \leq A_H$ and H is nonzero constant, we can apply the generalized maximum princple due to Omori [10] and Yau [14] to the function S by the similar way with one of the proof of Theorem 3. Hence we get

$$0 \geq \frac{1}{2}\triangle S.$$

Under our assumption $|\phi|^2 \leq A_H$, i.e., $-P_H(|\phi|) \geq 0$ we obtain

$$0 \geq \frac{1}{2}\triangle S \geq \sum_{i,j,k=1}^{n} h_{ijk}^2 + |\phi|^2\{-P_H(|\phi|)\} \geq 0.$$

Thus we get $\sum_{i,j,k=1}^{n} h_{ijk}^2 = 0$ and either $|\phi|^2 \equiv 0$ or $|\phi|^2 \equiv A_H$. This proves the part (i) of Theorem 4.

We now consider the part (ii). Since $\sum_{i,j,k=1}^{n} h_{ijk} = 0$ and it holds the equality of the right-hand side of Lemma 6, after renumbering if necessary, we can assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}.\lambda_1 \neq \lambda_n \text{ and } \lambda_i = \text{constant}.$$

From Remark 6 we obtain

$$\mu_1 = \mu_2 = \cdots = \mu_{n-1} = -\frac{1}{\sqrt{n(n-1)}}|\phi| \text{ and } \mu_n = \sqrt{\frac{n-1}{n}}|\phi|.$$

Hence we have

$$
\begin{aligned}
\lambda_n\lambda_1 &= (H - \sqrt{\frac{n-1}{n}}|\phi|)(H + \frac{1}{\sqrt{n(n-1)}}|\phi|) \\
&= H^2 - \frac{1}{n}|\phi|^2 - \frac{n-2}{\sqrt{n(n-1)}}H|\phi|.
\end{aligned}
$$

Since $|\phi|^2 \equiv A_H$, we get

$$n\lambda_n\lambda_1 = nH^2 - |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| = n,$$

that is, $\lambda_n\lambda_1 = 1$. On the other hand, from

$$\lambda_n = H - \mu_n = \frac{\lambda_n + (n-1)\lambda_1}{n} - \mu_n,$$

we have

$$(n-1)(\lambda_1 - \lambda_n) = n\mu_n.$$

Since $\mu_n > 0$, we get $\lambda_n < \lambda_1$. From (5) of Lemma 7 we suppose that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \nu = \frac{1/\rho^2}{\sqrt{1/\rho^2 + 1/r^2}} \text{ and } \lambda_n = \lambda = \sqrt{1/\rho^2 + 1/r^2}.$$

Since $\lambda_n\lambda_1 = 1$, we obtain $\rho^2 = 1$. However, it is contradiction in $\lambda < \lambda_1$. Thus we set

$$\lambda_n = \nu = \frac{1}{\sqrt{1 + 1/r^2}} \text{ and } \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda = \sqrt{1 + 1/r^2}.$$

Then it yields $\lambda < \lambda_1$. Therefore we get that M^n is isometric to $S^{n-1} \times H^(- 1/(r^2+1))$. If $r > 0$, then the mean curvature holds

$$H = \frac{\lambda_n + (n-1)\lambda_1}{n} = \frac{n(r^2+1) - 1}{nr\sqrt{r^2+1}} > 1,$$

which satisfies the assumption of Thoerem 4. Hence it must be $r > 0$.

This finishes our proof of Theorem 4.

Proof of Theorem 5. We prove the Thoerem 5 by the similar argument with the proof of Theorem 3. In the case of $p = 2$, since

$$\begin{aligned} \frac{1}{2}\triangle S_2 \geq & \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-(S_1 + S_2) \\ & - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} + n(H^2 - 1)\} \qquad (17) \\ = & \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-|\phi|^2 \end{aligned}$$

$$- \frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S - nH^2 - S_2} + n(H^2-1)\}$$
$$\geq \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S - nH^2} + n(H^2-1)\}$$
$$= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| + n(H^2-1)\}$$
$$= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-P_H(|\phi|)\}.$$

Under our assumption $|\phi|^2 \leq A_H$, i.e., $-P_H(|\phi|) \geq 0$ we get

$$\sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-P_H(|\phi|)\} \geq 0.$$

Moreover, by the similar way with one of the proof of Theorem 3, we obtain

$$0 \geq \frac{1}{2}\triangle S_2.$$

Hence we have

$$0 \geq \frac{1}{2}\triangle S_2 \geq \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-P_H(|\phi|)\} \geq 0.$$

Thus we get

$$\sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 = 0$$

on M^n. Since the equalities of (17) hold, we obtain

$$S_2 = 0$$

on M^n.

On the other hand, from Theorem 1 in [15], we get that M^n lies in a totally geodesic hypersurface $H^{n+1}(-1)$ of $H^{n+2}(-1)$ by the similar way with one of the proof of Theorem 3. Since H is constant and $S_2 = 0$, we can obtain the result by the similar argument with the proof of Theorem 4. This complets the proof of Theorem 5.

Proof of Theorem 6. In the case for $p \geq 3$, we know that $|\phi|^2$ and that M^n lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$ by the similar argument with the proof of Thoerem 3. This finishes our proof of Theorem 6.

References

[1] H. Alencar, M. do Carmo, Hypersurfaces with constant mean curvature in sphere, *Proc. Amer. Math. Soc.*, **120** (1994), 1223-1229.

[2] B.Y. Chen, *Totally Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore (1984).

[3] Q.M. Cheng, Submanifolds with constant scalar curvature, *Proc. Royal Society Edinbergh*, **132** (2002), 1163-1183.

[4] S.Y. Cheng, S.T. Yau, Hypersurfaces with constant scalar curvaature, *Math. Ann.*, **225** (1977), 195-204.

[5] S.S Chern, M. do Carmo, S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, *Functional Analysis and Related Fields* (1970), 59-75.

[6] H.B. Lawson, Jr., Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.*, **89** (1969), 187-197.

[7] A.M. Li, J.M. Li, An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.*, **58** (1992), 582-594.

[8] X. Liu, W. Su, Hypersurfaces with constant scalar curvature in a hyperbolic space, *Balkan J. of Geo. and Application*, **7** (2002), 121-132.

[9] K. Nomizu, B. Smyth, A formula for Simon's type and hypersurfaces, *J. Diff. Geom.*, **3** (1969), 367-377.

[10] H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc.*, Japan **19** (1967), 205-214.

[11] P.J. Ryan, Hypersurfaces with parallel Ricci tensor, *Osaka J. Math.*, **8** (1971), 251-259.

[12] M. Santos, Submanifods with parallel mean curvature vector in spheres, *Tohoku Math. J.*, **46** (1994), 403-415.

[13] Y. Uchida, Y. Matsuyama, Submanifolds with nonzero mean curvature in a euclidean sphere, *I.J. Pure and Appl. Math.*, **29** (2006), 119-130.

[14] S.T. Yau, Harmonic functions on complete riemannian manifolds, *Comm. Pure and Appl. Math.*, **28** (1975), 201-228.

[15] S.T. Yau, Submanifolds with constant mean curvature, *Amer. J. Math.*, **96** (1974), 346-366.