ON SUBMANIFOLDS WITH CONSTANT MEAN CURVATURE
IN A REAL SPACE FORM

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Abstract: The purpose of this paper is to classify submanifolds with constant mean curvature in a real space form. We put $S$ the squared norm of the second fundamental form and $|\phi|^2 = S - nH^2$. Denote by $A_H$ and $B_H$ the squares of the positive roots of the equations

$$x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c) = 0$$

and

$$\frac{3}{2} x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c) = 0,$$

respectively. We prove the following: First, let $M^n$ be a complete, connected and orientable submanifold with nonzero constant mean curvature $H$ in $S^{n+p}(c)(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic submanifold $S^{n+1}(c)$ of $S^{n+p}(c)$, and $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic. Next, let $M^n$ be a complete, connected and orientable hypersurface with constant mean curvature $H > 1$ in $H^{n+1}(-1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M^n$. Then (i) either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$. (ii) $|\phi|^2 \equiv A_H$ if and only if $M^n$ is isometric to $S^{n-r} \times H^1(-\frac{1}{r^2+1})$ for some $r > 0$. Moreover, we prove a generalization of this result of the hypersurface in a hyperbolic space.

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1. Introduction

Let $\tilde{M}^{n+p}(c)$ be an $(n+p)$-dimensional complete, connected and simply connected Riemannian manifold of constant sectional curvature $c$. We call it a space form. A space form $\tilde{M}^{n+p}(c)$ is one of the following:

(i) If $c > 0$, then $\tilde{M}^{n+p}(c)$ is an $(n+p)$-dimensional Euclidean sphere $S^{n+p}(c)$,

(ii) If $c = 0$, then $\tilde{M}^{n+p}(c)$ is an $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$,

(iii) If $c < 0$, then $\tilde{M}^{n+p}(c)$ is an $(n+p)$-dimensional hyperbolic sphere $\mathbb{H}^{n+p}(c)$.

Let $M^n$ be an $n$-dimensional, connected and orientable submanifold isometrically immersed in $\tilde{M}^{n+p}(c)$. Denote by $h_{ij}^\alpha$ the local component of the second fundamental form for each $1 \leq i, j \leq n, n+1 \leq \alpha \leq n+p$. We set

$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n (h_{ij}^\alpha)^2 \quad \text{and} \quad H := \frac{1}{n} \sqrt{\sum_{\alpha=n+1}^{n+p} \left( \sum_{i=1}^n h_{ii}^\alpha \right)^2}$$

be the squared norm of the second fundamental form and the mean curvature of $M^n$ in $\tilde{M}^{n+p}(c)$, respectively.

Now, we denote by $A_\alpha$ the $n \times n$ matrix of $h_{ij}^\alpha$ with respect to indices $i, j$. Define linear maps $\phi_\alpha : T_x M \rightarrow T_x M$ by

$$< \phi_\alpha X, Y > := \frac{1}{n} \text{trace} A_\alpha < X, Y > - < A_\alpha X, Y > \quad \text{for} \quad n+1 \leq \alpha \leq n+p,$$

where $<,>$ is the Riemannian metric of $M^n$. Moreover, we define the bilinear map $\phi : T_x M \times T_x M \rightarrow T_x^\perp$ by

$$\phi(X,Y) = \sum_{\alpha=n+1}^{n+p} < \phi_\alpha X, Y > e_\alpha,$$

where $\{e_{n+1}, \ldots, e_{n+p}\}$ denotes an orthonormal basis. It is easy to check that

$$\text{trace } \phi = 0 \quad \text{and that} \quad |\phi|^2 := \sum_{\alpha=n+1}^{n+p} \text{trace } \phi_\alpha^2 = S - nH^2.$$
Let
\[ P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c) \]
and
\[ Q_H(x) = \frac{3}{2} x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c) \]
be the polynomials for each real number \( H \in \mathbb{R} \). We put \( A_H \) the square of the positive root of \( P_H(x) = 0 \) and \( B_H \) one of \( Q_H(x) = 0 \).

Besides, in the case of \( p = 1 \), we denote by \( h_{ij} \) the local component of the second fundamental form for each \( i, j (1 \leq i, j \leq n) \) and by \( A \) the \( n \times n \) matrix of \( h_{ij} \) with respect to indices \( i, j \). We choose a local orthonormal frame field \( \{e_1, \ldots, e_n\} \) such that \( h_{ij} = \lambda_i \delta_{ij} \). Then we have
\[ H = \frac{1}{n} \left| \sum_{i=1}^{n} \lambda_i \right| \quad \text{and} \quad S = \sum_{i=1}^{n} \lambda_i^2. \]

In the hypersurface we may put \( \phi = \phi_{n+1} \). Then \( \phi : T_x M \to T_x M \) satisfies
\[ < \phi X, Y > := \frac{1}{n} \text{trace} \ A < X, Y > - < AX, Y >. \]

It easily check that \( \text{trace} \ \phi = 0 \) and that
\[ |\phi|^2 := \text{trace} \ \phi^2 = \frac{1}{2n} \sum_{i,j} (\lambda_i - \lambda_j)^2. \]

Hence we get that \( |\phi|^2 = 0 \) if and only if \( M^n \) is totally umbilic.

We study generalizations of the results of the following theorems. Moreover, we also study in the case of \( c = -1 \).

**Theorem 1.** (see Alencar and do Carmo [1]) Let \( M^n \) be a compact and orientable hypersurface with constant mean curvature \( H \) in \( S^{n+1}(1) \). Assume that \( |\phi|^2 \leq A_H \) for all \( x \in M \). Then:

(i) either \( |\phi|^2 \equiv 0 \) and \( M^n \) is totally umbilic or \( |\phi|^2 \equiv A_H \).

(ii) \( |\phi|^2 \equiv A_H \) if and only if:

(A) \( H = 0 \) and \( M^n \) is a Clifford torus in \( S^{n+1}(1) \), i.e., \( M^n \) is a product of spheres \( S^{n_1}(r_1) \times S^{n_2}(r_2) \), \( n_1 + n_2 = n \), of appropriate radii.

(B) \( H \neq 0, n \geq 3, \) and \( M^n = S^{n-1}(r) \times S^1(\sqrt{1 - r^2}) \) in \( S^{n+1}(1) \), where \( r^2 < \frac{n-1}{n} \).

(C) \( H \neq 0, n = 2, \) and \( M^2 = S^1(r) \times S^1(\sqrt{1 - r^2}) \) in \( S^3(1) \), where \( r^2 \neq \frac{1}{2} \).
Theorem 2. (see Uchida and Matsuyama [13]) Let $M^n$ be a complete, connected and orientable submanifold with nonzero constant mean curvature $H$ in $S^{n+2}(c)$. If $|\phi|$ satisfies $|\phi|^2 \leq A_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic hypersurface $S^{n+1}(c)$ of $S^{n+2}(c)$ and

(i) either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if:

(B) $n \geq 3$, and $M^n = S^{n-1}(r_1) \times S^1(r_2)$ in $S^{n+1}(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 < \frac{n-1}{nc}$.

(C) $n = 2$ and $M^2 = S^1(r_1) \times S^1(r_2)$ in $S^3(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 \neq \frac{1}{2c}$.

Theorem 3. Let $M^n$ be a complete, connected and orientable submanifold with nonzero constant mean curvature $H$ in $S^{n+p}(c)(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic submanifold $S^{n+1}(c)$ of $S^{n+p}(c)$, and $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic.

Theorem 4. Let $M^n$ be a complete, connected and orientable hypersurface with constant mean curvature $H > 1$ in $H^{n+1}(-1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M^n$. Then:

(i) either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if $M^n$ is isometric to $S^1(r) \times H^1(-\frac{1}{r^2 + 1})$ for some $r > 0$.

Theorem 5. Let $M^n$ be a complete, connected and orientable submanifold with constant mean curvature $H > 1$ in $H^{n+2}(-1)$. If $|\phi|$ satisfies $|\phi|^2 \leq A_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic hypersurface $H^{n+1}(-1)$ of $H^{n+2}(-1)$ and:

(i) either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.

(ii) $|\phi|^2 \equiv A_H$ if and only if $M^n$ is isometric to $S^1(r) \times H^1(-\frac{1}{r^2 + 1})$ for some $r > 0$.

Theorem 6. Let $M^n$ be a complete, connected and orientable submanifold with nonzero constant mean curvature $H > 1$ in $H^{n+p}(c)(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$, and $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic.
2. Preliminaries

Let $\tilde{M}^{n+p}(c)$ be an $(n+p)$-dimensional space form of constant curvature $c$ and $M^n$ an $n$-dimensional, complete, connected and orientable submanifold in $\tilde{M}^{n+p}(c)$. We choose a local field of an orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ adapted to the Riemannian metric of $\tilde{M}^{n+p}(c)$ and the dual coframe $\{\omega_1, \ldots, \omega_{n+p}\}$ are tangent to $M^n$. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq n+p, \quad 1 \leq i, j, k, \cdots \leq n,$$

$$n+1 \leq \alpha\beta, \gamma, \cdots \leq n+p.$$

Then the structure equations of $\tilde{M}^{n+p}(c)$ are given by

$$d\omega_A = -\sum_{B=1}^{n+p} \omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_{C=1}^{n+p} \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,C=1}^{n+p} K_{ABCD} \omega_C \wedge \omega_D,$$

with $K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$, where $K_{ABCD}$ denotes the component of the curvature tensor of $\tilde{M}^{n+p}(c)$. We restrict these forms to $M^n$. Then we get

$$\omega_\alpha = 0.$$

Since $0 = d\omega_\alpha = -\sum_{i=1}^{n} \omega_{\alpha i} \wedge \omega_i$, by Cartan’s Lemma, we may write

$$\omega_{\alpha i} = \sum_{j=1}^{n} h_{ij}^\alpha \omega_j, h_{ij}^\alpha = h_{ji}^\alpha.$$  \hfill (1)

From these formulas, we obtain the structure equations of $M^n$:

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,\ell}^{n} R_{ijk\ell} \omega_k \wedge \omega_\ell.$$
with

\[ R_{ijk\ell} = c(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p} (h_{i\ell}^{\alpha}h_{j\ell}^{\alpha} - h_{i\alpha}^{\alpha}h_{j\alpha}^{\alpha}), \]  

(2)

where \( R_{ijk\ell} \) is the component of the curvature tensor of \( M^n \). Denoting by \( R_{jk} \) the component of the Ricci curvature of \( M^n \), from (2), we have

\[ R_{jk} = (n-1)c\delta_{jk} + \sum_{\alpha=n+1}^{n+p} \left( \sum_{i=1}^{n} h_{i\ell}^{\alpha}h_{j\ell}^{\alpha} - \sum_{i=1}^{n} h_{i\ell}^{\alpha}h_{j\ell}^{\alpha} \right). \]  

(3)

We also have

\[ d\omega_{\alpha\beta} = -\sum_{\gamma=1}^{n+p} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2} \sum_{i,j=1}^{n} R_{\alpha\beta ij} \omega_i \wedge \omega_j, \]

where

\[ R_{\alpha\beta ij} = \sum_{\ell=1}^{n} (h_{i\ell}^{\alpha}h_{j\ell}^{\beta} - h_{i\ell}^{\alpha}h_{j\ell}^{\beta}). \]

The Riemannian connection of \( M^n \) is defined by \((\omega_{ij})\). The form \((\omega_{\alpha\beta})\) defines a connection in the normal bundle of \( M^n \). The second fundamental form \( \Pi \) and the mean curvature vector \( h \) of \( M^n \) are defined by

\[ \Pi := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h_{i\ell}^{\alpha}\omega_{j}\omega_{i}^{\ell} e_\alpha \text{ and } h := \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \left( \sum_{i=1}^{n} h_{ii}^{\alpha} \right) e_\alpha, \]

respectively. On the other hand, the mean curvature \( H \) of \( M^n \) is defined by

\[ H := \frac{1}{n} \sqrt{\sum_{\alpha=n+1}^{n+p} \left( \sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2} \]

(see Introduction). We take the exterior differentiation of (1) and define \( h_{ijk}^{\alpha} \) by

\[ \sum_{k=1}^{n} h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} - \sum_{k=1}^{n} h_{ik}^{\alpha}\omega_k - \sum_{k=1}^{n} h_{jk}^{\alpha}\omega_k - \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta}\omega_{\beta}. \]  

(4)

By straightforward computations we obtain the Codazzi equation

\[ h_{ijk}^{\alpha} = h_{ikj}^{\alpha}. \]  

(5)
Similarly, we take the exterior differentiation of (4) and define $h_{ijkl}^\alpha$ by
\[
\sum_{\ell=1}^{n} h_{ijkl}^\alpha \omega_{\ell} = dh_{ijk}^\alpha - \sum_{\ell=1}^{n} h_{ijkl}^\alpha \omega_{\ell} - \sum_{\ell=1}^{n} h_{ij\ell k}^\alpha \omega_{\ell j} - \sum_{\ell=1}^{n} h_{ij\ell k}^\alpha \omega_{\ell k} - \sum_{\beta=n+1}^{n+p} h_{ij\ell k}^\beta \omega_{\beta \alpha}.
\]
Then the Ricci formula for the second fundamental form is given by
\[
h_{ijkl}^\alpha - h_{ij\ell k}^\alpha = \sum_{m=1}^{n} h_{mj}^\alpha R_{kol\ell} + \sum_{m=1}^{n} h_{mjk}^\alpha R_{mij\ell} + \sum_{\beta=n+1}^{n+p} h_{ij}^\beta R_{\beta\alpha k\ell}. \tag{6}
\]
The Laplacian $\triangle h_{ij}^\alpha$ of $h_{ij}^\alpha$ is defined by
\[
\triangle h_{ij}^\alpha := \sum_{k=1}^{n} h_{ij\ell k}^\alpha.
\]
From the Codazzi equation (5) we obtain
\[
\sum_{k=1}^{n} h_{ij\ell k}^\alpha = \sum_{k=1}^{n} h_{kij}^\alpha
\]
for any $\alpha, n+1 \leq \alpha \leq n+p$. Moreover, using the Ricci formula (6), we have
\[
\triangle h_{ij}^\alpha = \sum_{k=1}^{n} h_{kij}^\alpha \tag{7}
\]
\[
= \sum_{k=1}^{n} h_{kij}^\alpha + \sum_{k,m=1}^{n} h_{km}^\alpha R_{mijk} + \sum_{k,m=1}^{n} h_{mk}^\alpha R_{mkij} + \sum_{k=1}^{n} \sum_{\beta=n+1}^{n+p} h_{kij}^\beta R_{\beta\alpha kij}
\]
\[
= \sum_{k=1}^{n} h_{kij}^\alpha + \sum_{k,m=1}^{n} h_{km}^\alpha R_{mijk} + \sum_{k,m=1}^{n} h_{mk}^\alpha R_{mkij} + \sum_{k=1}^{n} \sum_{\beta=n+1}^{n+p} h_{kij}^\beta R_{\beta\alpha kij}.
\]
Let
\[
S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^\alpha)^2
\]
denote the squared norm of the second fundamental form of $M^n$ (see Introduction). Assuming the mean curvature vector $h \neq 0$ on $M^n$, we know that $e_{n+1} = h/H$ is a normal vector field defined globally on $M^n$. We define $S_1$ and $S_2$ by
\[
S_1 := \sum_{i,j=1}^{n} (h_{ij}^{n+1} - H \delta_{ij})^2 \quad \text{and} \quad S_2 := \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^\alpha)^2,
\]
respectively. Then $S_1$ and $S_2$ are functions defined on globally and they are independent of the choice of the orthonormal frame $\{e_1, \ldots, e_n\}$. Also we have

$$|\phi|^2 = S - nH^2 = S_1 + S_2. \tag{8}$$

From the definition of the mean curvature vector $h$ we know that

$$nH = \sum_{i=1}^{n} h_{ii}^{n+1} \text{ and } \sum_{i=1}^{n} h_{ii}^{\alpha} = 0$$

for $n + 2 \leq \alpha \leq n + p$.

We establish the following lemmas for the proofs of theorems:

**Lemma 1.** (see Cheng [3]) Let $M^n$ be an $n$–dimensional submanifold with the mean curvature vector $h \neq 0$ in a space form $\tilde{M}^{n+p}(c)$. Then we have

$$\frac{1}{2} \triangle S_2 = \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 + nc \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^\alpha)^2$$

$$+ nH \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1} A_{\alpha}^2) - \sum_{\alpha=n+2}^{n+p} [\text{trace}(A_{n+1} A_{\alpha})]^2$$

$$+ \sum_{\alpha,\beta=n+2}^{n+p} \text{trace}(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p} [\text{trace}(A_{\alpha} A_{\beta})]^2$$

$$+ \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1} A_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1} A_{\alpha}^2).$$

**Lemma 2.** (see Cheng [3]) Let $a_i$ and $b_{ij}$ be real numbers satisfying

$$\sum_{i=1}^{n} a_i = 0, \sum_{i=1}^{n} b_{ii} = 0, \sum_{i,j=1}^{n} b_{ij}^2 = b$$

and $b_{ij} = b_{ji}$ for $i, j = 1, \ldots, n$. Then we obtain

$$-(\sum_{i=1}^{n} b_{ij} a_i)^2 + \sum_{i,j=1}^{n} b_{ij}^2 a_i a_j - \sum_{i,j=1}^{n} b_{ij}^2 a_i^2 \leq -\sum_{i=1}^{n} a_i^2 b.$$
Lemma 3. (see Cheng [3]) Let $b_i$ be real numbers such that $\sum_{i=1}^{n} b_i = 0$ and $\sum_{i=1}^{n} b_i^2 = B$ for $i = 1, \ldots, n$. Then we have

$$\sum_{i=1}^{n} b_i^4 - \frac{B^2}{n} \leq \frac{(n-2)^2}{n(n-1)} B^2.$$ 

Lemma 4. (see Cheng [3]) Let $a_i$ and $b_i$ be real numbers satisfying $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i^2 = a$ for $i = 1, \ldots, n$. Then we obtain

$$\sum_{i=1}^{n} a_i b_i^2 \geq -\sqrt{\sum_{i=1}^{n} b_i^4 - \frac{1}{n} (\sum_{i=1}^{n} b_i^2)^2} \sqrt{a}.$$ 

Lemma 5. (see Li and Li [7]) For symmetric matrices $A_1, \ldots, A_p (p \geq 2)$, we put

$$S_{\alpha\beta} = \text{trace}(A_\alpha B_\beta), S = \sum_{\alpha=1}^{p} S_{\alpha\alpha} \text{ and } \text{trace}A_\alpha^2 = \text{trace}(t A_\alpha A^\alpha).$$

Then we have

$$- \sum_{\alpha,\beta=1}^{p} \text{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 + \sum_{\alpha,\beta=1}^{p} S_{\alpha\beta}^2 \leq \frac{3}{2} S^2,$$

and the equality holds if and only if one of the following conditions holds:

1. $A_1 = A_2 = \cdots = A_p = 0$.

2. Only two of the matrices $A_1, \ldots, A_p$ are different from zero. Moreover, assuming $A_1 \neq 0, A_2 \neq 0$ and $A_3 = \cdots = A_p = 0$, then $S_{11} = S_{22}$ and there exists an orthonormal matrix $T$ such that

$$tTA_1T = \sqrt{\frac{1}{2} S_{11}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$
and

\[ T^*A_2T = \sqrt{\frac{1}{2} S_{11}} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \]

**Lemma 6.** (see Alencar and do Carmo [1]) Let \( \{\mu_i\}_{i=1}^n \) be a set of real numbers satisfying \( \sum_{i=1}^n \mu_i = 0 \) and \( \sum_{i=1}^n \mu_i^2 = \beta^2 \), where \( \beta \geq 0 \). Then we obtain

\[-\frac{n-2}{\sqrt{n(n-1)}} \beta^3 \leq \sum_{i=1}^n \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} \beta^3,\]

and the equality holds in the right-hand (resp. left-hand) side if and only if \( (n-1) \) of the \( \mu_i \)'s are non-positive and equal (resp. \( (n-1) \) of the \( \mu_i \)'s are non-negative and equal).

**Remark 6.** (see Alencar and do Carmo [1]) It is convenient to observe from the proof that the equality holds in the right-hand side if and only if \( (n-1) \mu_i \)'s are of the form \( -\frac{1}{\sqrt{n(n-1)}} \beta \) and the remaining one is \( \sqrt{\frac{n-1}{n}} \beta \).

In order to represent our theorems, we need some notations, for detail, see Lawson [6], Ryan [11] or Liu [8]. We give a description of the real hyperbolic space \( H^{n+1}(c) \) of constant curvature \( c(<0) \). For any two vectors \( x \) and \( y \) in \( R^{n+2} \), we set

\[ g(x, y) = \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2}. \]

\( (R^{n+2}, g) \) is the so-called Minkowski space. Denote \( \rho = \sqrt{-1/c} \). We define

\[ H^{n+1}(c) = \{ x \in R^{n+2} | g(x, x) = -\rho^2, x_{n+2} > 0 \}. \]

Then \( H^{n+1}(c) \) is a connected and simply connected hypersurface of \( R^{n+2} \). Hence we obtain models of a real hyperbolic space. We define

\[ M_1 = \{ x \in H^{n+1}(c) | x_1 = 0 \}, \]
\[ M_2 = \{ x \in H^{n+1}(c) | x_1 = r > 0 \}, \]
\[ M_3 = \{ x \in H^{n+1}(c) | x_{n+2} = x_{n+1} + \rho \}, \]
\[
M_4 = \{ x \in H^{n+1}(c) \mid \sum_{i=1}^{n+1} x_i^2 = r^2 > 0 \},
\]
\[
M_5 = \{ x \in H^{n+1}(c) \mid \sum_{i=1}^{k+1} x_i^2 = r^2 > 0, \sum_{j=k+2}^{n+2} x_j^2 - x_{n+2}^2 = -\rho^2 - r^2 \}.
\]

\(M_1, \ldots, M_5\) are often called the standard examples of complete hypersurfaces in \(H^{n+1}(c)\) with at most two distinct constant principal curvatures. It is obvious that \(M_1, \ldots, M_4\) are totally umbilic. In the sense of Chen [2], they are called the hyperspheres of \(H^{n+1}(c)\). \(M_3\) is called the horosphere and \(M_4\) the geodesic sphere of \(H^{n+1}(c)\). Ryan [11] obtained the following:

**Lemma 7.** (see Ryan [11]) Let \(M^n\) be a complete hypersurface in \(H^{n+1}(c)\). Suppose that, under a suitable choice of a local orthonormal tangent frame field of \(TM^n\), the shape operator over \(TM^n\) is expressed as a matrix \(A\). If \(M^n\) has at most two distinct constant principal curvatures, then it is congruent to one of the following:

1. In the case of \(M_1\), \(A = 0\) and \(M_1\) is totally geodesic. Hence \(M_1\) is isometric to \(H^n(c)\),
2. In the case of \(M_2\), \(A = \frac{1/\rho^2}{\sqrt{1/\rho^2 + 1/r^2}} I_n\), where \(I_n\) denotes the identity matrix of degree \(n\), and \(M_2\) is isometric to \(H^n(-1/(r^2 + \rho^2))\),
3. In the case of \(M_3\), \(A = \frac{1}{\rho} I_n\) and \(M_3\) is isometric to a Euclidean space \(R^n\),
4. In the case of \(M_4\), \(A = \sqrt{1/\rho^2 + 1/r^2} I_n\) and \(M_4\) is isometric to a round sphere \(S^n(r)\) of radius \(r\),
5. In the case of \(M_5\), \(A = \lambda I_k \oplus \mu I_{n-k}\), where

\[
\lambda = \sqrt{1/\rho^2 + 1/r^2} \quad \text{and} \quad \mu = \frac{1/\rho^2}{\sqrt{1/\rho^2 + 1/r^2}}
\]

and \(M_5\) is isometric to \(S^k(r) \times H^{n-k}(-1/(\rho^2 + r^2))\).

The following generalized maximum principle due to Omori [10] and Yau [14] will be used in order to prove our theorems:

**Generalized Maximum Principle.** (see Omori [10] and Yau [14]) Let \(M^n\) be a complete Riemannian manifolds whose Ricci curvature is bounded
from the below and \( f \in C^2(M) \) a function bounded from the above on \( M^n \). Then, for any \( \epsilon > 0 \), there exists a point \( p \in M^n \) such that
\[
f(p) \geq \sup f - \epsilon, \quad |\nabla f|(p) < \epsilon \quad \text{and} \quad \Delta f(p) < \epsilon.
\]

3. Proofs of Theorems

Proof of Theorem 3. We first compute the Laplacian \( \Delta S_2 \) and show that \( \Delta S_2 \geq 0 \). In the case of \( p \geq 3 \), from Lemma 1, we have
\[
\frac{1}{2} \Delta S_2 = \sum_{\alpha=n+1}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^{\alpha})^2 + nc \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2
\]
\[
+ nH \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}A_{\alpha}^2) - \sum_{\alpha=n+2}^{n+p} [\text{trace}(A_{n+1}A_{\alpha})]^2
\]
\[
+ \sum_{\alpha,\beta=n+2}^{n+p} \text{trace}(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})^2 - \sum_{\alpha,\beta=n+2}^{n+p} [\text{trace}(A_{\alpha}A_{\beta})]^2
\]
\[
+ \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}A_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}^2 A_{\alpha}^2).
\]

According to Lemma 5 and the definition of \( S_2 \), we obtain
\[
\sum_{\alpha,\beta=n+2}^{n+p} \text{trace}(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})^2 - \sum_{\alpha,\beta=n+2}^{n+p} [\text{trace}(A_{\alpha}A_{\beta})]^2 \geq -\frac{3}{2} S_2. \tag{10}
\]

Since \( e_{n+1} = h/H \), we get \( \text{trace}A_{\alpha} = 0 \) for \( \alpha = n+2, \ldots, n+p \) and \( \text{trace}A_{n+1} = nH \). Hence we have
\[
- \sum_{\alpha=n+2}^{n+p} [\text{trace}(A_{n+1}A_{\alpha})]^2 + \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}A_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}^2 A_{\alpha}^2)
\]
\[
= \sum_{\alpha=n+2}^{n+p} [-[\text{trace}(A_{n+1}A_{\alpha})]^2 + \text{trace}(A_{n+1}A_{\alpha})^2 - \text{trace}(A_{n+1}^2 A_{\alpha}^2)]
\]
\[
= \sum_{\alpha=n+2}^{n+p} [-[\text{trace}((A_{n+1} - HI)A_{\alpha})]^2 + H \text{trace}A_{\alpha}
\]
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\[ + \text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 + 2H\text{trace}(A_{n+1}A_\alpha^2) - H^2\text{trace}A_\alpha^2 \]

\[ - \text{trace}\{(A_{n+1} - HI)^2A_\alpha^2\} - 2H\text{trace}(A_{n+1}A_\alpha^2) + H^2\text{trace}A_\alpha^2 \]

\[ = \sum_{\alpha=n+2}^{n+p} [-\text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 + \text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 \]

\[ - \text{trace}\{(A_{n+1} - HI)^2A_\alpha^2\}] ,

where \( I \) denotes the identity matrix.

For a fixed \( \alpha, n + 2 \leq \alpha \leq n + p \), we can take a local orthonormal frame field \( \{e_1, \ldots, e_n\} \) such that \( h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij} \). Thus we obtain \( \text{trace}A_\alpha = \sum_{i=1}^{n} \lambda_i^\alpha = 0 \) and \( \text{trace}A_\alpha^2 = \sum_{i=1}^{n} (\lambda_i^\alpha)^2 \). Let \( B := A_{n+1} - HI = (b_{ij}) \). Then we get \( \sum_{i=1}^{n} b_{ii} = 0 \), \( \sum_{i,j=1}^{n} b_{ij}^2 = S_1 \) and \( b_{ij} = b_{ji} \) for any \( i, j = 1, \ldots, n \). Hence we have

\[ - [\text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 + \text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 \]

\[ - \text{trace}\{(A_{n+1} - HI)^2A_\alpha^2\} \]

\[ = - [\text{trace}(BA_\alpha)]^2 + \text{trace}(BA_\alpha)^2 - \text{trace}(B^2A_\alpha^2) \]

\[ = - (\sum_{i=1}^{n} b_{ii} \lambda_i^\alpha)^2 + \sum_{i,j=1}^{n} b_{ij}^2 \lambda_i^\alpha \lambda_j^\alpha - \sum_{i,j=1}^{n} b_{ij}^2 (\lambda_i^\alpha)^2 . \]

Since \( \lambda_i^\alpha \) and \( b_{ij} \) satisfy the conditions in Lemma 2 for \( i, j = 1, \ldots, n \), we get

\[ - [\text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 + \text{trace}\{(A_{n+1} - HI)A_\alpha\}^2 \]

\[ - \text{trace}\{(A_{n+1} - HI)^2A_\alpha^2\} \]

\[ \geq - S_1 \text{trace}A_\alpha^2 . \]

Therefore we obtain

\[ - \sum_{\alpha=n+2}^{n+p} [\text{trace}(A_{n+1}A_\alpha)]^2 + \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}A_\alpha)^2 \]

\[ - \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}^2A_\alpha^2) \]

(11)

\[ = \sum_{\alpha=n+2}^{n+p} [-[\text{trace}(A_{n+1}A_\alpha)]^2 + \text{trace}(A_{n+1}A_\alpha)^2 - \text{trace}(A_{n+1}^2A_\alpha^2)] \]
\[ \geq -S_1 \sum_{\alpha=n+2}^{n+p} \text{trace}A_\alpha^2 \]
\[ = -S_1 S_2. \]

By making use of the same assertion as above we have

\[ nH \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}A_\alpha^2) = nH \sum_{\alpha=n+2}^{n+p} \text{trace}\{(A_{n+1} - HI)A_\alpha^2\} \]
\[ + nH^2 \sum_{\alpha=n+2}^{n+p} \text{trace}A_\alpha^2 \]
\[ = nH \sum_{\alpha=n+2}^{n+p} \text{trace}\{(A_{n+1} - HI)A_\alpha^2\} + nH^2 S_2 \]
\[ = nH \sum_{\alpha=n+2}^{n+p} n b_{ii}(\lambda_i^\alpha)^2 + nH^2 S_2. \]

From Lemma 3 and Lemma 4 we obtain

\[ \sum_{i=1}^{n} b_{ii}(\lambda_i^\alpha)^2 \geq - \sqrt{\sum_{i=1}^{n}(\lambda_i^\alpha)^4 - \frac{1}{n} \sum_{i=1}^{n}(\lambda_i^\alpha)^2} \sum_{i=1}^{n} b_{ii}^2 \]
\[ \geq - \frac{n-2}{\sqrt{n(n-1)}} \sqrt{S_1 \text{trace}A_\alpha^2}. \]

Thus we get

\[ nH \sum_{\alpha=n+2}^{n+p} \text{trace}(A_{n+1}A_\alpha^2) \geq - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S_1 S_2} + nH^2 S_2. \] (12)

From (9)-(12) we have

\[ \frac{1}{2} \Delta S_2 \geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 + ncS_2 \]
\[ + S_2(nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S_1}) - S_1 S_2 - \frac{3}{2} S_2^2 \]
\[ = \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^\alpha)^2 + S_2\left(- \frac{3}{2}(S_1 + S_2) - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S_1}\right) \] (13)
\[ +n(H^2 + c) + \frac{1}{2}S_1 \] 
\[ = \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^{\alpha})^2 + S_2 \left\{ -\frac{3}{2} |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S - nH^2 - S_2} \right\} +n(H^2 + c) + \frac{1}{2}S_1 \] 
\[ \geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^{\alpha})^2 + S_2 \left\{ -\frac{3}{2} |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\phi| + n(H^2 + c) \right\} \]
\[ = \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^{\alpha})^2 + S_2 \left\{ -Q_H(|\phi|) \right\} \]

Under our assumption \(|\phi|^2 \leq B_H\), i.e., \(-Q_H(|\phi|) \geq 0\) we get
\[ \frac{1}{2} \Delta S_2 \geq 0. \]

Next, we show that the Ricci curvature \(R_{ii}\) of \(M^n\) is bounded from the below and \(S_2\) is bounded from the above, when \(|\phi|^2 \leq B_H\). From (3) and (8) we obtain
\[ R_{ii} = (n-1)c + \sum_{\alpha=n+1}^{n+p} \sum_{k=1}^{n} (h_{kk}^{\alpha} h_{ii}^{\alpha} - (h_{ik}^{\alpha})^2) \]
\[ = (n-1)c + \sum_{\alpha=n+1}^{n+p} \left( h_{ii}^{\alpha} \sum_{k=1}^{n} h_{kk}^{\alpha} \right) - \sum_{\alpha=n+1}^{n+p} \sum_{k=1}^{n} (h_{ik}^{\alpha})^2 \]
\[ = (n-1)c + h_{ii}^{n+1} \sum_{k=1}^{n} h_{kk}^{n+1} - \sum_{\alpha=n+1}^{n+p} \sum_{k=1}^{n} (h_{ik}^{\alpha})^2 \]
\[ = (n-1)c + nH h_{ii}^{n+1} - \sum_{\alpha=n+1}^{n+p} \sum_{k=1}^{n} (h_{ik}^{\alpha})^2 \]
\[ \geq (n-1)c + nH h_{ii}^{n+1} - \sum_{\alpha=n+1}^{n+p} \sum_{i,k=1}^{n} (h_{ik}^{\alpha})^2 \]
\[ = (n-1)c + nH h_{ii}^{n+1} - (|\phi|^2 + nH^2) \]
\[ \geq (n-1)c + nH h_{ii}^{n+1} - B_H - nH^2. \]

and \(S_2 = |\phi|^2 - S_1 \leq B_H - S_1 \leq B_H\), respectively. Since \(H\) is nonzero constant, there exist real numbers \(c_1, c_2\) such that
\( R_{ii} \geq c_1 \) and \( S_2 \leq c_2 \).

Since the Ricci curvature of \( M^n \) and \( S_2 \) satisfy the above conditions, we can apply the generalized maximum principle due to Omori [10] and Yau [14] to the function \( S_2 \). Then there exists a sequence \( \{p_\ell \} \) in \( M^n \) such that

\[
\lim_{\ell \to \infty} S_2(p_\ell) = \sup S_2 \quad \text{and} \quad \lim_{\ell \to \infty} \sup \triangle S_2(p_\ell) \leq 0.
\]

Hence we have

\[
0 \geq \frac{1}{2} \triangle S_2 \geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + S_2\{-Q_H(|\varphi|)\} \geq 0.
\]

Thus we get

\[
\sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 = 0
\]

on \( M^n \). Since the equalities of (13) hold, we obtain

\[
S_1 = S_2 = 0, \ \text{i.e.}, \ |\varphi|^2 = 0
\]

on \( M^n \).

On the other hand, from (4), we have

\[
\sum_{i,k=1}^n h_{ik}^\alpha \omega_k = \sum_{i=1}^n dh_{ii}^\alpha - \sum_{i,k=1}^n h_{ik}^\alpha \omega_{ki} - \sum_{i,k=1}^n h_{ik}^\alpha \omega_{ki} - \sum_{\beta=n+1}^{n+p} \sum_{i=1}^n h_{ii}^{\beta} \omega_{\beta \alpha}
\]

\[= -2 \sum_{i,k=1}^n h_{ik}^\alpha \omega_{ki} - \sum_{i=1}^n h_{ii}^{n+1} \omega_{(n+1)\alpha}
\]

\[= -nH \omega_{(n+1)\alpha}
\]

for any \( \alpha, n + 2 \leq \alpha \leq n + p \). From \( H \neq 0 \) and the above equality, we infer \( \omega_{(n+1)\alpha} = 0 \). Thus we know that \( e_{n+1} \) is parallel in the normal bundle \( T^\perp M \) of \( M^n \). Hence, if we denote by \( N_1 \) the normal subbundle spanned by \( e_\alpha \) of the normal bundle of \( M^n \), then \( M^n \) is totally geodesic with respect to \( N_1 \). Since the \( e_{n+1} \) is parallel in the normal bundle, we know that the normal subbundle \( N_1 \) is invariant under parallel translation with respect to the normal connection of \( M^n \). Then, from Theorem 1 in [15], we get that \( M^n \) lies in a totally geodesic submanifold \( S^{n+1}(c) \) of \( S^{n+p}(c) \). This completes the proof of Theorem 3.
Proof of Theorem 4. From (7) we have
\[ \frac{1}{2}S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i,j=1}^{n} h_{ij} \Delta h_{ij} \]
\[ = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i,j,k=1}^{n} h_{ij} h_{kkl} + \sum_{i,j,k,m=1}^{n} h_{ij} h_{km} R_{mijk} \]
\[ + \sum_{i,j,k,m=1}^{n} h_{ij} h_{mi} R_{mkjk}. \]

We choose a local orthonormal frame field \( \{ e_1, \ldots, e_n \} \) such that \( h_{ij} = \lambda_i \delta_{ij} \). Then, by a standard and direct computation, we get
\[ \frac{1}{2} \Delta S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i=1}^{n} \lambda_i (nH)_{ii} + \frac{1}{2} \sum_{i,j=1}^{n} R_{ijij} (\lambda_i - \lambda_j)^2, \tag{14} \]
where \( R_{ijij} = -1 + \lambda_i \lambda_j (i \neq j) \) denotes the sectional curvature of the section spanned by \( \{ e_i, e_j \} \).

We first compute the last term on the right-hand side of (14). Since \( R_{ijij} = -1 + \lambda_i \lambda_j \), we have
\[ \frac{1}{2} \sum_{i,j=1}^{n} \left[ -1 + \lambda_i \lambda_j \right] (\lambda_i - \lambda_j)^2 = -n \sum_{i=1}^{n} \lambda_i^2 + \left( \sum_{i=1}^{n} \lambda_i \right)^2 - \left( \sum_{i=1}^{n} \lambda_i^2 \right)^2 \]
\[ + \left( \sum_{j=1}^{n} \lambda_j \right) \left( \sum_{i=1}^{n} \lambda_i^3 \right) \]
\[ = -nS + n^2 H^2 - S^2 + nH \sum_{i=1}^{n} \lambda_i^3. \]

Let \( \{ e_1, \ldots, e_n \} \) be an orthonormal frame which diagonalizes \( \phi \) at each point of \( M^n \), that is, \( \phi e_i = \mu_i e_i \). By the definition of \( \phi \) we get \( \mu_i = H - \lambda_i \). Then we obtain
\[ |\phi|^2 = \sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} (H - \lambda_i)^2 = S - nH^2 \]
and
\[ \sum_{i=1}^{n} \lambda_i^3 = \sum_{i=1}^{n} (H - \mu_i)^3. \]
Therefore we have

$$\frac{1}{2} \sum_{i,j=1}^{n} R_{ij} (\lambda_i - \lambda_j)^2 = -nS + n^2 H^2 - S^2$$

$$+ nH(nH^3 + 3H|\phi|^2 - \sum_{i=1}^{n} \mu^3_i)$$

$$= -n(S - nH^2) + nH^2(S - nH^2) - (S - nH^2)^2$$

$$- nH \sum_{i=1}^{n} \mu^3_i$$

$$= -n|\phi|^2 + nH^2|\phi|^2 - |\phi|^4 - nH \sum_{i=1}^{n} \mu^3_i$$

$$= n(H^2 - 1)|\phi|^2 - |\phi|^4 - nH \sum_{i=1}^{n} \mu^3_i.$$ (15)

Putting (15) in (14), we obtain

$$\frac{1}{2} \Delta S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i=1}^{n} \lambda_i (nH)_{ii} + n(H^2 - 1)|\phi|^2 - |\phi|^4 - nH \sum_{i=1}^{n} \mu^3_i.$$ (16)

Since the assumption of Theorem 4 that the mean curvature $H$ is constant, we get

$$\frac{1}{2} \Delta S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + n(H^2 - 1)|\phi|^2 - |\phi|^4 - nH \sum_{i=1}^{n} \mu^3_i.$$ (16)

By using Lemma 6 in (16) we have

$$\frac{1}{2} \Delta S \geq \sum_{i,j,k=1}^{n} h_{ijk}^2 + n(H^2 - 1)|\phi|^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi|^3.$$
\[
\begin{align*}
&= \sum_{i,j,k=1}^{n} h_{ijk}^2 + |\phi|^2 \{ -|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\phi| + n(H^2 - 1) \} \\
&= \sum_{i,j,k=1}^{n} h_{ijk}^2 + |\phi|^2 \{ -P_H(\phi) \}.
\end{align*}
\]

Since \(|\phi|^2 \leq A_H\) and \(H\) is nonzero constant, we can apply the generalized maximum principle due to Omori [10] and Yau [14] to the function \(S\) by the similar way with one of the proof of Theorem 3. Hence we get

\[
0 \geq \frac{1}{2} \Delta S.
\]

Under our assumption \(|\phi|^2 \leq A_H\), i.e., \(-P_H(\phi) \geq 0\) we obtain

\[
0 \geq \frac{1}{2} \Delta S \geq \sum_{i,j,k=1}^{n} h_{ijk}^2 + |\phi|^2 \{ -P_H(\phi) \} \geq 0.
\]

Thus we get \(\sum_{i,j,k=1}^{n} h_{ijk}^2 = 0\) and either \(|\phi|^2 \equiv 0\) or \(|\phi|^2 \equiv A_H\). This proves the part (i) of Theorem 4.

We now consider the part (ii). Since \(\sum_{i,j,k=1}^{n} h_{ijk} = 0\) and it holds the equality of the right-hand side of Lemma 6, after renumbering if necessary, we can assume that

\[
\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}, \lambda_1 \neq \lambda_n \text{ and } \lambda_i = \text{constant}.
\]

From Remark 6 we obtain

\[
\mu_1 = \mu_2 = \cdots = \mu_{n-1} = -\frac{1}{\sqrt{n(n-1)}} |\phi| \text{ and } \mu_n = \sqrt{\frac{n-1}{n}} |\phi|.
\]

Hence we have

\[
\lambda_n \lambda_1 = (H - \sqrt{\frac{n-1}{n}} |\phi|)(H + \frac{1}{\sqrt{n(n-1)}} |\phi|) = H^2 - \frac{1}{n} |\phi|^2 - \frac{n-2}{\sqrt{n(n-1)}} H |\phi|.
\]
Since $|φ|^2 \equiv A_H$, we get

$$nλ_nλ_1 = nH^2 - |φ|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|φ| = n,$$

that is, $λ_nλ_1 = 1$. On the other hand, from

$$λ_n = H - μ_n = \frac{λ_n + (n-1)λ_1}{n} - μ_n,$$

we have

$$(n-1)(λ_1 - λ_n) = nμ_n.$$

Since $μ_n > 0$, we get $λ_n < λ_1$. From (5) of Lemma 7 we suppose that

$$λ_1 = λ_2 = \cdots = λ_{n-1} = ν = \frac{1/ρ^2}{\sqrt{1/ρ^2 + 1/r^2}} \text{ and } λ_n = λ = \sqrt{1/ρ^2 + 1/r^2}.$$

Since $λ_nλ_1 = 1$, we obtain $ρ^2 = 1$. However, it is contradiction in $λ < λ_1$. Thus we set

$$λ_n = ν = \frac{1}{\sqrt{1+1/r^2}} \text{ and } λ_1 = λ_2 = \cdots = λ_{n-1} = λ = \sqrt{1+1/r^2}.$$n

Then it yields $λ < λ_1$. Therefore we get that $M^n$ is isometric to $S^{n-1} \times H(- 1/(r^2 + 1))$. If $r > 0$, then the mean curvature holds

$$H = \frac{λ_n + (n-1)λ_1}{n} = \frac{n(r^2 + 1) - 1}{n r \sqrt{r^2 + 1}} > 1,$$

which satisfies the assumption of Theorem 4. Hence it must be $r > 0$.

This finishes our proof of Theorem 4.

Proof of Theorem 5. We prove the Theorem 5 by the similar argument with the proof of Theorem 3. In the case of $p = 2$, since

$$\frac{1}{2} \triangle S_2 \geq \sum_{i,j,k=1}^n (h_{ijk}^{n+2})^2 + S_2\{- (S_1 + S_2)$$

$$\quad - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S_1 + n(H^2 - 1)} \} \quad (17)$$

$$= \sum_{i,j,k=1}^n (h_{ijk}^{n+2})^2 + S_2\{-|φ|^2$$
\[
-\frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S - nH^2 - S_2 + n(H^2 - 1)}
\geq \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{S - nH^2 + n(H^2 - 1)}\}
= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\phi| + n(H^2 - 1)\}
= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-P_H(|\phi|)\}.
\]

Under our assumption $|\phi|^2 \leq A_H$, i.e., $-P_H(|\phi|) \geq 0$ we get

\[
\sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-P_H(|\phi|)\} \geq 0.
\]

Moreover, by the similar way with one of the proof of Theorem 3, we obtain

\[
0 \geq \frac{1}{2} \Delta S_2.
\]

Hence we have

\[
0 \geq \frac{1}{2} \Delta S_2 \geq \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2\{-P_H(|\phi|)\} \geq 0.
\]

Thus we get

\[
\sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 = 0
\]
on $M^n$. Since the equalities of (17) hold, we obtain

\[
S_2 = 0
\]
on $M^n$.

On the other hand, from Theorem 1 in [15], we get that $M^n$ lies in a totally geodesic hypersurface $H^{n+1}(-1)$ of $H^{n+2}(-1)$ by the similar way with one of the proof of Theorem 3. Since $H$ is constant and $S_2 = 0$, we can obtain the result by the similar argument with the proof of Theorem 4. This completes the proof of Theorem 5.

**Proof of Theorem 6.** In the case for $p \geq 3$, we know that $|\phi|^2$ and that $M^n$ lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$ by the similar argument with the proof of Theorem 3. This finishes our proof of Theorem 6.
References


