MATCHING NUMBER AND EDGE COVERING NUMBER
ON KRONECKER PRODUCT OF $C_n$

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Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching number and edge covering number, respectively. The Kronecker Product $G_1 \otimes G_2$ of graph $G_1$ and $G_2$ has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. In this paper, let $G$ is a simple graph with order $m$, we prove that

$$\alpha'(C_n \otimes G) = \max \{n\alpha'(G), m\left\lceil \frac{n}{2} \right\rceil \} \text{ and } \beta'(C_n \otimes G) = \min \{n\beta'(G), m\left\lfloor \frac{n}{2} \right\rfloor \}.$$ 

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let $G_1$ and $G_2$ be graphs. The Kronecker product $G_1 \otimes G_2$ of graph $G_1$ and $G_2$, denote by $G_1 \otimes G_2$, be the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. 

Next, we give the definitions about some graph parameters. A subset of the edge set $E$ of $G$ is said to be matching or an independent edge set of $G$, if no two distinct edges in $M$ have a common vertex. A matching $M$ is maximum matching in $G$ if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality

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of maximum matching of $G$ is called the matching number of $G$, denoted by $\alpha'(G)$.

An edge of graph $G$ is said to cover the two vertices incident with it, and an edge cover of a graph $G$ is a set of edges covering all the vertices of $G$. The minimum cardinality of an edge cover of a graph $G$ is called the edge covering number of $G$, denoted by $\beta'(G)$.

By definitions of matching number, edge covering number, clearly that $\alpha'(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\beta'(C_n) = \lceil \frac{n}{2} \rceil$.

In [1], there are some properties about Kronecker product of graph. We recall here.

**Proposition 1.** Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then:
(i) $n(V(H)) = n(V(G_1))n(V(G_2))$;
(ii) $n(E(H)) = 2n(E(G_1))n(E(G_2))$;
(iii) for every $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Note that for any graph $G$, we have $G_1 \otimes G_2 \cong G_2 \otimes G_1$.

**Theorem 2.** Let $G_1$ and $G_2$ be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if $G_1$ or $G_2$ contains an odd cycle.

**Theorem 3.** Let $G_1$ and $G_2$ be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.

Next we get that general form of graph of Kronecker Product of $C_n$ and a simple graph.

**Proposition 4.** Let $G$ be connected graph order $m$, the graph of $C_n \otimes G$ is

$$( \bigcup_{i=1}^{n-1} H_i ) \cup H_n$$

where $V(H_i) = W_i \cup W_{i+1}$ for $i = 1, 2, ..., n - 1$; $W_i = \{(i, 1), (i, 2), ..., (i, m)\}$; $E(H_i) = \{(i, u)(i + 1, v) / uv \in E(G)\}$ and $V(H_n) = W_n \cup W_{n+1}$; $E(H_n) = \{(n, u) / uv \in E(G)\}$ Moreover, if $G$ has no odd cycle then for each $H_1$ and $H_n$ has exactly two connected components isomorphic to $G$.

**Example.**

2. Matching Number of the Graph of $C_n \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 7 that show character of matching for each $H_i$. 
Definition 5. Given a matching $M$, an $M$-alternating path is a path that alternates between edges in $M$ and edges not in $M$. An $M$-alternating path whose endpoints are unsaturated by $M$ is an $M$-augmenting path.

Theorem 6. A matching $M$ in a graph $G$ is a maximum matching in $G$ if and only if $G$ has no $M$-augmenting path.

Next, we giving Lemma 7 which show character of matching for each $H_i$.

Lemma 7. Let $C_n \otimes G = \bigcup_{i=1}^{n-1} H_i \cup H_n$. For each $H_i$ and $H_n$, then $\alpha'(H_i) = \alpha'(H_n) = 2\alpha'(G)$.

Proof. Suppose $G$ has no odd cycle, by proposition 1.4 we get $H_i = 2G$. So $\alpha'(H_i) = 2\alpha'(G)$. If $G$ has odd cycle, for each $H_i$, vertex $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} H_i = C_n \otimes (G - \tau)$ when $\tau$ is an edge in odd cycle, $M$ be the maximum matching of
We get $H_i = 2(G - e)$ then

$$\alpha'(H_i) = 2\alpha'(G - e) = \begin{cases} 
2[\alpha'(G) - 1], & \text{if } e \text{ is in } M, \\
2\alpha'(G), & \text{otherwise.}
\end{cases}$$

When we add $e$ comeback, we get $\alpha'(H_i) = \alpha'(H_i) + 1$. Hence $\alpha'(H_i) = 2\alpha'G$. Similarly, $\alpha'(H_n) = 2\alpha'G$.

Next, we establish Theorem 8 for a matching number of $C_n \otimes G$

**Theorem 8.** Let $G$ be connected graph order $m$, then

$$\alpha'(C_n \otimes G) = \max\{na'(G), m[n/2]\}.$$  

**Proof.** Let $V(C_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(C_n \otimes G)/j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$ and since $\alpha'(C_n) = [n/2]$.

Let $\alpha'(G) = k$, assume that the maximum matching of $C_n, G$ be

$$M_1 = \{u_1u_2, u_3u_4, ..., u_{2[\frac{n}{2}]-1}u_{2[\frac{n}{2}]}\},$$

$$M_2 = \{v_jv_{j+1}/j = 1, 3, ..., 2k - 1\},$$

respectively.

By Lemma 2.2 we have $\alpha'(H_i) = 2\alpha'(G)$. Since $C_n \otimes G$ is $\bigcup_{i=1}^{n-1} H_i \cup H_n$ which have matching in $H_1, H_3, ..., H_{2[\frac{n}{2}]-1}$, then $\alpha'(C_n \otimes G) \geq na'(G)$.

By definition of matching, we get another matching of $C_n \otimes G$ be set of edges such that incident with vertices in $S_i$ and $S_{i+1}$, $i = 1, 3, ..., 2[\frac{n}{2}] - 1$. So $\alpha'(C_n \otimes G) \geq m[\frac{n}{2}]$.

Hence $\alpha'(C_n \otimes G) \geq \max\{na'(G), m[\frac{n}{2}]\}$.

If $na'(G) > m[\frac{n}{2}]$, suppose that $\alpha'(C_n \otimes G) > na'(G)$, then there exist a matching $M$ is a augmenting path. That is not true because each vertices in $C_n \otimes G$ always incident with edges in

$$M = \bigcup_{i=1,3,2[\frac{n}{2}]-1}^{n-1} \{(u_i, v_j)(u_{i+1}, v_{j+1})/j = 1, 3, ..., 2k - 1\}$$

$$\cup \bigcup_{i=1,3, ..., 2[\frac{n}{2}]-1}^{n-1} \{(u_i, v_j)(u_{i+1}, v_{j-1})/j = 2, 4, ..., 2k\}$$

and another edges which are not in $M$:
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Figure 2: The Matching $M$ when $n\alpha'(G) > m \lfloor \frac{n}{2} \rfloor$ and $n$ is odd

$$N = \left\{ \bigcup_{i=2,4,2\lfloor \frac{n}{2} \rfloor} \{(u_i, v_j)(u_{i+1}, v_{j+1})/j = 1, 3, ..., 2k - 1\}\right\}$$
$$\cup \left[ \bigcup_{i=2,4,2\lfloor \frac{n}{2} \rfloor} \{(u_i, v_j)(u_{i+1}, v_{j-1})/j = 2, 4, ..., 2k\}\right]$$
$$\cup \{(u_1, v_j)(u_n, v_{j+1})/j = 1, 3, ..., 2k - 1\} \cup \{(u_1, v_j)(u_n, v_{j-1})/j = 2, 4, ..., 2k\},$$

so the endpoints of $M$ are saturated by $M$.

If $n\alpha'(G) < m \lfloor \frac{n}{2} \rfloor$, suppose that $\alpha'(C_n \otimes G) > m \lfloor \frac{n}{2} \rfloor$, it is not true because every $S_i$ have $|S_i| = m$.

Hence $\alpha'(C_n \otimes G) = \max\{n\alpha'(G), m \lfloor \frac{n}{2} \rfloor\}$. $\square$

3. Edge Covering number of the graph of $C_n \otimes G$

We begin this section by giving Lemma 9 that shows a relation of matching number and edge covering number and Lemma 10 that show character of edge cover number for each $H_i$.

**Lemma 9.** Let $G$ be a simple graph with order $n$. Then $\alpha'(G) + \beta'(G) = n$
Lemma 10. Let \( C_n \otimes G = \bigcup_{i=1}^{n-1} H_i \cup H_n \). For each \( H_i \) and \( H_n \) then
\[
\beta'(H_i) = \beta'(H_n) = 2\beta'(G)
\]

Proof. Suppose \( G \) has no odd cycle, by proposition 1.4, we get \( H_i = 2G \). So \( \beta'(H_i) = 2\beta'(G) \).

If \( G \) has odd cycle, for each \((u_{i+1}, v) \in W_i, \ (u_i, v) \in W(i+1) \) in \( V(H_i) \) and \((u_n, v) \in W_n \) in \( V(H_n) \) have \( d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v) = d_{H_n}((u_n, v)) = d_{H_n}(u_1, v)) \). Let \( \bigcup_{i=1}^{n-1} \overline{H_i} = C_n \otimes (G - \overline{e}) \) when \( \overline{e} \) is an edge in odd cycle, \( C \) be the minimum edge covering set of \( G \). We get \( \overline{H_i} = 2(G - \overline{e}) \) then

\[
\beta(\overline{H_i}) = \begin{cases} 
2[\beta(G) + 2], & \text{if } \overline{e} = xy \in C \text{ with } d(x) > 1 \text{ and } d(y) > 1, \\
2[\beta(G) - 1], & \text{if } \overline{e} = xy \in C \text{ with } d(x) \geq 1 \text{ or } d(y) \geq 1, \\
2\beta(G), & \text{otherwise.}
\end{cases}
\]
When we add $\overline{e}$ comeback, in the case $\beta'(G - \overline{e}) = \beta'(G) - 1$, we get $\beta'(H_i) = \beta'((H_i) + 1$. And in the case $\beta'(G - \overline{e}) = \beta'(G) + 2$, we get $\overline{e} = xy \in C$ of $G$ replace edges $ux, yv$ (edge cover of $G - \overline{e}$), so $\beta'(G - \overline{e}) = \beta'(G) - 2$.

Hence $\beta'(H_i) = 2\beta'(G)$. Similarly, $\beta'(H_n) = 2\beta'(G)$. □

Next, we establish Theorem 11 for a minimum edge covering number of $C_n \otimes G$.

**Theorem 11.** Let $G$ be connected graph order $m$, then $\beta'(C_n \otimes G) = \min\{n\beta'(G), m\lceil \frac{n}{2} \rceil\}$

*Proof.* Let $V(C_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(C_n \otimes G) / j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$ and since $\beta'(C_n) = \lceil \frac{n}{2} \rceil$. Let $\beta'(G) = k$, assume that the maximum matching of $G$ be $M_2$, and minimum edge covering set of $C_n, G$ be

$$C_1 = \begin{cases} \{u_1 u_2, u_3 u_4, ..., u_{n-1} u_n\} & \text{where } n \text{ is even}, \\ \{u_1 u_2, u_3 u_4, ..., u_{n-2} u_{n-1}, u_n u_1\} & \text{where } n \text{ is odd,} \end{cases}$$

$$C_2 = M_2 \cup \{v_j v / j = 2k + 1, 2k + 2, ..., m \text{ and } v \text{ is endvertex of matching in } M_2\},$$

respectively.
By Lemma 3.2 we have $\beta'(H_i) = 2\beta'(G)$. Since $C_n \otimes G$ is $(\bigcup_{i=1}^{n-1} H_i) \cup H_n$ which have edge cover in $H_1, H_3, ..., H_{2\left[\frac{n}{2}\right]-1}$, then $\beta'(C_n \otimes G) \leq n\beta'(G)$.

Since definition of edge cover, we get another edge cover of $C_n \otimes G$ be set of edges, such that incident with vertices in $S_i$ and $S_{i+1}$, $i = 1, 3, ..., 2\left[\frac{n}{2}\right] - 1$. So $\beta'(C_n \otimes G) \leq m\left[\frac{n}{2}\right]$.

Hence $\beta'(C_n \otimes G) \leq \min\{n\beta'(G), m\left[\frac{n}{2}\right]\}$.

If $n\beta'(G) < m\left[\frac{n}{2}\right]$, suppose that $\beta'(C_n \otimes G) < n\beta'(G)$, then there exist edges $xy$ in edge covering of each $H_1, H_3, ..., H_{2\left[\frac{n}{2}\right]-1}$, which is endvertex $x$ and $y$ incident with another edges in edge covering of each $H_1, H_3, ..., H_{2\left[\frac{n}{2}\right]-1}$, it not impossible.

If $n\beta'(G) > m\left[\frac{n}{2}\right]$ , suppose that $\beta'(C_n \otimes G) > m\left[\frac{n}{2}\right]$ , that is not true because every $S_i$ have $|S_i| = m$.

Hence $\beta'(C_n \otimes G) = \min\{n\beta'(G), m\left[\frac{n}{2}\right]\}$.

By Theorem 2.3 and Lemma 3.1, we can also show that:

$$\alpha'(C_n \otimes G) + \beta'(C_n \otimes G) = mn,$$

Figure 5: The edge cover when $n\beta'(G) > m\left[\frac{n}{2}\right]$ and $n$ is odd
\[
\max\{n\alpha'(G), m\lfloor \frac{n}{2} \rfloor\} + \beta'(C_n \otimes G) = mn,
\]

\[
\beta'(C_n \otimes G) = mn - \max\{n\alpha'(G), m\lfloor \frac{n}{2} \rfloor\}
\]

\[
= mn + \min\{-n\alpha'(G), -m\lfloor \frac{n}{2} \rfloor\}
\]

\[
= \min\{n(m - \alpha'(G)), m(n - \lfloor \frac{n}{2} \rfloor)\}
\]

\[
= \min\{n\beta'(G)), m\lfloor \frac{n}{2} \rfloor\}.
\]

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References


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