International Journal of Pure and Applied Mathematics
Volume 72 No. 4 2011, 565-571

UPPER AND LOWER BOUNDS FOR PRODUCTS OF MULTIPLICATION OPERATOR AND HAUSDORFF MATRIX ON BLOCK WEIGHTED SEQUENCE SPACES

Kuldip Raj[1 §], Sunil K. Sharma[2], Ajay K. Sharma[3]
[1,2,3]School of Mathematics
Shri Mata Vaishno Devi University
Katra, 182320, INDIA

Abstract: In this paper, we compute upper and lower bounds for products of multiplication operator and Hausdorff matrix on block weighted sequence spaces.

AMS Subject Classification: 47B33, 46E10, 30D55
Key Words: multiplication operator, Hausdorff matrix, block weighted sequence space

1. Introduction and Preliminaries

Let l_p $(0 < p < \infty)$ denote the space of all sequences for which $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

For $p \geq 1$, l_p is a Banach space with the norm

$$||x||_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}.$$

If $0 < p < 1$, then $||.||_p$ does not satisfies the triangle inequality and therefore, it is not a norm. Abusing notation, we will write

$$||x||_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}$$

even if $0 < p < 1$. The space l_p for $0 < p < 1$ is an F space.

Received: August 24, 2011
§Correspondence author

Let $u : \mathbb{N} \to \mathbb{C}$ be a measurable function. Then a bounded linear transformation, $M_u : L^p(\mathbb{N}, \mathbb{C}) \to L^p(\mathbb{N}, \mathbb{C})$ defined by $(M_u f)(k) = (u.f)(k) = u(k)(f(k))$ is called a multiplication operator induced by u.

For a decreasing non-negative sequence $w = \{w_n\}$ and $0 < p < \infty$. We define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) = \Big\{ \{x_n\} : ||x||_{w,p} = \Big(\sum_{n=1}^{\infty} w_n |x_n|^p \Big)^{\frac{1}{p}} < \infty \Big\}.$$

Let F be a partition of positive integers. If $F = (F_n)$, where each F_n is a finite interval of positive integers and

$$\max F_n < \min F_{n+1} \qquad (n = 1, 2, \cdots),$$

then we define the block weighted sequence space $l_p(w, F)$ as

$$l_p(w, F) = \Big\{ x = \{x_n\} : \sum_{n=1}^{\infty} w_n |\langle x, F_n \rangle|^p \text{ is finite } \Big\},$$

where $\langle x, F_n \rangle = \sum_{j \in F_n} x_j$. The norm on $l_p(w, F)$ denoted by $||.||_{p,w,F}$ is defined as follows

$$||x||_{p,w,F} = \Big(\sum_{n=1}^{\infty} w_n |\langle x, F_n \rangle|^p \Big)^{\frac{1}{p}}.$$

For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$ is a partition of positive integers, $l_p(w, I) = l_p(w)$ and also $||x||_{p,w,I} = ||x||_{p,w}$.

Let $A : l_p(w, I) \to l_p(w, F)$ be a linear operator. The problem of the norm of matrix operators on $l_p(w)$ and $l_p(w, F)$ is considered in [1]and [5]. We consider upper and lower bounds U and L of the form

$$L||x||_{p,w,I} \le ||Ax||_{p,w,F} \le U||x||_{p,w,I}$$

for all decreasing non-negative sequences x.

We consider the Hausdorff matrix operator $H(\mu) = (h_{j,k})$, such that

$$h_{j,k} = \begin{cases} \begin{pmatrix} j-1 \\ k-1 \end{pmatrix} \Delta^{j-k} a_k, & \text{if } 1 \le k \le j \\ 0, & \text{if } k > j, \end{cases}$$

where Δ is the difference operator, that is,

$$\Delta a_k = a_k - a_{k+1},$$

and $\{a_k\}$ is a sequence of real numbers, normalized so that $a_1 = 1$.

If

$$a_k = \int_0^1 \theta^{k-1} d\mu(\theta), \quad (k = 1, 2, \cdots),$$

where μ is a probability measure on $[0, 1]$, then for all $j, k = 1, 2, \cdots$, we have

$$h_{j,k} = \begin{cases} \begin{pmatrix} j-1 \\ k-1 \end{pmatrix} \int_0^1 \theta^{k-1}(1-\theta)^{j-k} d\mu(\theta), & \text{if } 1 \le k \le j \\ 0, & \text{if } k > j \end{cases}$$

The Hausdorff matrix contains following classes of matrices:

1. If $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$, then $H(\mu)$ reduces to the Cesaro matrix of order α, which we denote by $T(\alpha)$;

2. If $d\mu(\theta) =$ point evaluation at $\theta = \alpha$, then $H(\mu)$ reduces to the Euler matrix of order α, which we denote by $E(\alpha)$;

3. If $d\mu(\theta) = \frac{|\log(\theta)|^{\alpha-1}}{\Gamma(\alpha)} d(\theta)$, then $H(\mu)$ reduces to Holder matrix of order α, which we denote by $S(\alpha)$;

4. If $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$, then $H(\mu)$ reduces to Gamma matrix of order α, which we denote by $\Gamma(\alpha)$;

The Cesaro matrix, Holder matrix and Gamma matrix have non-negative entries, whenever $\alpha > 0$. Also the Euler matrix is non-negative, whenever $0 \le \alpha \le 1$.

Recently, R.Lashkaripour and D.Foroutannia have computed lower bounds for matrices on block weighted sequence space I see [8], whereas the study of the multiplication operator M_u studied by many authors (see [6],[7]).

In this paper we compute lower bounds of the operator $M_u H(\mu)$. Throughout this paper, constants are denoted by c, they are positive and may differ from one occurrence to the other. The notation $a \sim b$ means that there is a positive constant c such that $\frac{1}{c}a \le b \le ca$.

2. Lower Bounds of $M_u H(\mu)$

In this section, we are computing the lower bound problem of $M_u H(\mu)$, where M_u be a multiplication operator and $H(\mu)$ is the Hausdorff matrix operator.

Theorem 2.1. *Let $0 < p \leq 1$ and $u : \mathbb{N} \to \mathbb{C}$ be a measurable function. Then*

$$\inf |u(x)|\Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big) \leq ||M_u H^t(\mu)||_{p,w,I} \leq \sup |u(x)|\Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big),$$

for every non negative sequence x .

Proof. If $u(k) = 0$ for some $k \in \mathbb{N}$ then the inequality is obvious. Otherwise, let $E(\alpha)$ be the Euler matrix of order α. Since the row sums of $E^t(\alpha)$ are all $\frac{1}{\alpha}$ and column sums are all 1, applying the proposition 2.2 of [8], we have

$$L_{p,w,I}(E^t(\alpha)) \geq \alpha^{1-p/p}.$$

we now applying the Minkowski's inequality, we have

$$\begin{aligned}
||M_u H^t(\mu)x||_{p,w,I} &= \Big(\sum_{n=1}^{\infty} w_n |\langle M_u H^t(\mu)x, I_n\rangle|^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{n=1}^{\infty} w_n |\sum_{j\in I_n} (M_u H^t(\mu)x)_j|^p\Big)^{\frac{1}{p}} \\
&\geq \inf_{k\in\mathbb{N}} |u(k)|\Big(\sum_{n=1}^{\infty} w_n \Big(\sum_{k=1}^{\infty} h_{k,n}x_k\Big)^p\Big)^{\frac{1}{p}} \\
&= \inf_{k\in\mathbb{N}} |u(k)|\Big(\sum_{n=1}^{\infty} w_n \Big(\int_0^1 \sum_{k=1}^{\infty} e_{k,n}x_k d\mu(\alpha)\Big)^p\Big)^{\frac{1}{p}} \\
&\geq \inf_{k\in\mathbb{N}} |u(k)| \int_0^1 \Big(\sum_{n=1}^{\infty} w_n \Big(\sum_{k=1}^{\infty} e_{k,n}x_k\Big)^p\Big)^{\frac{1}{p}} d\mu(\alpha) \\
&= \inf_{k\in\mathbb{N}} |u(k)| \int_0^1 ||E^t(\alpha)x||_{p,w,I} d\mu(\alpha) \\
&\geq \inf_{k\in\mathbb{N}} |u(k)|\Big(\int_0^1 \alpha^{\frac{1-p}{p}} d\mu(\alpha)\Big)||x||_{p,w,I}.
\end{aligned}$$

Therefore, for any real number $\alpha > 0$, we have

$$||M_u H^t(\mu)x||_{p,w+\alpha,I} \geq \inf_{k\in\mathbb{N}} |u(k)|\Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big)||x||_{p,w+\alpha,I} \tag{1}$$

for all non-negative sequence x in $l_p(w, I)$. Thus we have

$$||M_u H^t(\mu)||_{p,w+\alpha,I} \geq \inf_{k\in\mathbb{N}} |u(k)|\Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big)$$

Let $\rho > \frac{1}{p}$ and n be a fixed integer such that $n \geq \rho$. we define x by

$$x_k = \begin{cases} 0, & \text{if } k < n \\ \dfrac{\begin{pmatrix} k-\rho \\ k-n \end{pmatrix}}{\begin{pmatrix} k \\ n \end{pmatrix}}, & \text{if } k \geq n \end{cases}$$

Since

$$x_k = \frac{(k-\rho)\cdots(n+1-\rho)}{k\cdots(n+1)} \sim k^{-\rho},$$

where $x \sim y$ means that there exist a constant $c > 0$ such that $\frac{1}{c}c \leq y \leq cx$. When $k \to \infty$, it follows that $||x||_p < \infty$ and $||x||_p \to \infty$ when $\rho \to \frac{1}{p}$. Since w is decreasing and also for all k, $w_{k+\alpha} \geq \alpha$ then we have,

$$\alpha^{1/p}||x||_p \leq ||x||_{p,w+\alpha,I} \leq (w_1+\alpha)^{1/p}||x||_p.$$

So $||x||_{p,w+\alpha,I} < \infty$ and $||x||_{p,w+\alpha,I} \to \infty$ when $\rho \to \frac{1}{\rho}$.

Moreover, for all $m > n$, we have

$$(M_u H^t(\mu)x)_m = u(m)\int_0^1 \theta^{p-1} d\mu(\theta).$$

Hence

$$\begin{aligned} ||M_u H^t(\mu)x||^p_{p,w+\alpha,I} &\leq \sup_k |u(k)|\Big(\sum_{m=1}^{n}(w_m+\alpha)\Big(|M_u\sum_{k=1}^{\infty}h_{k,m}(x_k)|\Big)^p \\ &+ \sum_{m=n+1}^{\infty}(w_m+\alpha)(|H^t(\mu)M_u x|)^p_m\Big) \\ &\leq \sup_k |u(k)|\Big(n(w_1+\alpha)\sup_{k,m}|h_{k,m}|^p||x||_1^p \\ &+ \Big(\int_0^1 \theta^{\rho-1}d\mu(\theta)\Big)^p ||x||^p_{p,w+\alpha,I}\Big) \end{aligned}$$

and also

$$||M_u H^t(\mu)||^p \leq \frac{\sup_k |u(k)| n(w_1+\alpha)\sup_{k,m}|h_{k,m}|^p||x||_1^p}{||x||^p_{p,w+\alpha,I}}$$

$$+\sup_k |u(k)|\Big(\int_0^1 \theta^{\rho-1} d\mu(\theta)\Big)^p.$$

If $\rho \to \frac{1}{p}$, then

$$L_{p,w+\alpha,I}(H^t(\mu)M_u) \leq \sup_k |u(k)|\Big(\int_0^1 \theta^{\frac{1}{p}-1} d\mu(\theta)\Big)^p.$$

Corollary 2.2. *Let $p > 0$ and $u : \mathbb{N} \to \mathbb{C}$ be a measurable function. Then*

$$||H^t(\mu)M_u(x)||_{p,w,I} \geq \inf |u(x)|\Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big)||x||_{p,w,I}$$

for every non-negative sequence x. The constant is the best possible.

Proof. Since $0 < p* < 1$, by applying above theorem, we establish the statement.

Corollary 2.3. *Suppose $p > 0$ and let $H(\mu)$ be the Hausdorff matrix operator. Then*

$$||H^t(\mu)(x)||_{p,w,I} \geq \Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big)||x||_{p,w,I}$$

for every non-negative sequence x. The constant is the best possible.

Proof. Follows by taking $|u(x)| = x$ in corollary 2.2.

Corollary 2.4. *Let $0 < p \leq 1$ and $u : \mathbb{N} \to \mathbb{C}$ be a measurable function. Then*

$$||H^t(\mu)M_u||_p \geq \inf |u(x)|\Big(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\Big)||x||_p$$

for every non - negative sequence x. The constant is the best possible.

Proof. By taking $w_n = 1$ for all n in corollary 2.2, we have the above result.

Corollary 2.5. *Suppose $0 < p \leq 1$ and $H(\mu)$ is the Hausdorff matrix operator. Then*

$$||H^t(\mu)(x)||_p \geq \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)||x||_p$$

for every non -negative sequence x. The constant is the best possible.

Proof. Again proof follows by taking $|u(x)| = x$ in corollary 2.4.

References

[1] Ajay K. Sharma, Kuldip Raj, Sunil K. Sharma, Inequalities involving upper and lower bounds for certain operators on l_p spaces, *Advances in Theoretical and Applied Mathematics*, **5**, No. 3 (2010), 339-345.

[2] G. Bennett, Factorizing the classical inequalities, *Mem. Am. Math. Soc.*, **576** (1996), 1-30.

[3] G. Bennett, *Inequalities Complimentary to Hardy*, *Quart. J. Math. Oxford*, **49**, No. 2 (1998), 395-432.

[4] G.H. Hardy, *Divergent Series*, 2-nd Edition, American Mathematical Society (2000).

[5] G.J.O. Jameson, R. Lashkaripour, Norms of certain operators on weighted l_p spaces and Lorentz sequence spaces, *J. Inequal. Pure Appl. Math.*, **3**, No. 1 (2002), 1-17.

[6] B.S. Komal, Kuldip Raj, Multiplication operators induced by operator valued maps, *Int. J. Contemp. Math. Sciences*, **3**, No. 14 (2008), 667-673.

[7] B.S. Komal, S. Gupta, Multiplication operators between Orlicz spaces, *Integr. Eqns. Oper. Theory*, **41** (2001), 324-330.

[8] R. Lashkaripour, D. Foroutannia, Zahedan, Lower bounds for matrices on block weighted sequence spaces I, *Czechoslovak Mathematical Journal*, **59**, No. 1 (2009), 81-94.

[9] J. Pecari, I. Peric, R. Roki, On bounds for weighted norms for matrices and integral operators, *Linear Algebra and Appl.*, **326** (2001), 121-135.