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ASYMPTOTIC EXPRESSIONS FOR THE EIGENVALUES AND EIGENVECTORS OF A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH A TURNING POINT (EXTENSION-1)

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Abstract: Consider the system of second order differential equation

$$y^{''}(x) + (\lambda^2 R(x) + Q(x))y(x) = 0, \quad 0 \le x \le \pi,$$

where $y(x) = (y_1(x), y_2(x))^T$, $Q(x) = \begin{pmatrix} p(x) & r(x)) \\ r(x) & q(x) \end{pmatrix}$, $R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}$, $s(x) = x^m s_1(x)$, $t(x) = x^m t_1(x)$, $s_1(x) > 0$, $t_1(x) > 0$ and $p(x)$, $q(x)$, $r(x)$, $s_1(x)$, $t_1(x)$ are real-valued functions having continuous second order derivatives at x, $0 \le x \le \pi$, m being a real constant and λ, a real parameter.

In the present paper we consider m, a negative real number and determine the asymptotic solutions alongwith their derivatives for such a system for large values of the parameter λ and apply these to determine the asymptotic expressions for the distribution of the eigenvalues and the normalized eigenvectors under the Dirichlet boundary conditions.

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1. Introduction

In the paper of Sengupta [4], the system of second order differential equations

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considered was

$$y''(x) + (\lambda^2 R(x) + Q(x))y(x) = 0, \quad 0 \le x \le \pi, \tag{1}$$

where $y(x) = (y_1(x), y_2(x))^T$, $Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}$, $R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}$, $p(x)$, $q(x)$, $r(x)$, are real-valued functions having continuous second order derivatives at x, $0 \le x \le \pi$, and $s(x) = x^m s_1(x)$, $t(x) = x^m t_1(x)$, $s_1(x) > 0$, $t_1(x) > 0$ have continuous second order derivatives at x, $0 \le x \le \pi$, m being positive constant and λ, a real parameter.

In the present paper for the system (1) we consider m as a negative constant and in particular we choose m = -1 i.e., we consider the system (1) where

$$s(x) = \frac{s_1(x)}{x}, \quad t(x) = \frac{t_1(x)}{x}, \quad 0 \le x \le \pi, \tag{2}$$

where $s_1(x)$, $t_1(x) > \rho^2 > 0$ are continuous functions of x, $0 \le x \le \pi$.

The boundary conditions considered in the problem are the Dirichlet boundary conditions

$$y_1(0) = y_2(0) = y_1(\pi) = y_2(\pi) = 0 \tag{3}$$

satisfied by the solution $y(x) = (y_1(x), y_2(x))^T$ of the system (1)-(2) at $x = 0, x = \pi$.The problem (1)-(3) may be termed after Langer [2], the singular second order system with a simple turning point. Following Neumark [3], we choose R(x) in (1) as a diagonal matrix. As stated explicitly in Dorodnicyn [1], the comparing equations considered for the problem (1)-(2) are the system of Bessel equations

$$\begin{aligned} v_1''(z) + \frac{v_1(z)}{z} &= 0, \\ v_2''(z) + \frac{v_2(z)}{z} &= 0, \end{aligned} \tag{4}$$

whose solutions are

$$\begin{aligned} v_1(z) &= -z^{\frac{1}{2}} Y_1(2z^{\frac{1}{2}}), \\ v_2(z) &= z^{\frac{1}{2}} J_1(2z^{\frac{1}{2}}), \end{aligned} \tag{5}$$

$J_1(.)$, $Y_1(.)$ being the Bessel functions of order one of the first and second kind respectively, have the Wronskian $W(J_1(z), Y_1(z)) = \frac{2}{z}$ (see Watson [5]).

Hence,

$$W(v_1, v_2) = 1, \tag{6}$$

$$\begin{aligned} \text{At } z = 0,\ v_1(0) &= 0 \text{ (in the sense of limit } v_1(z) \text{ as } z \to 0) \\ \text{and } v_2(0) &= 0. \end{aligned} \tag{7}$$

It may be noted that $v_1^{'}(0)$ does not exist; in fact $v_1^{'}(z) \to \infty$, as $z \to 0$.

In the present paper, for the problem (1)-(3), we determine the asymptotic expressions of the solutions, their derivatives and apply these to determine the asymptotic expressions for the n-th eigenvalue λ_n and the corresponding eigenvector $\Psi(x, \lambda_n)$, for large values of the parameter λ.

As the present work is an extension of the work of Sengupta [4], we mention explicitly only those which are considerably different.

2. Preliminaries

Let us substitute

$$\begin{aligned} w_1(x) &= (\frac{1}{2}\int_0^x \sqrt{s(z)}dz)^2, \\ w_2(x) &= (\frac{1}{2}\int_0^x \sqrt{t(z)}dz)^2. \end{aligned} \tag{8}$$

Then

$$\begin{aligned} \sqrt{s(x)} &= \frac{w_1^{'}(x)}{\sqrt{w_1(x)}}, \\ \sqrt{t(x)} &= \frac{w_2^{'}(x)}{\sqrt{w_2(x)}}, \end{aligned} \tag{9}$$

and $w_i^{'}(0) \neq 0$, $i = 1, 2$, since $x = 0$ is a zero of order one of $w_i(x)$, $i = 1, 2$.

The auxiliary equation is

$$y^{''}(x) + (\lambda^2 R(x) + Q_0(x))y(x) = 0 \tag{10}$$

where R(x) is given in (1) and

$$Q_0(x) = \begin{pmatrix} p_0(x) & 0 \\ 0 & q_0(x) \end{pmatrix},$$

$$
\begin{aligned}
p_0(x) &= -\sqrt{w_1'(x)}D^2(w_1'(x))^{-\frac{1}{2}}. \\
q_0(x) &= -\sqrt{w_2'(x)}D^2(w_2'(x))^{-\frac{1}{2}}, D \equiv \frac{d}{dx}
\end{aligned} \tag{11}
$$

we assume further that

$$|p(x) - p_0(x)|, |q(x) - q_0(x)|, |r(x)| \leq c.x^k \tag{12}$$

where $k > 0$ and c is a non-zero constant. Then adopting the technique used to prove Theorem-1 of Sengupta[4], the following theorem which establishes the existence of a particular solution of the system (10) is proved.

Theorem 1. *Let*

$$y_i(x) = C_{i1}(w_i'(x))^{-\frac{1}{2}}v_1(\lambda^2 w_i(x)) + C_{i2}(w_i'(x))^{-\frac{1}{2}}v_2(\lambda^2 w_i(x)), \quad i = 1, 2, \tag{13}$$

where $C_{ij}, i, j = 1, 2$ are constants, $(v_1(x), v_2(x))^T$ is the solution of the system (4), satisfy at x = 0 the conditions(7) and $w_i(x), i = 1, 2$ are those given in (8).

Then $y(x) = (y_1(x), y_2(x))^T$ satisfies the system (10).

3. Asymptotic Estimates of Certain Integrals

Let $\alpha_i(x) = (\alpha_{i1}(x), \alpha_{i2}(x))^T$, $i = 1, 2$, where

$$\alpha_{ij}(x) = (w_j'(x))^{-\frac{1}{2}}.v_i(\lambda^2 w_j(x)), \quad i, j = 1, 2 \tag{14}$$

Then for j = 1, 2; $\alpha_{1j}(0) = (w_j'(0))^{-\frac{1}{2}}$ (in the sense of limit $\alpha_{1j}(x)$ as $x \to 0$), $\alpha_{2j}(0) = (w_j'(0))^{-\frac{1}{2}}$, $\alpha_{2j}'(0) = (w_j'(0))^{-\frac{1}{2}}$ but $\alpha_{1j}'(0)$ does not exist.

As $(v_1(x), v_2(x))^T$ are the solutions of (4), following Theorem 1, it follows that $\alpha_1(x)$, $\alpha_2(x)$ are the two linearly independent solutions of the system (10).

However, the bilinear concomitant $[\alpha_1(x), \alpha_2(x)]$ is independent of x and is a function of λ alone and

$$[\alpha_1(x), \alpha_2(x)] = 2\lambda^2 = \frac{1}{\delta} \quad \text{(say)} \tag{15}$$

Let

$$f(x) = Q(x) - Q_0(x) = (f_{ij}(x)), i, j = 1, 2 \tag{16}$$

where $f_{ij}(x), i, j = 1, 2$ satisfy the conditions (12).

Using the conditions (12), we settle the convergence of the integrals

$$\int_0^x \frac{f_{11}(z)}{\sqrt{s(z)}}dz, \quad \int_0^x \frac{f_{12}(z)}{\sqrt[4]{s(z)t(z)}}dz$$

and

$$\int_0^x \frac{f_{22}(z)}{\sqrt{t(z)}}dz.$$

Following Dorodnicyn [1] and Watson [5] we obtain the following asymptotic relations for large λ :

$$\int_0^x \alpha_{11}^2(z)f_{11}(z)dz, \int_0^x \alpha_{21}^2(z)f_{11}(z)dz, \int_0^x \alpha_{11}(z)\alpha_{21}(z)f_{11}(z)dz$$
$$= c\lambda. \int_0^x \frac{f_{11}(z)}{\sqrt{s(z)}}dz + 0(\lambda^{-1}),$$

$$\int_0^x \alpha_{12}^2(z)f_{22}(z)dz, \int_0^x \alpha_{22}^2(z)f_{22}(z)dz, \int_0^x \alpha_{12}(z)\alpha_{22}(z)f_{22}(z)dz$$
$$= c\lambda. \int_0^x \frac{f_{22}(z)}{\sqrt{s(z)}}dz + 0(\lambda^{-1}),$$

$$\int_0^x \alpha_{12}\alpha_{21}(z)f_{12}(z)dz, \int_0^x \alpha_{22}\alpha_{11}(z)f_{12}(z)dz, \int_0^x \alpha_{11}\alpha_{12}(z)f_{12}(z)dz,$$
$$\int_0^x \alpha_{21}\alpha_{22}(z)f_{12}(z)dz = c\lambda. \int_0^x \frac{f_{12}(z)}{\sqrt[4]{s(z)t(z)}}dz + 0(z^{-1}), \quad (17)$$

where c represents different numerical constants.

4. Asymptotic Relations for the Solutions

Let

$$\alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{21}(z) \\ \alpha_{12}(z) & \alpha_{22}(z) \end{pmatrix}, \quad \alpha_i(z) = (\alpha_{i1}(z), \alpha_{i2}(z))^T, \quad i = 1, 2... \quad (18)$$

and

$$y(x,z) = \begin{pmatrix} y_{11}(x,z) & y_{21}(x,z) \\ y_{12}(x,z) & y_{22}(x,z) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{21}(z)\alpha_{11}(x) - \alpha_{11}(z)\alpha_{21}(x) & \alpha_{22}(z)\alpha_{11}(x) - \alpha_{12}(z)\alpha_{21}(x) \\ \alpha_{21}(z)\alpha_{12}(x) - \alpha_{11}(z)\alpha_{22}(x) & \alpha_{22}(z)\alpha_{12}(x) - \alpha_{12}(z)\alpha_{22}(x) \end{pmatrix}, \quad (19)$$

where $\alpha_{ij}(.)$ are given by (14).

Further let

$$\beta(z) = \begin{pmatrix} \beta_{11}(z) & \beta_{21}(z) \\ \beta_{12}(z) & \beta_{22}(z) \end{pmatrix}, \quad \beta_i(z) = (\beta_{i1}(z), \beta_{i2}(z))^T, \quad i = 1, 2... \quad (20)$$

Then following Sengupta [4], the following theorem is proved.

Theorem 2. *Let the elements of the matrix f(x), defined by (16), satisfy the conditions (12). Then the two linearly independent solutions of (1) are given by*

$$y_i(x) = (y_{i1}(x), y_{i2}(x))^T = \alpha_i(x) + \beta_i(x), \quad i = 1, 2..., \quad (21)$$

where $\alpha_i(x), i = 1, 2$, given by (14), are the linearly independent solutions of (10) and $\beta_i; i = 1, 2$ defined by (20) is the unique solution of the integral equation

$$\beta(x) = \delta \int_0^x y(x,z) f(z)(\alpha(z) + \beta(z))dz \quad (22)$$

y(x,z), δ being those defined in (19),(15) respectively.

By using the asymptotic relations as given in Section 3, we obtain for large λ that

$$\beta(x) = \lambda^{-1}\alpha(x)MP(x) + 0(\lambda^{-2}||\alpha(x)||), \quad (23)$$

where $M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\pi} \\ -\frac{\pi}{2} & -\frac{1}{2} \end{pmatrix}$, and

$$P(x) = \int_0^x \left[\frac{f_{11}(z)}{\sqrt{s(z)}} + 2\frac{f_{12}(z)}{\sqrt[4]{s(z)t(z)}} + \frac{f_{22}(z)}{\sqrt{t(z)}} \right] dz \quad (24)$$

Hence,

$$y(x) = \alpha(x) + \lambda^{-1}\alpha(x)MP(x) + 0(\lambda^{-2}||\alpha(x)||) \quad (25)$$

Differentiating with respect to x first and then using the asymptotic estimates of the integrals as given in Section 3, it follows from (22) that for large λ,

$$\beta^{'}(x) \cong \delta y(x,x) f(x) \alpha(x) + \lambda^{-1} \alpha'(x) MP(x) + 0(\lambda^{-2} ||\alpha(x)||), \tag{26}$$

where

$$y(x,x) = (\alpha_{11}(x)\alpha_{22}(x) - \alpha_{12}(x)\alpha_{21}(x)).I_1, I_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{27}$$

and δ, M, p(x) are those given before. Hence,

$$y^{'}(x) = \alpha^{'}(x) + \delta y(x,x) f(x) \alpha(x) + \lambda^{-1} \alpha^{'}(x) MP(x) + 0(\lambda^{-2}.||\alpha^{'}(x)||) \tag{28}$$

5. Asymptotic Representation for the Eigenvalues

The four linearly independent solutions of the system (1)-(2) are

$$\begin{aligned} U_1(x) &= (U_{11}(x), U_{12}(x))^T = (\alpha_{11}(x) + B_{11}(x), B_{12}(x))^T, \\ U_2(x) &= (U_{21}(x), U_{22}(x))^T = (\alpha_{21}(x) + B_{21}(x), B_{22}(x))^T, \\ \overline{U}_3(x) &= (\overline{U}_{11}(x), \overline{U}_{12}(x))^T = (\overline{B}_{11}(x), \alpha_{12}(x) + \overline{B}_{12}(x))^T \\ & \qquad and \\ \overline{U}_4(x) &= (\overline{U}_{21}(x), \overline{U}_{22}(x))^T = (\overline{B}_{21}(x), \alpha_{22}(x) + \overline{B}_{22}(x))^T, \end{aligned} \tag{29}$$

where the $U_{ij}(x)$, $\overline{U}_{ij}(x)$, $B_{ij}(x)$, $\overline{B}_{ij}(x)$ involving the $\alpha_{ij}(x)$, i, j = 1, 2 are defined in the same way as given in Section-5 of Sengupta [4].

Following Sengupta [4], it readily follows that as $\lambda_n \to \infty$,

$$U_{1j}(0, \lambda_n), \overline{U}_{1j}(0, \lambda_n) = (w_j^{'}(0))^{-\frac{1}{2}} + 0(\lambda_n^{-\frac{1}{2}}),$$

$$U_{2j}(0, \lambda_n), \overline{U}_{2j}(0, \lambda_n) = 0(\lambda_n^{-2}),$$

$$\begin{aligned} & U_{1j}(\pi, \lambda_n), \overline{U}_{1j}(\pi, \lambda_n) \\ & \qquad = \lambda_n^{\frac{1}{2}} (w_j(\pi))^{\frac{1}{4}} (w_j^{'}(\pi))^{-\frac{1}{2}} (\pi)) \pi^{\frac{1}{2}} Cos(2\lambda_n (w_j(\pi))^{\frac{1}{2}} - \frac{\pi}{4}) + 0(\lambda_n^{-\frac{1}{2}}), \end{aligned}$$

$$\begin{aligned} & U_{2j}(\pi, \lambda_n), \overline{U}_{2j}(\pi, \lambda_n) \\ & \qquad = \lambda_n^{\frac{1}{2}} (w_j(\pi))^{\frac{1}{4}} (w_j^{'}(\pi))^{-\frac{1}{2}} (\pi))^{\frac{1}{2}} Sin(2\lambda_n (w_j(\pi))^{\frac{1}{2}} - \frac{\pi}{4}) + 0(\lambda_n^{-\frac{1}{2}}), \end{aligned}$$

where $i, j = 1, 2.$ and

$$\beta_{ij}(0), \overline{\beta}_{ij}(0), \beta_{ij}(\pi), \overline{\beta}_{ij}(\pi) = 0(1), i, j = 1, 2 \tag{30}$$

Hence from the general solution

$$V(x) = AU_1(x) + BU_2(x) + CU_3(x) + DU_4(x),$$

A, B, C, D constants, of the system (1) by using the Dirichlet boundary conditions (3) and the relations (30) it follows in the same way as in Sengupta [4] that the asymptotic expressions for the eigenvalues (λ_n) are given by

$$\lambda_n = \frac{2(n + \frac{1}{4})\pi}{\int_0^\pi \sqrt{s(x)}dx} + 0(n^{-2}) \tag{31}$$

or,

$$\lambda_n = \frac{2(n + \frac{1}{4})\pi}{\int_0^\pi \sqrt{t(x)}dx} + 0(n^{-2}) \tag{32}$$

as $n \to \infty$, according as

$$\int_0^\pi \sqrt{s(x)}dx > or < \int_0^\pi \sqrt{t(x)}dx$$

It is to be noted that the asymptotic expressions of the eigenvalues λ_n in the present problem with Neumann boundary conditions (i.e., $y_1^{'}(0) = y_2^{'}(0) = y_1^{'}(\pi) = y_2^{'}(\pi) = 0$ satisfied by the solution $y(x) = (y_1(x), y_2(x))^T$ of the system (1) at x = 0, $x = \pi$) cannot be derived, as already shown that $v_1^{'}(x) \to \infty$ as $x \to 0$. This has also been pointed out by Dorodnicyn [1], (p.32).

6. Asymptotic Representation of the Normalized Eigenvector

Arguing in the same way as in Sengupta [4], it follows that the normalized eigenvector $\Psi(x, \lambda_n)$ with normalizing constant An is given by

$$\Psi(x, \lambda_n) = A_n^{-\frac{1}{2}}[a_{2n}U_1(x, \lambda_n) - a_{1n}\overline{U_3}(x, \lambda_n)] \tag{33}$$

where

$$A_n = a_{1n}a_{2n}(a_{1n} + a_{2n}),$$

$$a_{1n} = \int_0^\pi s(x)U_{11}^2(x)dx$$
$$a_{2n} = \int_0^\pi t(x)\overline{U}_{12}^2(x)dx$$

Since $U_1(x), \overline{U}_3(x), j = 1, 2$ behave asymptotically in the same way as $y_j(x), j = 1, 2$ of Theorem 2 it follows that as $\lambda \to \infty$.

$$a_{1n} = \lambda_n.2\pi(w_1(\pi))^{\frac{1}{2}} + 0(1),$$
$$a_{2n} = \lambda_n.2\pi(w_2(\pi))^{\frac{1}{2}} + 0(1) \quad (34)$$

Also, from (14), $\alpha_{2j} = 0(\lambda_n)^{\frac{1}{2}}$, for $v_2(z) \cong z^{\frac{1}{4}}.\pi^{-\frac{1}{2}}$, follows from the asymptotic formula for the Bessel function $J_1(.)$. Hence,

$$A_n^{-\frac{1}{2}} = \lambda_n^{-\frac{3}{2}}(2\pi)^{-\frac{3}{2}}(w_1(\pi)w_2(\pi))^{-\frac{1}{4}}(w_1^{\frac{1}{2}}(\pi) + w_2^{\frac{1}{2}}(\pi))^{-\frac{1}{2}}(1 + 0(\lambda_n^{-1})) \quad (35)$$

Thus, the asymptotic expressions for normalized eigenvector corresponding to the eigenvalue n for the Dirichlet boundary conditions is given by

$$\Psi(x, \lambda_n) = \lambda_n^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}(w_1(\pi)w_2(\pi))^{-\frac{1}{4}}.(w_1^{\frac{1}{2}}(\pi) + w_2^{\frac{1}{2}}(\pi))$$
$$(w_2^{\frac{1}{2}}(\pi)\alpha_{11}(x), -w_1^{\frac{1}{2}}(\pi)\alpha_{12}(x))(1 + 0(\lambda_n^{-1}), \quad (36)$$

as $\lambda_n \to \infty$.

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Corrigendum

Certain typographical mistakes are communicated in the paper of Sengupta [4]:

(i) On page 385: The equation $y_i(x) =$... in the third line will be numbered (15).

(ii) On page 396: In the last line:

Choose $\phi^*(x, \lambda_n) = b_{2n}U_1(x, \lambda_n) - b_{1n}(x, \lambda_n)\overline{U_3}(x, \lambda_n)$ an eigenvector.