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ON SOME OPERATOR RELATED TO TRI-LAPLACE EQUATION

Wanchak Satsanit

Department of Mathematics
Faculty of Science
Maejo University
Chiang Mai, 50290, THAILAND

Abstract: In this paper, we study the operator $\otimes_w^k$ where $\otimes_w^k$ is the operator iterated k times and is defined by

$$\otimes_w^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right)^k,$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$. At first we study the elementary solution or the Green function of the operator $\otimes_w^k$ and then such a solution is related to the solution of the Laplace equation. We found that the relationship of such solutions depending on the condition p,q and k. Finally, we applied the elementary solution finding solution equation $\otimes_w^k u(x)=f(x)$. where $u(x)$ is an unknown function for $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, $f(x)$ is the generalized function, k is a positive integer.

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1. Introduction

The n-dimensional ultra-hyperbolic operator $\Box^k$ iterated k-times is defined by

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$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \tag{1.1}$$

where $p+q=n$(the dimension of the space $\mathbb{R}^n$) and k is nonnegative integer.

Consider the linear differential equation of the form

$$\square^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized function and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Gelfand and Shilov, see [2], pp. 279-282, have first introduced the elementary solution of (1.2) which is of complicated form. Later, Trione (see [8]) has shown that the Generalized function $R_{2k}(x)$ which is defined by (2.2) with $\beta = 2k$ is the unique elementary solution of (1.2) and Tellez (see [6], pp. 147-149) also proved that $R_{2k}(x)$ exists only for the case p is odd with n odd or even and $p+q=n$.

Next, The operator $\diamondsuit^k$ was studied firstly by A. Kananthai (see [3]) and is named as the Diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k, \quad p+q=n \tag{1.3}$$

is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer.

Actually the operator $\diamondsuit^k$ is an extension of the ultra-hyperbolic operator and the Laplacian operator. So the operator $\diamondsuit^k$ can be expressed as the product of the the operator $\square^k$ and $\triangle^k$, that is $\diamondsuit^k = \square^k \triangle^k = \triangle^k \square^k$ where

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{1.4}$$

is the Laplacian operator iterated k times and the operator $\square^k$ is the ultra-hyperbolic operator iterated k-times which is deined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

A. Kananthai (see [3], Theorem 3.1, p. 33) has shown that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is an elementary solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) = \delta(x), \tag{1.5}$$

where $\delta(x)$ is Dirac-delta distribution and the function $R^e_{2k}(x)$ and $R^H_{2k}(\upsilon)$ are defined by (2.6) and (2.2) respectively with $\alpha = \beta = 2k, k$ is nonnegative integer.

Furthermore, W. Satsanit has been first introduced the operator $\otimes^k_w$ and is defined by

$$
\begin{aligned}
\otimes^k_w &= \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right)^k \\
&= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \\
&\quad \times \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= (\square)^k(\triangle^2 - \frac{1}{4}(\triangle + \square)(\triangle - \square))^k \\
&= (\frac{3}{4}\diamondsuit\triangle + \frac{1}{4}\square^3)^k, \qquad (1.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\triangle &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}, \\
\square &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}, \\
\diamondsuit &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2.
\end{aligned}
$$

Now, the purpose of this work is to study the operator $\otimes^k_w$ where is defined by (1.6).

Firstly, we can find the elementary solution $G(x)$ of the operator $\otimes^k_w$, that is

$$\otimes^k_w G(x) = \delta, \qquad (1.7)$$

where δ is the Dirac-delta distribution. Moreover, we can find the relationship between $G(x)$ and the elementary solution of the Laplace operator. After that we study the equation

$$\otimes^k_w u(x) = f(x). \qquad (1.8)$$

This equation is the generalization of the ultra-hyperbolic equation and it can be applied to the Laplace equation.

Let $K_{\alpha,\beta}(x)$ be a distributional family and is defined by

$$K_{\alpha,\beta}(x) = R^e_\alpha(x) * R^H_\beta(\upsilon), \tag{1.9}$$

where $R^e_\alpha(x)$ is called the elliptic kernel defined by (2.5)and $R^H_\beta(\upsilon)$ is called the ultra-hyperbolic kernel defined by (2.2) and α, β are the complex parameters.

The family $K_{\alpha,\beta}(x)$ is well defined and is tempered distribution, since $R^e_\alpha(x) * R^H_\beta(\upsilon)$ is a tempered (see [1], Lemma 2.2) and $R^H_\beta(\upsilon)$ has a compact support. We can show that

$$\begin{aligned} u(x) = ((-1)^{2k} K_{4k,4k}(x) * (R^H_{2(k-1)}(\upsilon))^{(m)} \\ + (-1)^{2k} K_{4k,6k}(x) * f(x)) * (S^{*k}(x))^{*-1}, \end{aligned} \tag{1.10}$$

is a solution of (1.8) with $m = \frac{n-4}{2}, \geq 4$ and n is even number, $(S^{*k}(x))$ is defined by (2.15) and $(S^{*k}(x))^{*-1}$ is an inverse of $(S^{*k}(x))$in the convolution algebra.$u(x)$ is a solution of(1.8) and $K_{4k,4k}(x)$ is defined by (1.9) with $\alpha = \beta = 4k$. Moreover, we can show that the solution related to the solution of Laplace operator $\triangle^{3k}$ defined by(1.4).

Before going that point the followings definitions and some important concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Denoted by

$$\upsilon = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \tag{2.1}$$

the nondegenerated quadratic form and $p+q = n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma}_+$ denotes it closure. For any complex number β, define the function

$$R^H_\beta(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\beta-n}{2}}}{K_n(\beta)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\beta)$ is given by the formula

$$K_n(\beta) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\beta-n}{2})\Gamma(\frac{1-\beta}{2})\Gamma(\beta)}{\Gamma(\frac{2+\beta-p}{2})\Gamma(\frac{p-\beta}{2})}. \tag{2.3}$$

The function $R^H_\alpha(\upsilon)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki (see [5]).

It is well known that $R^H_\beta(\upsilon)$ is an ordinary function if $Re(\beta) \geq n$ and is a distribution of β if $Re(\alpha) < n$. Let supp $R^H_\alpha(\upsilon)$ denote the support of $R^H_\alpha(\upsilon)$ and suppose supp $R^H_\alpha(\upsilon) \subset \bar{\Gamma}_+$, that is supp $R^H_\alpha(\upsilon)$ is compact.

From S.E. Trione (see [5], p. 11), $R^H_{2k}(\upsilon)$ is an elementary solution of the operator $\Box^k$ that is

$$\Box^k R^H_{2k}(\upsilon) = \delta(x). \tag{2.4}$$

Definition 2.2. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and $|x| = x_1^2 + x_2^2 + ... + x_n^2$ the function $R^e_\alpha(x)$ denoted the elliptic kernel of Marcel Riesz and is defined by

$$R^e_\alpha(x) = \frac{|x|^{\frac{\alpha-n}{2}}}{W_n(\alpha)} \tag{2.5}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \tag{2.6}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k\triangle^k\delta(x)$ where $\triangle^k$ is defined by (1.3). It follows that $R^e_0(x) = \delta(x)$ (see [3]).

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$ that is

$$\triangle^k((-1)^k R^e_{2k}(x) = \delta(x), \tag{2.7}$$

(see [3], Lemma 2.4, p. 31).

Lemma 2.1. *Given P is a hyper-function then*

$$P\delta^{(k)}(p) + k\delta^{(k-1)}(p) = 0,$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.

Proof. (see [2], p. 233).

Lemma 2.2. *(Convolution of $R^e_\alpha(x)$ and $R^H_\alpha(x)$). Let $R^e_\alpha(x)$ and $R^H_\alpha(x)$ defined by (2.5) and (2.2) respectively, then we obtain:*

(1) $R^e_\alpha(x) * R^e_\beta(x) = R^e_{\alpha+\beta}(x)$ where α and β are complex parameters.

(2) $R^H_\alpha(\upsilon) * R^H_\beta(\upsilon) = R^H_{\alpha+\beta}(\upsilon)$ for α and β are both integers and except only the case both α and β are both integers.

Proof. (see [4]).

Lemma 2.3. *Given the equation*

$$\triangle^k u(x) = 0, \tag{2.8}$$

where $\triangle^k$ is defined by(1.4)and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ then

$$u(x) = (R^e_{2(k-1)}(x))^{(m)}$$

is a solution of (2.8) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension. The generalized function $(R^e_{2(k-1)}(x))^{(m)}$ is defined by (2.5) with m - derivatives and $\alpha = 2(k-1)$.

Proof. We first to show that the generalized function $u(x) = \delta^{(m)}(r^2)$ where $r^2 = |x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$ is a solution of

$$\triangle u(x) = 0, \tag{2.9}$$

where $\triangle = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2})$ is a Laplace operator, Now

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2) = 2x_i\delta^{(m+1)}(r^2)$$
$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2) = 2\delta^{(m+1)}r^2 + 4x_i^2\delta^{(m+2)}(r^2).$$

Thus

$$\begin{aligned}\triangle\delta^{(m)}(r^2) &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2)\\ &= 2n\delta^{(m+1)}r^2 + 4r^2\delta^{(m+2)}(r^2)\\ &= 2n\delta^{(m+1)}r^2 - 4(m+2)\delta^{(m+1)}(r^2).\end{aligned}$$

By Lemma 2.1 with $P = r^2$ we have

$$\begin{aligned}\triangle\delta^{(m)}(r^2) &= (2n - 4(m+2))\delta^{(m+1)}(r^2)\\ &= 0 \;\; if \;\; 2n - 4(m+2) = 0\end{aligned}$$

or $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of(2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Now

$$\triangle^k u(x) = \triangle(\triangle^{k-1} u(x)) = 0$$

then from the above proof $\triangle^{k-1} u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{k-1} R^e_{2(k-1)}(x)$ we obtain

$$(-1)^{k-1} R^e_{2(k-1)}(x) * \triangle^{k-1} u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * \delta^{(m)}(r^2)$$

$$\triangle^{k-1}((-1)^{k-1} R^e_{2(k-1)}(x)) * u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * \delta^{(m)}(r^2)$$

$$\delta * u(x) = u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * \delta^{(m)}(r^2).$$

Now from (2.5)

$$\begin{aligned} R^e_{2(k-1)}(x) &= \frac{|x|^{2(k-1)-n}}{W_n(\alpha)} \\ &= \frac{(|x|^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \\ &= \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Hence

$$\begin{aligned} R^e_{2(k-1)}(x) * \delta^{(m)}(r^2) &= \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} * \delta^{(m)}(r^2) \\ &= \left(\frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \right)^{(m)} \\ &= (R^e_{2(k-1)}(x))^{(m)} \end{aligned}$$

It follows that $u(x) = (-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)}$ is a solution of (2.8) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension of R^n.

Lemma 2.4. *Given the equation*

$$\Box^k u(x) = 0, \tag{2.10}$$

where the operator $\square^k$ *is defined by(1.1)and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *then* $u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}$ *is a solution of(2.10) with* $m = \frac{n-4}{2}, n \geq 4$ *and* n *is even dimension and* υ *is defined by Definition(2.1). The function* $(R^H_{2(k-1)}(\upsilon))^{(m)}$ *is defined by (2.2) with* m *- derivatives and* $\beta = 2(k-1)$

Proof. We first to show that the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \ldots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2$ $, p + q = n$ is a solution of the equation

$$\square u(x) = 0, \tag{2.11}$$

which $\square$ is defined by (1.1) with $k = 1$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2 - s^2) = 2x_i\delta^{(m+1)}(r^2 - s^2)$$

$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) = 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2\delta^{(m+2)}(r^2 - s^2)$$

$$\begin{aligned}
\square\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&+ 4s^2\delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) - 4(m + 2)\delta^{(m+1)}(r^2 - s^2) \\
&+ 4s^2\delta^{(m+2)}(r^2 - s^2) \\
&= (2p - 4(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

By Lemma 2.1 with $P = r^2 - s^2$.Similarly,

$$\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2 - s^2) = (-2q + 4(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2).$$

Thus

$$\square\delta^{(m)}(r^2 - s^2) = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2)$$

$$\begin{aligned}
&= (2(p+q) - 8(m+2))\delta^{(m+1)}(r^2 - s^2) \\
&\quad - 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&= (2n - 8(m+2))\delta^{(m+1)}(r^2 - s^2) + 4(m+2)\delta^{(m+1)}(r^2 - s^2) \\
&= (2n - 4(m+2))\delta^{(m+1)}(r^2 - s^2).
\end{aligned}$$

If $2n - 4(m+2) = 0$, we have $\Box\delta^{(m)}(r^2 - s^2) = 0$. That is

$$u(x) = \delta^{(m)}(r^2 - s^2)$$

is a solution of (2.11) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Now

$$\Box^k u(x) = \Box(\Box^{k-1} u(x)) = 0,$$

then from the above proof we have

$$\Box^{k-1} u(x) = \delta^{(m)}(r^2 - s^2)$$

with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension. Convolving the above equation by $R^H_{2(k-1)}(\upsilon)$, we obtain

$$R^H_{2(k-1)}(\upsilon) * \Box^{k-1} u(x) = R^H_{2(k-1)}(\upsilon) * \delta^{(m)}(r^2 - s^2)$$

$$\Box^{k-1}(R^H_{2(k-1)}(\upsilon)) * u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}, \textit{where} \quad \upsilon = (r^2 - s^2)$$

$$\delta * u(x) = u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}$$

by (2.3) and $\upsilon = r^2 - s^2$ is defined by Definition (2.1)

Thus $u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}$ is a solution of (2.10)with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Lemma 2.5. *Let L be the operator defined by*

$$L^k = (\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2)^k \tag{2.12}$$

where $\triangle$ and $\Box$ is defined by (1.2) and (1.1) respectively.Then we obtain $H(x)$ where

$$H(x) = \left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * \left(S^{*k}(x)\right)^{*-1}. \tag{2.13}$$

and by (1.7) we obtain

$$H(x) = (-1)^{2k} K_{4k,4k}(x) * \left(S^{*k}(x)\right)^{*-1}, \tag{2.14}$$

where

$$S(x) = \frac{3}{4}R_4^H(\upsilon) + \frac{1}{4}(-1)^2 R_4^e(x) \tag{2.15}$$

is an elementary solution of the operator L^k iterated k-times $S^{*k}(x)$ denotes the convolution of S it self k-times, $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra. Moreover $H(x)$ is a tempered distribution.

Proof. From (3.1), we have

$$\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^k H(x) = \delta(x)$$

or we can write

$$\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} H(x) = \delta(x).$$

Convolving both sides of the above equation by $R_4^H(\upsilon) * (-1)^2 R_4^e(x)$,

$$\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)\left(R_4^H(\upsilon) * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} H(x)$$
$$= \delta(x) * R_4^H(\upsilon) * (-1)^2 R_4^e(x)$$

and

$$\left(\frac{3}{4}\triangle^2(R_4^H(\upsilon) * (-1)^2 R_4^e(x)) + \frac{1}{4}\Box^2(R_4^H(\upsilon) * (-1)^2 R_4^e(x))\right).$$
$$.\left(R_4^H(\upsilon) * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} H(x) = \delta(x) * R_4^H(\upsilon) * (-1)^2 R_4^e(x)$$

and

$$\left(\frac{3}{4}\triangle^2((-1)^2 R_4^e(x)) * R_4^H(\upsilon) + \frac{1}{4}(\Box^2 R_4^H(\upsilon)) * (-1)^2 R_4^e(x))\right).$$
$$.\left(R_4^H(\upsilon) * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} H(x) = \delta(x) * R_4^H(\upsilon) * (-1)^2 R_4^e(x)$$

By (2.4) and (2.8)

$$\left(\frac{3}{4}\delta(x) * R_4^H(\upsilon) + \frac{1}{4}\delta * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} H(x)$$

$$= \delta(x) * R^H_4(\upsilon) * (-1)^2 R^e_4(x).$$

Thus

$$\left(\frac{3}{4}R^H_4(\upsilon) + \frac{1}{4}(-1)^2 R^e_4(x)\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} H(x) = R^H_4(\upsilon) * (-1)^2 R^e_4(x)$$

keeping on convolving both sides of the above equation by $R^H_4(\upsilon) * (-1)^2 R^e_4(x)$ up to $k-1$ times, we obtain

$$S^{*k}(x) * H(x) = \left(R^H_4(\upsilon) * (-1)^2 R^e_4(x)\right)^{*k}$$

the symbol $*k$ denotes the convolution of itself k-times. By properties of $R_\alpha(\upsilon)$, we have

$$\left(R^H_4(\upsilon) * (-1)^2 R^e_4(x)\right)^{*k} = R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x).$$

Putting in the above equation, we obtain

$$S^{*k}(x) * H(x) = \left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right)$$

$$H(x) = \left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * (S^{*k}(x))^{*-1}$$

is an elementary solution of the operator L^k.

Lemma 2.6. *Given the equation*

$$\otimes^k_w u(x) = 0, \tag{2.16}$$

where $\otimes^k_w$ is the operator iterated k times defined by (1.6) and $u(x)$ is an unknown generalized function. Then

$$u(x) = (R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)} \tag{2.17}$$

and by (1.7), we obtain

$$u(x) = ((-1)^{2k} K_{4k,4k}(x)) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)} \tag{2.18}$$

is solution of (2.16) and $(R^H_{2(k-1)}(\upsilon))^{(m)}$ is a function with m-derivatives defined by (2.2) and υ is defined by definition 2.1.and $S(x)$ defined by (2.15)

Proof. Now,

$$\otimes^k_w u(x) = \Box^k(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2)^k u(x) = 0.$$

By Lemma 2.4, we have

$$(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2)^k u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}. \tag{2.19}$$

Convolving both sides by $\left(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(\upsilon)\right) * (S^{*k}(x))^{*-1}$, we have

$$\left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * (S^{*k}(x))^{*-1} * (\frac{3}{4}\triangle^2 + \frac{1}{2}\Box^2)^k u(x) = \left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)})(\upsilon))^{(m)}$$

By Lemma 2.5,

$$\delta(x) * u(x) = \left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)}. \tag{2.20}$$

It follows that

$$u(x) = \left(R^H_{4k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)} \tag{2.21}$$

and by (1.7) we obtain

$$u(x) = \left((-1)^{2k} K_{4k,4k}(x)\right) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)} \tag{2.22}$$

as a homogeneous solution of (2.16). That complete this proof.

3. Main Results

Theorem 3.1. *Given the equation*

$$\otimes^k_w G(x) = \delta(x) \tag{3.1}$$

then

$$G(x) = \left(R^H_{6k}(u) * (-1)^{2k} R^e_{4k}(\upsilon)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{3.2}$$

and by (1.7), we obtain

$$G(x) = \left((-1)^{2k} K_{4k,6k}(x)\right) * (S^{*k}(x))^{*-1} \tag{3.3}$$

is a Green function for the operator $\otimes^k_w$ iterated k-times where $\otimes$ is defined by (1.6), and

$$S(x) = \frac{3}{4} R^H_4(\upsilon) + \frac{1}{4}(-1)^2 R^e_4(x) \tag{3.4}$$

$S^{*k}(x)$ denotes the convolution of S itself k-times, $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra. Moreover $G(x)$ is a tempered distribution.

Proof. From (3.1), we have

$$\otimes_w^k G(x) = \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^k G(x) = \delta(x)$$

or we can write

$$\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $R_6^H(\upsilon) * (-1)^2 R_4^e(x)$,

$$\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)\left(R_6^H(\upsilon) * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^{k-1} G(x)$$
$$= \delta(x) * R_6^H(\upsilon) * (-1)^2 R_4^e(x)$$

or

$$\left(\frac{3}{4}\Box\left(R_2^H(\upsilon)\right) * \triangle^2\left((-1)^2 R_4^e(x)\right) + \frac{1}{4}\Box^3 R_6^H(\upsilon) * (-1)^2 R_4^e(x)\right)\cdot$$
$$\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^{k-1} G(x) = \delta(x) * R_6^H(\upsilon) * (-1)^2 R_4^e(x).$$

By (2.4) and (2.8), we obtain

$$\left(\frac{3}{4}\delta * \left(\triangle^2 R_4^H(\upsilon)\right) + \frac{1}{4}\delta * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^{k-1} G(x) = \delta(x) * R_6^H(\upsilon).$$

Thus

$$\left(\frac{3}{4}R^H(\upsilon) + \frac{1}{4}(-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3\right)^{k-1} G(x) = R_6^H(\upsilon) * (-1)^2 R_4^e(x).$$

keeping on convolving both sides of the above equation by $R_6^H(\upsilon) * (-1)^2 R_4^e(x)$ up to $k-1$ times, we obtain

$$S^{*k}(x) * G(x) = \left(R_6^H(\upsilon) * (-1)^2 R_4^e(x)\right)^{*k} \tag{3.5}$$

the symbol $*k$ denotes the convolution of itself k-times. By properties of $R_\alpha(\upsilon)$, we have

$$\left(R_6^H(\upsilon) * (-1)^2 R_4^e(x)\right)^{*k} = R_{6k}^H(\upsilon) * (-1)^{2k} R_{4k}^e(x).$$

Putting into (3.5), we obtain

$$S^{*k}(x) * G(x) = R^H_{6k}(\upsilon) * (-1)^{2k} R^e_{4k}(x). \tag{3.6}$$

Now, consider the function $S^{*k}(x)$, since $R^H_6(\upsilon) * (-1)^2 R^e_4(x)$ is a tempered distribution. Thus $S(x)$ defined by (2.15) is a tempered distribution, we obtain $S^{*k}(x)$ is a tempered distribution. Now, $R^H_{6k}(\upsilon)*(-1)^{2k} R^e_{4k}(x) \in \mathcal{S}'$, the space of tempered distribution. Choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution.

Thus $R^H_{6k}(\upsilon) * (-1)^{2k} R^e_{4k}(x) \in \mathcal{D}'_{\mathcal{R}}$. It follow that $R^H_{6k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)$ is an element of convolution algebra, since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence Zemanian (see [9]), the equation (3.6) has a unique solution

$$G(x) = \left(R^H_{6k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{3.7}$$

or

$$G(x) = \left((-1)^{2k} K_{4k,6k}(x)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{3.8}$$

where $\left(S^{*k}(x)\right)^{*-1}$ is an inverse of $S^{*k}(x)$ in the convolution algebra, $G(x)$ is called the Green function of the operator $\otimes^k_w$. Since $R^H_{6k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)$ and $\left(S^{*k}(x)\right)^{*-1}$ are lie in S', then by [9, p.152] again, we have

$$\left(R^H_{6k}(\upsilon) * (-1)^{2k} R^e_{4k}(x)\right) \left(S^{*k}(x)\right)^{*-1} \in S'.$$

Hence $G(x)$ is a tempered distribution.

Theorem 3.2. *Given the equation*

$$\otimes^k_w u(x) = f(x), \tag{3.9}$$

where $\otimes^k_w$ is the operator iterated k times defined by (1.6), $f(x)$ is a generalized function, $u(x)$is an unknown function and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,the $n$dimensional Euclidean space and n is even, then (3.9)has the general solution

$$u(x) = ((-1)^{2k} K_{4k,4k}(x) * (R^H_{2(k-1)}(\upsilon))^{(m)} + (-1)^{2k} K_{4k,6k} * f(x)) * (S^{*k}(x))^{*-1}, \tag{3.10}$$

where $(R^H_{2(k-1)}(\upsilon))^{(m)}$ is a function m-derivatives defined by (2.2) and

$$K_{\alpha,\beta}(x) = R^e_\alpha(x) * R^H_\beta(\upsilon).$$

If we put $q = 0,\ k = 1$, we obtain solution related to the Laplace equation.

Proof. From (3.9), we have

$$\otimes_w^k u(x) = f(x),$$

Convolving both sides of the above equation by (3.8),we obtain

$$(-1)^{2k} K_{4k,6k}(x) * (S^{*k}(x))^{*-1} * \otimes_w^k u(x) = (-1)^{2k} K_{4k,6k}(x) * (S^{*k}(x))^{*-1} * f(x).$$

By properties of convolution

$$\otimes_w^k ((-1)^{2k} K_{4k,6k}(x) * (S^{*k}(x))^{*-1}) * u(x) = (-1)^{2k} K_{4k,6k}(x) * (S^{*k}(x))^{*-1} * f(x).$$

By Theorem 3.1, we obtain

$$\delta(x) * u(x) = u(x) = (-1)^{2k} K_{4k,6k}(x) * (S^{*k}(x))^{*-1} * f(x). \tag{3.11}$$

Next, we consider a homogeneous equation

$$\otimes_w^k u(x) = 0,$$

we have a solution[see,Lemma 2.6]

$$u(x) = (-1)^{2k} K_{4k,4k}(x) * (S^{*k})^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)}). \tag{3.12}$$

Thus the general solution of (3.9) is

$$u(x) = (-1)^{2k} K_{4k,4k}(x) * (S^{*k}(x))^{*-1} * (R^H_{2(k-1)}(\upsilon))^{(m)} + (-1)^{2k} K_{4k,6k}(x) * (S^{*k}(x))^{*-1} * f(x)$$

or

$$u(x) = ((R^H_{2(k-1)}(\upsilon))^{(m)} + R^H_{2k}(\upsilon) * f(x)) * (-1)^{2k} K_{4k,4k}(x) * (S^{*k}(x))^{*-1}. \tag{3.13}$$

In particular, if $q = 0$ the equation (3.9) is the Laplace equation

$$\triangle^{3k} u(x) = f(x), \tag{3.14}$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even. Now, from (2.17) for $q = 0$ we have

$$\triangle^{3k} u(x) = 0 \quad or \quad \triangle^{2k} \left(\triangle^k u(x) \right) = 0.$$

By lemma 2.3, we have

$$\triangle^{2k} u(x) = (-1)^{k-1} \left(R^e_{2(k-1)}(x) \right)^{(m)}.$$

Convolving both sides the above equation by $(-1)^{2k}R^e_{4k}(x)$, we obtain

$$\begin{aligned} u(x) &= (-1)^{2k}R^e_{4k}(x) * (-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)} \\ &= (-1)^{2k}R^e_{4k}(x) * (-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)} \\ &= (-1)^{3k-1}(R^e_{6k-2}(x))^{(m)}, \end{aligned} \tag{3.15}$$

is a homogeneous solution of (3.14). Next we are finding particular solution. Convolving both sides of (3.14) by $(-1)^{3k}R^e_{6k}(x)$, we obtain

$$(-1)^{3k}R^e_{6k}(x) * \triangle^{3k}u(x) = (-1)^{3k}R^e_{6k}(x) * f(x)$$

or

$$\triangle^{3k}\left((-1)^{3k}R^e_{6k}(x)\right) * u(x) = (-1)^{3k}R^e_{6k}(x) * f(x)$$

By (2.7), we obtain

$$\delta(x) * u(x) = u(x) = (-1)^{3k}R^e_{6k}(x) * f(x). \tag{3.16}$$

By (3.15) and (3.18), we obtain the general solution of equation (3.14) is

$$u(x) = (-1)^{3k-1}(R^e_{6k-2}(x))^{(m)} + (-1)^{3k}R^e_{6k}(x) * f(x) \tag{3.17}$$

for $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$ and p is even.

It follows that (3.17) is the general solution of the Laplace equation

$$\triangle^{3k}u(x) = f(x)$$

where $\triangle^{3k}$ is the Laplace operator iterated $3k$-times defined by (1.3)for $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$ and p is even and if we put $k = 1$, in (3.18), we obtain

$$u(x) = (-1)^2(R^e_4(x))^{(m)} + (-1)^3R^e_6(x) * f(x) \tag{3.18}$$

is the general solution of the equation

$$\triangle^3u(x) = f(x),$$

which is related to the Laplace equation. That completes this proof

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References

[1] W.F. Donoghue, *Distributions and Fourier Transform*, Academic Press (1969).

[2] I.M. Gelfand, G.E. Shilov, *Generalized Function*, Academic Press, New York (1964).

[3] A. Kananthai, *On the Solution of the n-Dimensional Diamond Operator*, Applied Mathematics and Computational, Elsevier Science Inc, New York.

[4] A. Kananthai, On the ditribution related to the ultra-hyperbolic equation, *Applied Mathematics and Computational*, Elsevier Science Inc, New York, **84** (1997), 101-106.

[5] Y.Nozaki, On Riemann-Liouville integral of Ultrahyperbolic type, *Kodai Mathematical Seminar Reports*, **6**, No. 2 (1964), 69-87.

[6] M.A. Tellez, *The Distribution Hankel Tranform of Marcel's Ultra-Hyperbolic Kernel*, Studies in Applied Mathematics, Volume 93, Massachusetts Institute of Technology, Elsevier, Amsterdam (1994), 133-162.

[7] S.E. Trione, On Marcel Riesz: On the Ultra-hyperbolic kernel, *Trabajos de Mathematica*, **116** (1987).

[8] S.E. Trione, *On Marcel Riesz's Ultra-Hyperbolic Kernel*, Studies in Applied Mathematics, Volume 79, Massachusetts Institute of Technology, Elsevier, Cambridge, Massachusetts, USA (1988), 185-191.

[9] A.H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill (1965), 150-151.