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# APPROXIMATION ORDER OF NONSTATIONARY TIGHT WAVELET FRAMES IN SOBOLEV SPACES

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**Abstract:** In this paper, approximation order of the nonstationary tight frame approximation operators in Sobolev spaces is studied. We first give two lemmas and then obtain the main theorem. The final result shows that the functions in a large class of Sobolev spaces can be approximated by its approximation operators under the Sobolev norm.

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**Key Words:** nonstationary, frames, Sobolev spaces, approximation order

## 1. Introduction and Preliminaries

As a redundant wavelet system, wavelet frames are easier to design and provide more flexibilities in applications. Because of this, wavelet frames have been extensively studied in the literature. In particular, wavelet frames obtained from refinable functions are of interest, due to the associated multiresolution structure and fast frame algorithms. Constructions of tight wavelet frames from a refinable function can be done by the unitary extension principle(UEP)[7]; Later, more general oblique extension principle(OEP) was independently developed by [3-4]. For the stationary case, it is impossible to obtain MRA-based compactly supported tight wavelet frames in $L^2(R)$ whose generators are in $C^\infty(R)$. In recent years, nonstationary tight wavelet frames have been systematically studied in [1,2,5,6]. In this paper, we mainly study the approximation order of the frame approximation operators in Sobolev spaces. Section 2 is devoted to some lemmas and the main result is given in part 3.

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For $2\pi$-periodic trigonometric polynomials masks $\widehat{a_j}, j \in N$, their associated nonstationary refinable functions are defined by

$$\widehat{\phi_{j-1}}(\xi) =: \widehat{a_j}(\frac{\xi}{2})\widehat{\phi_j}(\frac{\xi}{2}) = \prod_{n=1}^{\infty} \widehat{a_{n+j-1}}(2^{-n}\xi), \xi \in R^d, j \in N; \tag{1.1}$$

Wavelet functions $\psi_{j-1}^{\ell}, j \in N$ are defined by

$$\widehat{\psi_{j-1}^{\ell}}(\xi) =: \widehat{b_j^{\ell}}(\frac{\xi}{2})\widehat{\phi_j}(\frac{\xi}{2}),\ \ell = 1, 2, \cdots, L. \tag{1.2}$$

For a real number $s$, we denote by $H^s(R^d)$ the Sobolev space consisting of all tempered distributions $f$ such that

$$\|f\|_{H^s(R^d)}^2 =: \frac{1}{(2\pi)^d} \int_{R^d} |\hat{f}(\xi)|^2 (1 + \|\xi\|^2)^s d\xi < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm in $R^d$. $H^s(R^d)$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{H^s(R^d)} =: \frac{1}{(2\pi)^d} \int_{R^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1 + \|\xi\|^2)^s d\xi, \quad f, g \in H^s(R^d).$$

The Sobolev seminorm $|f|_{H^s(R^d)}$ is defined to be

$$|f|_{H^s(R^d)} =: \int_{R^d} \|\xi\|^{2s} |\hat{f}(\xi)|^2 d\xi.$$

Denote $\mathbb{N}_0 =: \mathbb{N} \cup \{0\}$. For given $\phi_0, \psi_j^{\ell}(j \in \mathbb{N}_0,\ \ell = 1, 2, \cdots, L)$, wavelet system

$$X(\phi_0;\ \{\psi_j^{\ell}\}_{j\in\mathbb{N}_0,\ \ell\in\{1,2,\cdots,L\}}) =: \{\phi_0(\cdot - k) :\ k \in \mathbb{Z}^d\} \cup \{\psi_{j;j,k}^{\ell} :\ j \in \mathbb{N}_0,\ \ell = 1, 2, \cdots, L\}$$

with $\psi_{j;j,k}^{\ell} =: 2^{j(\frac{d}{2})}\psi_j^{\ell}(2^j \cdot -k)$ is a nonstationary tight wavelet frame in $L^2(R^d)$ if

$$\|f\|_{L^2(R^d)}^2 = \sum_{k\in\mathbb{Z}^d} |\langle f, \phi_{0;0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{j;j,k}^{\ell}\rangle|^2,\ f \in L^2(R^d).$$

Furthermore, we define the frame approximation operators

$$Q_n(f) =: \sum_{k\in\mathbb{Z}^d} \langle f, \phi_0(\cdot - k)\rangle \phi_0(\cdot - k) + \sum_{\ell=1}^{L}\sum_{j=0}^{n-1}\sum_{k\in\mathbb{Z}^d} \langle f, \psi_{j;j,k}^{\ell}\rangle \psi_{j;j,k}^{\ell}.$$

## 2. Lemmas

For a $2\pi-$ periodic trigonometric polynomial $\hat{a}$ in $d-$ variables, we denote $deg(\hat{a})$ the smallest nonnegative integer such that its Fourier coefficients vanish outside $[-deg(\hat{a}), deg(\hat{a})]^d$.

**Lemma 2.1.** (see [6]) *Let* $\widehat{a_j},\ j \in \mathbb{N}$ *be* $2\pi-$ *periodic trigonometric polynomials such that* $|\widehat{a_j}(\xi)|^2 + |\widehat{a_j}(\xi+\pi)|^2 \le 1$. *If* $\sum_{j=1}^{\infty} 2^{-j} deg(\widehat{a_j}) < \infty$ *and* $\widehat{a_j}(0) = 1$ *hold, then the infinite product in* (1.1) *converges uniformly on every compact set of* $R^d$ *and all* $\phi_j,\ j \in \mathbb{N}_0$ *are well-defined compactly supported functions in* $L^2(R)$. *Moreover,* $\sum_{k\in Z} |\widehat{\phi}_j(\xi + 2k\pi)|^2 \le 1,\ a.e. \xi \in R,\ j \in N_0$.

For two functions $f, g:\ R^d \to \mathbb{C}$, define

$$[f,g]_t(\xi) =: \sum_{k\in\mathbb{Z}^d} f(\xi+2k\pi)\overline{g(\xi+2k\pi)}(1+\|\xi+2k\pi\|^2)^t,\ \xi \in R^d,\ t \in R.$$

Furthermore, for our use, we define $\nu(\phi) =: \sup\{s \in R:\ [\widehat{\phi}_j, \widehat{\phi}_j]_s \le (1+|\xi|^{2s}),\ j \in \mathbb{N}_0\}$.

**Lemma 2.2.** *Let* $\phi \in H^s(R)$ *and* $\upsilon \ge s \ge 0$. *Define a linear operator* $P$ *by*

$$P(f) =: \sum_{k\in Z} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k),\ f \in H^s(R),$$

*then* $\|f - P(f)\|_{H^s(R)} \le C_\phi |f|_{H^\upsilon(R)}$ *for all* $f \in H^\upsilon(R)$ *with a positive constant*

$$C_\phi =: \sqrt{\max\{2c_1, 2c_3\} + \max\{4c_2, 4c_4\} + 4}$$

*provided that there exist positive constants* $c_1, c_2, c_3$ *and* $c_4$ *such that for almost every* $\xi \in [-\pi, \pi]$, *the following inequalities hold*

$$(1+|\xi|^{2s})|1 - |\widehat{\phi}(\xi)|^2|^2 \le c_1|\xi|^{2\upsilon}$$

$$|\widehat{\phi}(\xi)|^2 \sum_{k\ne 0}(1+|\xi+2k\pi|^{2s})|\widehat{\phi}(\xi+2k\pi)|^2 \le c_2|\xi|^{2\upsilon}$$

$$(1+|\xi|^{2s})|\widehat{\phi}(\xi)|^2 \sum_{k\ne 0} |\xi+2k\pi|^{-2\upsilon}|\widehat{\phi}(\xi+2k\pi)|^2 \le c_3$$

$$\sum_{k\ne 0} |\xi+2k\pi|^{-2\upsilon} \sum_{\ell\ne 0}(1+|\xi+2\ell\pi|^{2s})|\widehat{\phi}(\xi+2\ell\pi)|^2|\widehat{\phi}(\xi+2k\pi)|^2 \le c_4.$$

*Proof.* Note that

$$\|f-P(f)\|^2_{H^s(R)}=\int_R(1+|\xi|^{2s})|[P(f)-f]^\wedge(\xi)|^2d\xi$$

$$=\int_R(1+|\xi|^{2s})|\widehat{\phi}(\xi)\sum_k\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}-\widehat{f}(\xi)|^2d\xi.$$

We look at the integrals over $C=[-\pi,\pi]$ and $R\setminus C$ separately.

$$\int_C(1+|\xi|^{2s})|\widehat{\phi}(\xi)\sum_k\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}-\widehat{f}(\xi)|^2d\xi$$

$$=\int_C(1+|\xi|^{2s})|(|\widehat{\phi}(\xi)|^2-1)\widehat{f}(\xi)+\widehat{\phi}(\xi)\sum_{k\neq0}\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}|^2d\xi$$

$$\le2\int_C(1+|\xi|^{2s})\{|(|\widehat{\phi}(\xi)|^2-1)\widehat{f}(\xi)|^2+|\widehat{\phi}(\xi)|^2|\sum_{k\neq0}\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}|^2\}d\xi$$

$$\le2\int_C(1+|\xi|^{2s})\{|(|\widehat{\phi}(\xi)|^2-1)\widehat{f}(\xi)|^2$$

$$+|\widehat{\phi}(\xi)|^2\sum_{k\neq0}|\xi+2k\pi|^{-2v}|\widehat{\phi}(\xi+2k\pi)|^2\sum_{k\neq0}|\xi+2k\pi|^{2v}|\widehat{f}(\xi+2k\pi)|^2\}d\xi$$

$$\le2c_1\int_C|\xi|^{2v}|\widehat{f}(\xi)|^2d\xi+2c_3\int_C\sum_{k\neq0}|\xi+2k\pi|^{2v}|\widehat{f}(\xi+2k\pi)|^2d\xi$$

$$\le\max\{2c_1,2c_3\}\int_C\sum_k|\xi+2k\pi|^{2v}|\widehat{f}(\xi+2k\pi)|^2d\xi$$

$$=\max\{2c_1,2c_3\}\int_R|\xi|^{2v}|\widehat{f}(\xi)|^2d\xi.$$

On the other hand, we have

$$\int_{R\setminus C}(1+|\xi|^{2s})|\widehat{\phi}(\xi)\sum_k\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}-\widehat{f}(\xi)|^2d\xi$$

$$=\sum_{\beta\neq0}\int_C(1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)\sum_k\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}-\widehat{f}(\xi+2\beta\pi)|^2d\xi$$

$$\leq 2\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})\{|\widehat{\phi}(\xi+2\beta\pi)\sum_k \widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}|^2+|\widehat{f}(\xi+2\beta\pi)|^2\}d\xi$$

$$\leq 4\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)|^2|\widehat{f}(\xi)|^2|\widehat{\phi}(\xi)|^2 d\xi +$$

$$4\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)|^2|\sum_{k\neq 0}\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}|^2 d\xi +$$

$$2\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{f}(\xi+2\beta\pi)|^2 d\xi.$$

Furthermore, we obtain that the first item

$$4\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)|^2|\widehat{f}(\xi)|^2|\widehat{\phi}(\xi)|^2 d\xi \leq 4c_2\int_C |\xi|^{2v}|\widehat{f}(\xi)|^2 d\xi;$$

The second item

$$4\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)|^2|\sum_{k\neq 0}\widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}|^2 d\xi$$

$$= 4\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)|^2|$$

$$\sum_{k\neq 0}\widehat{f}(\xi+2k\pi)(\xi+2k\pi)^{v}(\xi+2k\pi)^{-v}\overline{\widehat{\phi}(\xi+2k\pi)}|^2 d\xi$$

$$\leq 4\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{\phi}(\xi+2\beta\pi)|^2\sum_{k\neq 0}|(\xi+2k\pi)|^{-2v}|\widehat{\phi}(\xi+2k\pi)|^2$$

$$\sum_{k\neq 0}|(\xi+2k\pi)|^{2v}|\widehat{f}(\xi+2k\pi)|^2 d\xi$$

$$\leq 4c_4\int_C\sum_{k\neq 0}|(\xi+2k\pi)|^{2v}|\widehat{f}(\xi+2k\pi)|^2 d\xi;$$

The third item

$$2\sum_{\beta\neq 0}\int_C (1+|\xi+2\beta\pi|^{2s})|\widehat{f}(\xi+2\beta\pi)|^2 d\xi \leq 4\sum_{\beta\neq 0}\int_C |\xi+2\beta\pi|^{2s}|\widehat{f}(\xi+2\beta\pi)|^2 d\xi$$

$$\leq 4\sum_{\beta\neq 0}\int_C |\xi+2\beta\pi|^{2\upsilon}|\widehat{f}(\xi+2\beta\pi)|^2 d\xi \leq 4\int_R |\xi|^{2\upsilon}|\widehat{f}(\xi)|^2 d\xi.$$

Therefore, we obtain

$$\int_{R\backslash C}(1+|\xi|^{2s})|\widehat{\phi}(\xi)\sum_k \widehat{f}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}-\widehat{f}(\xi)|^2 d\xi$$

$$\leq (\max\{4c_2,4c_4\}+4)\int_R |\xi|^{2\upsilon}|\widehat{f}(\xi)|^2 d\xi.$$

Finally, $\|f-P(f)\|^2_{H^s(R)} \leq C^2_\phi |f|^2_{H^\upsilon(R)}$ with the constant

$$C_\phi =: \sqrt{\max\{2c_1,2c_3\}+\max\{4c_2,4c_4\}+4}.$$

Define the linear operators $P_n(f) =: \sum_{k\in Z}\langle f,\phi_{n;n,k}\rangle\phi_{n;n,k}$ with $\phi_{n;n,k} =: 2^{\frac{n}{2}}\phi_n(2^n\cdot -k)$.

**Lemma 2.3.** *Let $\widehat{a_j}$, $j\in\mathbb{N}$ be $2\pi-$ periodic trigonometric polynomials satisfying the conditions of Lemma 2.1 and $\upsilon\geq s\geq 0$. If for $n\in N$,*

$$(1+|\xi|^{2s})(1-|\widehat{\phi_n}(\xi)|^2)\leq C_{\phi_n}|\xi|^{2\upsilon}, \quad a.e.\ \xi\in[-\pi,\pi],$$

*then the linear operators $P_n$ satisfies*

$$\|f-P_n(f)\|_{H^s(R)}\leq \max\{\sqrt{8C_{\phi_n}},4\}2^{n(s+\frac{1}{2})}2^{-\upsilon n}|f|_{H^\upsilon(R)},\ f\in H^\upsilon(R).$$

*Proof.* By Lemma 2.1, we know $|\widehat{\phi_n}(\xi)|\leq 1$, then

$$(1+|\xi|^{2s})|1-|\widehat{\phi_n}(\xi)|^2|^2 \leq (1+|\xi|^{2s})(1-|\widehat{\phi_n}(\xi)|^2)\leq C_{\phi_n}|\xi|^{2\upsilon}.$$

$$|\widehat{\phi_n}(\xi)|^2\sum_{k\neq 0}(1+|\xi+2k\pi|^{2s})|\widehat{\phi_n}(\xi+2k\pi)|^2 \leq \sum_{k\neq 0}(1+|\xi+2k\pi|^{2s})|\widehat{\phi_n}(\xi+2k\pi)|^2$$

$$=[\widehat{\phi_n},\widehat{\phi_n}]_s-(1+|\xi|^{2s})|\widehat{\phi_n}(\xi)|^2 \leq (1+|\xi|^{2s})-(1+|\xi|^{2s})|\widehat{\phi_n}(\xi)|^2\leq C_{\phi_n}|\xi|^{2\upsilon}.$$

On the other hand, for $\xi\in[-\pi,\pi]$ and $k\neq 0$, it is obvious that $|\xi+2k\pi|^{-2\upsilon}\leq 1$. Therefore,

$$(1+|\xi|^{2s})|\widehat{\phi_n}(\xi)|^2\sum_{k\neq 0}|\xi+2k\pi|^{-2\upsilon}|\widehat{\phi_n}(\xi+2k\pi)|^2\leq 2,\ 0\leq|\xi|\leq 1.$$

For $1 \le |\xi| \le \pi$, since $|\xi + 2k\pi| \ge |\xi|$, we have

$$(1+|\xi|^{2s})|\widehat{\phi_n}(\xi)|^2 \sum_{k\neq 0} |\xi+2k\pi|^{-2\upsilon}|\widehat{\phi_n}(\xi+2k\pi)|^2 \le (1+|\xi|^{2s})|\xi|^{-2\upsilon} \le 2.$$

Similarly, for $0 \le |\xi| \le 1$, we obtain

$$\sum_{k\neq 0} |\xi+2k\pi|^{-2\upsilon} \sum_{\ell\neq 0}(1+|\xi+2\ell\pi|^{2s})|\widehat{\phi_n}(\xi+2\ell\pi)|^2|\widehat{\phi_n}(\xi+2k\pi)|^2$$

$$\le \sum_{\ell\neq 0}(1+|\xi+2\ell\pi|^{2s})|\widehat{\phi_n}(\xi+2\ell\pi)|^2 \le 1+|\xi|^{2s} \le 2.$$

For $1 \le |\xi| \le \pi$, $\sum\limits_{k\neq 0} |\xi+2k\pi|^{-2\upsilon} \sum\limits_{\ell\neq 0}(1+|\xi+2\ell\pi|^{2s})|\widehat{\phi_n}(\xi+2\ell\pi)|^2|\widehat{\phi_n}(\xi+2k\pi)|^2$

$$\le \sum_{k\neq 0} |\xi+2k\pi|^{-2\upsilon}|\widehat{\phi_n}(\xi+2k\pi)|^2(1+|\xi|^{2s}) \le (1+|\xi|^{2s})|\xi|^{-2\upsilon} \le 2.$$

Therefore, $\phi_n$ satisfies the conditions of Lemma 2.2 with $c_1 = c_2 = C_{\phi_n},\ c_3 = c_4 = 2$. Moreover, the constant $C_\phi$ in Lemma 2.2 is $\max\{\sqrt{8C_{\phi_n}}, 4\}$.

Finally, it is easy to verify that $P_n(f)(x) = P(f(2^{-n}\cdot))(2^n x)$. Furthermore, we obtain

$$[P_n(f)]^\wedge(\xi) = 2^{-n}[P(f(2^{-n}\cdot))](2^{-n}\xi).$$

$$\|f - P_n(f)\|^2_{H^s(R)} = \int_R (1+|2^n\xi|^{2s})|[f(2^{-n}\cdot)]^\wedge(\xi) - [P(f(2^{-n}\cdot))](\xi)|^2 d\xi$$

$$\le 2^{2ns}\|f(2^{-n}\cdot) - P(f(2^{-n}\cdot))\|^2_{H^s(R)} \le 2^{2ns}[\max\{\sqrt{8C_{\phi_n}}, 4\}]^2|f(2^{-n}\cdot)|^2_{H^\upsilon(R)}.$$

Therefore, we have

$$\|f - P_n(f)\|_{H^s(R)} \le 2^{ns}\max\{\sqrt{8C_{\phi_n}}, 4\}|f(2^{-n}\cdot)|_{H^\upsilon(R)}$$
$$= 2^{ns}\max\{\sqrt{8C_{\phi_n}}, 4\}2^{\frac{n}{2}}2^{-\upsilon n}|f|_{H^\upsilon(R)}.$$

## 3. Main Result

**Theorem 3.1.** *Let $\widehat{a_j}, j \in \mathbb{N}$ be $2\pi-$ periodic trigonometric polynomials, which satisfy the conditions of Lemma 2.1. Suppose that $\widehat{\phi_j}$, $\widehat{\psi_j^\ell}(j \in \mathbb{N}_0)$ are defined as in* (1.1) *and* (1.2)*. Moreover, the following conditions holds:*

(1)

$$|\widehat{a_j}(\xi)|^2 + \sum_{\ell=1}^{L} |\widehat{b_j^\ell}(\xi)|^2 = 1$$

$$\overline{\widehat{a_j}(\xi)}\widehat{a_j}(\xi+\pi) + \sum_{\ell=1}^{L} \overline{\widehat{b_j^\ell}(\xi)}\widehat{b_j^\ell}(\xi+\pi) = 0;$$

(2) *for* $s \geq 0$, *there is* $\nu(\phi) > s$;

(3) *In addition, there exists a* $\upsilon \in \frac{1}{2}N,\ \upsilon > s + \frac{1}{2}$ *and a positive integer* $N$ *such that*

$$|\widehat{a_j}(\xi)|^2 = 1 + O(|\xi|^{2\upsilon}),\quad \xi \to 0,\ j \geq N.$$

*Moreover, for some* $\alpha \geq 0, 0 \leq \beta < \frac{\upsilon - s - \frac{1}{2}}{\upsilon}$, *there have*

$$deg(\widehat{a_j}) = O(j^\alpha 2^{\beta j}),\quad j \to \infty.$$

*Then there exists a* $C > 0$ *independent of* $f$ *and* $n$ *such that*

$$\|f - Q_n(f)\|_{H^s(R)} \leq C 2^{n(s+\frac{1}{2})} n^{\upsilon\alpha} 2^{-\upsilon(1-\beta)n} |f|_{H^\upsilon(R)},\ n \geq N.$$

*Proof.* It has been proved in [6] that

$$X(\phi_0;\ \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ \ell\in\{1,2,\cdots,L\}}) =: \{\phi_0(\cdot - k):\ k \in \mathbb{Z}^d\} \cup \{\psi_{j;j,k}^\ell:\ j \in \mathbb{N}_0,\ \ell = 1,2,\cdots,L\}$$

is a nonstationary tight wavelet frame in $L^2(R^d)$. Now, let us first estimate the constant $C_{\phi_n}$ in Lemma 2.3. Define $\widehat{d_j}(\xi) =: |\widehat{a_j}(\xi)|^2$, then by the same method as [6], we obtain

$$C_{\phi_n} =: \frac{1+\pi^{2s}}{(2\upsilon)!} \sum_{\ell=1}^{\infty} 2^{-2\upsilon\ell} [deg(\widehat{d_{\ell+n}})]^{2\upsilon}.$$

By the condition (3), there exists a $C_1 > 0$ such that

$$deg(\widehat{a_j}) \leq C_1 j^\alpha 2^{\beta j},\ j \in N.$$

Therefore,

$$2^{-2\upsilon\ell}[deg(\widehat{d_{\ell+n}})]^{2\upsilon} \leq 2^{2\upsilon} C_1^{2\upsilon} (\ell+n)^{2\upsilon\alpha} 2^{2\upsilon\beta(\ell+n)} 2^{-2\upsilon\ell}$$

$$= 2^{2\upsilon} C_1^{2\upsilon} n^{2\upsilon\alpha} 2^{2\upsilon\beta n} (1+\frac{\ell}{n})^{2\upsilon\alpha} 2^{-2\upsilon(1-\beta)\ell} \leq 2^{2\upsilon} C_1^{2\upsilon} n^{2\upsilon\alpha} 2^{2\upsilon\beta n} (1+\ell)^{2\upsilon\alpha} 2^{-2\upsilon(1-\beta)\ell}.$$

Furthermore,

$$C_{\phi_n} \leq (1+\pi^{2s}) \frac{2^{2\upsilon} C_1^{2\upsilon}}{(2\upsilon)!} n^{2\upsilon\alpha} 2^{2\upsilon\beta n} \sum_{\ell=1}^{\infty} (1+\ell)^{2\upsilon\alpha} 2^{-2\upsilon(1-\beta)\ell} =: C_2 n^{2\upsilon\alpha} 2^{2\upsilon\beta n}.$$

Finally, by Lemma 2.3, we know

$$\begin{aligned} \|f - Q_n(f)\|_{H^s(R)} &= \|f - P_n(f)\|_{H^s(R)} \\ &\leq \max\{\sqrt{8C_2}, 4\} 2^{n(s+\frac{1}{2})} n^{\upsilon\alpha} 2^{-\upsilon(1-\beta)n} |f|_{H^\upsilon(R)}, \ n \geq N \end{aligned}$$

with $C =: \max\{\sqrt{8C_2}, 4\}$ independent of $f$ and $n$.

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