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SPECTRAL ANALYSIS OF THE SEMIGROUP ASSOCIATED TO A MIXED FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract: In this article we present an spectral analysis of the semigroup associated to certain mixed functional differential equation. The associated semigroup to the solution of this equation and its correspondig infinitesimal generator are defined on a closed subspace of $C^{\infty}_{[-1,1]}$. Expressions for the resolvent associated to the infinitesimal generator and its point spectrum are presented so as the analysis of the asymptotic distribution of the eigenvalues of the infinitesimal generator associated to the semigroup.

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1. Introduction

Mixed functional differential equations (MFDE) are a class of functional differential equations where the time derivative depends on both past and future values of the variable. They are also referred in literature as forward-backward equations. Interest in MFDEs is motivated by problems in optimal control (see [12]) and applications also arise in nerve conduction [5], economic dynamics (see [13], [10]) and physics (see [4], [1]). The study of other problems for mixed differential equations can be also found in (see [8], [9], [11], [3] and [7]) and the references therein. MFDE are, in general, ill-posed as initial value problems (see for example [12] and [8]), but there are also cases (see [7], [9] and [11]) where a unique solution exists.

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The goal of this paper is to present an spectral analysis of the semigroup associated to certain mixed functional differential equation. The associated semigroup to the solution of this equation and its correspondig infinitesimal generator are defined on a closed subspace of $C^{\infty}_{[-1,1]}$. Expressions for the resolvent associated to the infinitesimal generator and its point spectrum are presented so as the analysis of the asymptotic distribution of the eigenvalues of the infinitesimal generator associated to the semigroup.

In our previous work [9] we have studied the particular case of the scalar equation

$$x'(t) = x(t-1) + x(t+1), \tag{1.1}$$

where $t \geq 0$ and $x : [-1, +\infty) \to \mathbb{C}$, is differentiable in $[0, +\infty)$. In order to obtain a solution of (1.1), the function $x(t)$ is defined for $t \in [-1, 1]$ as

$$x(t) = \varphi(t) = \begin{cases} \varphi_1, & t \in [-1, 0] \\ \varphi_2, & t \in [0, 1], \end{cases} \tag{1.2}$$

where the function φ is taken in $C^{\infty}_{[-1,1]}$. Then using the step-by-step differentiation method we have constructed the solution which may be written via the following iterative formula on each arbitrary interval $(m, m+1)$:

$$x(t) = \sum_{i=0}^{m} c_{1i}\varphi^{(i)}(t-m) + \sum_{i=0}^{m} c_{2i}\varphi^{(i)}(t-m-1), \tag{1.3}$$

where c_{1i} and c_{2i} are constants not necessarily all different of zero. The proof is done by induction. We also proved the existence, uniqueness and smoothness of the solution. For the details of the proofs see Theorem 3.1, Corollary 3.1 and Theorem 3.2 from [9]. We recall these results:

[*i*] *The solution $x(t)$ of (1.1) satisfying the initial condition (1.2) where φ belongs to $C^{\infty}_{[-1,1]}$, exists and it is differentiable, if and only if the following relationship holds*

$$\varphi^{(n+1)}(0) = \varphi^{(n)}(-1) + \varphi^{(n)}(1) \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

[*ii*] *If for the initial function $\varphi \in C^{\infty}_{[-1,1]}$ there exists a differentiable solution $x(t)$ for $t \geq 0$ of equation (1.1) with the initial condition (1.2), then this solution belongs to the space $C^{\infty}_{[-1,+\infty)}$.*

[*iii*] *Let $\varphi \in C^{\infty}_{[-1,1]}$. If the solution $x(t)$ of equation (1.1) with the initial condition (1.2) exists and it is differentiable, then it is unique.*

Now in this article we analyze properties of the semigroup associated to the solutions of equation (1.1). A particular feature in this work is that the researched semigroup is defined on a topological space which is not a Banach space.

2. The Semigroup Associated to the Solutions

Consider the space $C^{\infty}_{[-1,1]}$. It is known that it is a topological space endows with the induced topology by the countable system of seminorms: $p_k(f) = \max_{x\in[-1,1]} |\partial^k f(x)|$ for $f \in C^{\infty}_{[-1,1]}$ and k a non-negative integer. The convergence in this topology means the uniform convergence of the function and each of its derivatives of any order. We denote $\|f_n - f\|_k = \sum_{i=0}^{k} \max_{[-1,1]} |f_n^{(i)}(x) - f^{(i)}(x)|$, for $k = 0, 1, ...$ Now consider the following closed subspace of $C^{\infty}_{[-1,1]}$

$$M = \{\varphi \in C^{\infty}_{[-1,1]} : \varphi^{(n)}(0) = \varphi^{(n-1)}(-1) + \varphi^{(n-1)}(1)\,,\ n = 1, 2, \dots\}.$$

For each $t \geq 0$ we define the operator T_t on the solutions $x(t)$ of equation (1.1) as follows: $T_t x(\theta) = x(t+\theta)\,,\ \theta \in [-1,1]$.

Theorem 2.1. *The family of operators $\{T_t : t \geq 0\}$ defines a strongly continuous semigroup in the space M.*

Proof. Straightforward from definition of T_t follow $T_0 = I$ (I =identity) and $T_{t+s} = T_t T_s$ for each $t, s \geq 0$. Since $x(\theta)/_{\theta\in[-1,1]} = \varphi(\theta) \in M$, the domain of T_t is M. On the other hand the function $y_t(\theta) := T_t x(\theta) = x(t+\theta)$ belongs to M for each $t \geq 0$, because if $x(t) \in C^{\infty}_{[-1,+\infty)}$ then $y_t(\theta) \in C^{\infty}_{[-1,1]}$ for each fixed $t \geq 0$. Besides $y_t^{(n)}(0) = y_t^{(n-1)}(-1) + y_t^{(n-1)}(1)$ or equivalently $x^{(n)}(t) = x^{(n-1)}(t-1) + x^{(n-1)}(t+1)$ for $n = 1, 2, \dots$ and for each $t \geq 0$. But this equality is obtained after differentiating $(n-1)$ times the equation (1.1) . To show that T_t is continuous for each fixed t, we shall prove that exist n_k for each k and some constant $c \geq 0$ such that $\|T_\tau x(\theta)\|_k \leq c \sum_{i=0}^{n_k} \|x(\theta)\|_i$, for each fixed $\tau \geq 0$. In view of

$$p_k(T_\tau x(\theta)) \leq \max_{[[\tau]-1,[\tau]]} |x^{(k)}(t)| + \max_{[[\tau],[\tau]+1]} |x^{(k)}(t)| + \max_{[[\tau]+1,[\tau]+2]} |x^{(k)}(t)|$$

and (1.3) with $m = [\tau]$ it follows $\max_{[-1,1]} |x^{(k)}(\tau+\theta)| \leq c \sum_{i=-1}^{i=1} \|\varphi\|_{m+k+i}$. Finally, $T_t x \xrightarrow[t\to t_0]{} T_{t_0} x \iff \|T_{t_0+\tau} x(\theta) - T_{t_0} x(\theta)\|_k \xrightarrow[\tau\to 0]{} 0$ for $t_0 \geq 0$ and $k = 0, 1, \dots$.

Therefore assuming that $0 \leq |\tau| \leq 1$ and taking into account the uniform continuity of $x^{(k)}(t)$ in the closed interval $[a,b] := [t_0 - 2, t_0 + 2]$ $(t, t' \in [a,b]\,)$, it follows

$$\max_{[-1,1]} |x^{(k)}(t_0 + \tau + \theta) - x^{(k)}(t_0 + \theta)| \leq \max_{|t-t'| \leq |\tau|} |x^{(k)}(t) - x^{(k)}(t')| \xrightarrow[\tau \to 0]{} 0. \quad \square$$

Remark 2.1. Since the solution $x(t)$ may be extended to the left, the semigroup T_t is actually a group.

We consider the space $D(A) = \{x \in C^{\infty}_{[-1,1]} : \lim_{t\to 0} \frac{T_t x - x}{t}$ exists$\}$ and we define the operator $A : D(A) \to C^{\infty}_{[-1,1]}$ as follows

$$Ax(\theta) = \lim_{t\to 0} \frac{T_t x(\theta) - x(\theta)}{t} = \lim_{t\to 0} \frac{x(t+\theta) - x(\theta)}{t},$$

where the limit is taking in the topology of $C^{\infty}_{[-1,1]}$. We call A the **infinitesimal generator** associated to the semigroup T_t.

Theorem 2.2. *The infinitesimal generator A maps M in M and $Ax = x'$.*

Proof. First we will prove $Ax(\theta) = x'(\theta)$. Assuming $x \in D(A)$ then we should have $\lim_{t\to 0} \max_{[-1,1]} |\frac{x(t+\theta) - x(\theta)}{t} - Ax(\theta)| = 0$. But this implies that for each $\theta \in [-1,1]$ there is the pointwise limit

$$\lim_{t\to 0} |\frac{x(t+\theta) - x(\theta)}{t} - Ax(\theta)| = 0 \;\text{ i.e., }\; Ax(\theta) = x'(\theta).$$

Now, we will prove that the domain of the operator A is M. Due the Main Value's Theorem, for each $n = 1, 2, \ldots$ there exists a number ξ_t with $0 < \xi_t < t$ such that

$$|\frac{x^{(n-1)}(t+\theta) - x^{(n-1)}(\theta)}{t} - x^{(n)}(\theta)| = |x^{(n)}(\theta + \xi_t) - x^{(n)}(\theta)|.$$

Since $x^{(n)}(\theta)$ is continuous in the closed interval $[-2, 2]$ and therefore uniform continuously in this interval, then for each $\varepsilon > 0$ exists $\delta > 0$ such that if $|\xi| < \delta$, $\max_{[-1,1]} |x^{(n)}(\theta + \xi) - x^{(n)}(\theta)| \xrightarrow[\xi \to 0]{} 0$. Since $\xi_t \xrightarrow[t \to 0]{} 0$ and applying limit of composition of functions

$$\max_{[-1,1]} |\frac{x^{(n-1)}(t+\theta) - x^{(n-1)}(\theta)}{t} - x^{(n)}(\theta)| \xrightarrow[t \to 0]{} 0 \;\text{ for each } n = 1, 2, \ldots$$

Now to check $Ax(\theta) \in M$ for each $x \in M$, we define $f(\theta) := Ax(\theta)$. Since $x \in C^{\infty}_{[-1,+\infty)}$ then $f = x' \in C^{\infty}_{[-1,+\infty)}$ and $f(\theta) = x'(\theta)$. Therefore due to $x \in M$ $f'(0) - f(-1) - f(1) = x''(0) - x'(-1) - x'(1) = 0$. Analogously, for each $n = 2, 3, \ldots$ $f^{(n-1)}(\theta) := x^{(n)}(\theta)$ and since $x \in M$ then $f^{(n)}(0) - f^{(n-1)}(-1) - f^{(n-1)}(1) = x^{(n+1)}(0) - x^{(n)}(-1) - x^{(n)}(1) = 0$. This means $f(\theta) \in M$. □

3. Spectral Analysis of the Infinitesimal Generator

In order to determine the resolvent associated to the infinitesimal generator we consider the equation $Ax(\theta) = \lambda x(\theta) + f(\theta)$, where f and x belong to M. Since $Ax(\theta) = x'(\theta)$ then the former equation can be rewritten as

$$x'(\theta) = \lambda x(\theta) + f(\theta). \tag{3.1}$$

and we consider it with the initial conditions $x^{(n+1)}(0) = x^{(n)}(-1) + x^{(n)}(1)$, $n = 0, 1, \ldots$. Therefore its solution is

$$x(\theta) = ce^{\lambda\theta} + \int_0^{\theta} e^{\lambda(\theta - s)} f(s)\, ds. \tag{3.2}$$

We choose the constant c such that the solution satisfies the infinity conditions $x^{(n+1)}(0) = x^{(n)}(-1) + x^{(n)}(1)$, $n = 0, 1, \ldots$. Since $x(\theta)$ belong to M then it satisfies the condition $x'(0) = x(-1) + x(1)$. Substituting in this condition the expressions coming from (3.1) for $\theta = 0$, and from (3.2) for $\theta = -1, 1$ we obtain

$$c = \frac{\int_0^{-1} e^{\lambda(-1-s)} f(s)\, ds + \int_0^{1} e^{\lambda(1-s)} f(s)\, ds - f(0)}{\lambda - e^{-\lambda} - e^{\lambda}}\,. \tag{3.3}$$

Clearly $x \in C^{\infty}_{[-1,1]}$ and using induction's method we can verify that $x(\theta)$ belongs to M if $f \in M$. Finally in view of $(A - \lambda I)x = f$ we have $x(\theta) = R(\lambda, A)f(\theta)$, where $R(\lambda, A)$ denotes the resolvent associated to A. From this identity, equation (3.2) and c given by (3.3) we get the expression for **the resolvent associated to** A:

$$R(\lambda, A)f(\theta) = ce^{\lambda\theta} + \int_0^{\theta} e^{\lambda(\theta - s)} f(s)\, ds. \tag{3.4}$$

We observe that the operator R is defined only for λ such that $\lambda - e^{-\lambda} - e^{\lambda}$ is different from cero.

Theorem 3.1. *The eigenvalues of A satisfy the equation $g(\lambda) = \lambda - e^{-\lambda} - e^{\lambda} = 0$ and there is no other spectrum.*

Proof. There exists $x \neq 0$ such that $Ax(\theta) = \lambda x(\theta) \Longleftrightarrow x(\theta) = ce^{\lambda\theta}$. We will prove that $x(\theta)$ belongs to M. For do it, we replace $t = 0$ in (1.1) obtaining $x'(0) = x(-1) + x(1)$ which implies, taking out c, $\lambda = e^{\lambda} + e^{-\lambda}$. Similarly we obtain for the nth derivative $x^{(n)}(0) = x^{(n-1)}(-1) + x^{(n-1)}(1) \Longleftrightarrow c\lambda^n = c\lambda^{n-1}e^{\lambda} + c\lambda^{n-1}e^{-\lambda} \Longleftrightarrow \lambda = e^{\lambda} + e^{-\lambda}$. Therefore, the point spectrum of operator A is the set of $\lambda \in \mathbb{C}$ such that satisfy the equation $\lambda - e^{-\lambda} - e^{\lambda} = 0$. Now we will prove there is no other spectrum which means that if $\lambda - e^{-\lambda} - e^{\lambda} \neq 0$, then λ belongs to the resolvent. It is enough to prove that the resolvent is a continuous operator for such λ. We help us using the following proposition [6]:

Let X and Y be vector spaces with the topologies defined by the families of seminorms $\{p_\alpha\}_{\alpha \in A}$ and $\{g_\beta\}_{\beta \in B}$ respectively, where A and B are certain set of indexes. Let $R : X \longmapsto Y$ be linear. Then R is continuous if and only if for all $\beta \in B$ exist $\alpha_1, \dots \alpha_k \in A$ and $c > 0$ such that $g_\beta(Rx) \leq c\sum_{j=1}^{k} p_{\alpha_j}(x)$. We rewrite the resolvent given by (3.4) in the form

$$R(\lambda, A)f(\theta) = c_\lambda R_1 f(\theta) + c_\lambda R_2 f(\theta) + c_\lambda R_3 f(\theta) + R_4 f(\theta),$$

where $c_\lambda = (\lambda - e^{-\lambda} - e^{\lambda})^{-1}$, $R_1 f(\theta) = c_\lambda \int_0^{-1} e^{\lambda(-1-s)} e^{\lambda\theta} f(s)\, ds$, $R_2 f(\theta) = c_\lambda \int_0^{1} e^{\lambda(1-s)} e^{\lambda\theta} f(s)\, ds$, $R_3 f(\theta) = c_\lambda f(0) e^{\lambda\theta}$ and $R_4 f(\theta) = \int_0^{\theta} e^{\lambda(\theta-s)} f(s)\, ds$. Now we will prove that $R_i f(\theta)$ are continuous for $i = 1, 2, 3, 4$. For $R_1 f(\theta)$ with $e^{\lambda(-1-s)} e^{\lambda\theta} = K_1(\theta, s)$ we have

$$|(R_1 f)^{(i)}(\theta)| \leq |c_\lambda| \max_{[-1,1]\times[0,-1]} |\partial_\theta^{(i)} K_1(\theta, s)| \max_{[0,-1]} |f(s)| \leq |c_\lambda| c_i \|f\|_0.$$

Hence $\|R_1 f(\theta)\|_l = \sum_{i=0}^{l} \max_{[-1,1]} |(R_1 f)^{(i)}(\theta)| \leq |c_\lambda| \sum_{i=0}^{l} c_i \|f\|_0$, where

$$c_i = \max_{[-1,1]\times[0,-1]} |\partial_\theta^{(i)} K_1(\theta, s)|$$

and $l = 0, 1, \ldots$.

Similarly doing $e^{\lambda(1-s)} e^{\lambda\theta} = K_2(\theta, s)$, we obtain

$$\|R_2 f(\theta)\|_l = \sum_{i=0}^{l} \max_{[-1,1]} |(R_2 f)^{(i)}(\theta)| \leq |c_\lambda| \sum_{i=0}^{l} c_i \|f\|_0\,,$$

where $c_i = \max_{[-1,1]\times[0,1]} |\partial_\theta^{(i)} K_2(\theta, s)|$ and $l = 0, 1, \ldots$.

$$\|R_3 f(\theta)\|_l = \sum_{i=0}^{l} \max_{[-1,1]} |(R_3 f)^{(i)}(\theta)| \leq |c_\lambda| \sum_{i=0}^{l} c_i \|f\|_0, \text{ where}$$

$c_i = \max\limits_{[-1,1]} |e^{\lambda\theta}||\lambda|^i$, and $l = 0, 1, \ldots$ and $\|R_4 f(\theta)\|_l \le \sum_{i=0}^{l} c_i \|f\|_{l-1}$, where

$c_i = \max\limits_{[-1,1]\times[-1,1]} |e^{\lambda(\theta-s)}||\lambda|^i + \sum\limits_{k=0}^{i-1} |\lambda|^{i-1-k}$ and $l = 0, 1, \ldots$ □

Theorem 3.2. *For initial functions $\varphi(\theta) = x_\lambda(\theta) = e^{\lambda\theta}$ for $\theta \in [-1,1]$ and $\lambda \in \sigma_p(A)$, we have $T_t e^{\lambda\theta} = e^{\lambda(\theta+t)}$ for $t \in \mathbb{R}$.*

Proof. Clearly the function $e^{\lambda\tau}$ for $\tau \in \mathbb{R}$, represents the smooth solution of equation $x'(t) = x(t-1) + x(t+1)$ if and only if $\lambda - e^{-\lambda} - e^{\lambda} = 0$, i.e., $\lambda \in \sigma_p(A)$. In view of $T_t x(\theta) = x(t+\theta)$, where $x(t+\theta)$ is the solution of equation $x'(t) = x(t-1) + x(t+1)$ with the initial function $\varphi(\theta)$, $\theta \in [-1,1]$, we have $T_t e^{\lambda\theta} = e^{\lambda(\theta+t)}$ for $t \in \mathbb{R}$. □

4. Asymptotic Distribution of the Eigenvalues

Multipling by $(-e^{\lambda})$ the equation $g(\lambda) = \lambda - e^{-\lambda} - e^{\lambda} = 0$, we obtain

$$g(\lambda) = 1 - \lambda e^{\lambda} + e^{2\lambda} = 0, \tag{4.1}$$

which is a particular case of $g(\lambda) = \sum_{i=0}^{n} p_i \lambda^{m_i} e^{\beta_i \lambda}$, for $0 = \beta_0 < \beta_1 < \ldots < \beta_n$, a more general equation having polynomial coefficients, where $p_i \ne 0$ and m_i denote the degree of the polynomial in λ. In (4.1) we have $\beta_0 = 0,\ \beta_1 = 1,\ \beta_2 = 2,\ \ m_0 = 0,\ m_1 = 1,\ m_2 = 0,$ and $p_0 = 1,\ p_1 = -1,\ p_2 = 1$.

In the following we construct the distribution diagram of the quasi polynomial which give us the information concerning the geometrical distribution of the zeros of $g(\lambda)$. To do it, we plotting the points P_i with coordinates (β_i, m_i). In our equation the points $P_i(\beta_i, m_i)$ have the following coordinates: $P_0(0,0), P_1(1,1), P_2(2,0)$. The polygonal line $L = [P_0, P_1] \cup [P_1, P_2]$ have two succesive segments which we denote as L_1 and L_2 resp. The line through P_0 and P_1 has angular coefficient $\mu = tan\frac{\pi}{4} = 1$ whereas for the line through P_1 and P_2 $\mu = tan\frac{3\pi}{4} = -1$. In [2] has been proven there exists a number $c_1 > 0$ such that $V_1 : |Re(\lambda + \ln\lambda)| \le c_1$, $V_2 : |Re(\lambda - \ln\lambda)| \le c_1$, where V_1 is the strip of retarded type and V_2 is the strip of advanced type. Each of these strips is bounded by curves of type $Re(\lambda + \mu \ln\lambda) = c$, $\mu \ne 0$. The zeros of $g(\lambda)$ lie within one of these strips and they coincide asymptotically with the zeros of the functions $g_1(\lambda) = p_0\lambda^{m_0}e^{\beta_0\lambda} + p_1\lambda^{m_1}e^{\beta_1\lambda} = 1 - \lambda e^{\lambda}$ and $g_2(\lambda) = p_1\lambda^{m_1}e^{\beta_1\lambda} + p_2\lambda^{m_2}e^{\beta_2\lambda} = -\lambda e^{\lambda} + e^{2\lambda}$. There are no zeros of $g(\lambda)$ outside of the sets V_1 and V_2 (see Theorem 12.10 in [2]). For the function $g_1(\lambda)$ we have the proportion relation: $m_i = m\beta_i$ with $m = 1$, whereas the equality

$m_i = m\beta_i$ is not satisfied for $g_2(\lambda)$. Therefore we rewrite $g_2(\lambda) = -\lambda e^{\lambda} + e^{2\lambda} = \lambda e^{\lambda}(-1 + \lambda^{-1}e^{\lambda})$. Appling Theorem 12.10 from [2] it follows that in any subregion of V_2 in which the variable λ is uniformly bounded away from all zeros of $g_2(\lambda)$, $|g_2(\lambda)\lambda^{-1}e^{-\lambda}|$ is also uniformly bounded from zero. For the function $\tilde{g_2}(\lambda) = -1 + \lambda^{-1}e^{\lambda}$ we have $m_i = m\beta_i$ with $m = -1$. On the other hand $g_1(\lambda) = 1-\lambda e^{\lambda} = 1-e^{\lambda+\ln\lambda} = 0$, means that $e^z = 1$ for $z = \lambda+\ln\lambda$. Since $e^z = 1$ then $z = \log 1 = \ln|1|+i\arg 1 = 2\pi ki$ and so $\lambda+\ln\lambda = 2\pi ki$, $k = 0, \pm1, \pm2, \ldots$. Similarly for $g_2(\lambda) = \lambda e^{\lambda}\tilde{g_2}(\lambda)$ and $\tilde{g_2}(\lambda) = -1 + \lambda^{-1}e^{\lambda} = -1 + e^{\lambda-\ln\lambda} = 0$ we obtain $\lambda - \ln\lambda = 2\pi ki$, $k = 0, \pm1, \pm2, \ldots$, for $z = \lambda - \ln\lambda$. Since $\lambda_k + m\ln\lambda_k = 2\pi ki$, $k = 0, \pm1, \pm2, \ldots$ the roots λ_k belong to the curves $Re(\lambda + m\ln\lambda) = 0$. In summary, the zeros of $g(\lambda)$ lie asymptotically along of two curves: $Re(\lambda + \ln\lambda) = 0$ and $Re(\lambda - \ln\lambda) = 0$.

These curves have the following properties (with little changes the proofs follow from Lemma 12.3 in [2]) *i) Each of these curves is symmetric with respect to the real axis. ii) If* $\lambda = x + iy$ *lies on the curve, then* $|\frac{y}{x}| \xrightarrow[|\lambda|\longrightarrow\infty]{} \infty$ *and* $|\arg\lambda| \xrightarrow[|\lambda|\longrightarrow\infty]{} \frac{\pi}{2}$. *iii) The curve* $Re(\lambda+m\ln\lambda) = 0$ *is asymptotic to the curve* $x + m\ln|y| = 0$ *for* $m = \pm1$. *iv) The curve* $Re(\lambda + \ln\lambda) = 0$ *lies in the left half plane* $Re\lambda < 0$ *and* $Re\lambda \xrightarrow[|\lambda|\to\infty]{} -\infty$ *while the curve* $Re(\lambda - \ln\lambda) = 0$ *lies in the right half plane* $Re\lambda > 0$ *and* $Re\lambda \xrightarrow[|\lambda|\to\infty]{} +\infty$. Finally, for $\lambda = x + iy$ we get $Re(\lambda + m\ln\lambda) = x + m\ln|\lambda| = m\ln|1| + o(1)$ and $Im(\lambda + m\ln\lambda) = y + m\arg\lambda = m(\arg 1 + 2\pi r) + o(1)$, $r \in \mathbb{Z}$. Since $\arg\lambda \xrightarrow[|\lambda|\longrightarrow\infty]{} \pm\frac{\pi}{2}$ then we have $y = m(\arg 1 + 2\pi r \mp \frac{\pi}{2}) + o(1)$. On the other hand, $|\lambda| = |y|(1 + o(1)) \Longrightarrow x + m\ln|\lambda| = x + m\ln|y|$ and so $x + m\ln|y| = m\ln|1| + o(1)$. So we obtain $x = m(-\ln|2\pi rm \mp \frac{\pi m}{2}|) + o(1)$ and $y = m(2\pi r \mp \frac{\pi}{2}) + o(1)$. Hence the roots of large modulus of the equation $\lambda - e^{-\lambda} - e^{\lambda} = 0$, lie within one the strips V_1 or V_2. In other words in V_1 lie $x = -\ln|2\pi r \mp \frac{\pi}{2}| + o(1)$, $y = 2\pi r \mp \frac{\pi}{2} + o(1)$ and in V_2 lie $x = \ln|-2\pi r \pm \frac{\pi}{2}| + o(1)$, $y = -2\pi r \pm \frac{\pi}{2} + o(1)$, where $r \in \mathbb{Z}$.

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