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APPLICATION OF DOUBLE-POISSON PROCESS IN BRITTLE FRACTURE

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Abstract: The paper models a particular mode of slow crack growth in brittle composites, namely a sequence of microscopic jumps of random length, each one followed by an arrest of random duration. The jump lengths are modeled by a non-homogeneous Poisson process (with a space coordinate in place of what ordinarily is time), and the arrest durations are modeled by a homogeneous Poisson process whose intensity is related to random energy barriers at the arrest points and, consequently, depends on the crack arrest location. The transition probability for the resulting random process of crack growth is shown to satisfy a hyperbolic PDE. The model predicts scatter, in identical experiments, of critical loads and critical crack lengths. It also captures both scatter and 'scale effect' for macroscopic fracture parameters, including fracture toughness K_C and critical energy release rate G_C. There are indications that the model is capable of simulating the Paris law. An illustrative numerical example is considered.

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1. Introduction

In the paper, we model an observed mode of slow crack growth in brittle ma-

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terials, namely, a sequence of microscopic jumps of random length, each one followed by an arrest of random duration. The jump lengths are modeled by a non-homogeneous Poisson process with a space coordinate in place of what ordinarily is time (Sec. 2), and the arrest durations are modeled by a homogeneous Poisson process whose intensity is related to random energy barriers at the arrest points and, consequently, depends on the crack arrest location (Sec. 4). In Sec. 3, the notions of stable and unstable specimen-loading configurations are defined; it is stated that the paper is devoted to the more common unstable case. In Sec. 5, crack growth is described as a random process and its transition probability is shown to satisfy a hyperbolic PDE (in contrast to the parabolic Kolmogoroff equations, which govern the transition probabilities in [1]). In Sec. 6, illustrative numerical examples are considered. These include prediction how crack depth distribution changes in time and demonstration of the effect of 'microstructure' on life time scatter. The last example in Sec. 6 indicates that the model is capable of simulating the Paris law. Section 7 contains a brief discussion of future developments.

2. Modeling a Crack Jump; Crack Propagator

Let us consider crack formation in a loaded two-dimensional solid. In this paper, we are concerned only with crack propagation along a straight line, say the x-axis (see Figure 1). "Crack jump" stands for an event during which a previously arrested crack instantaneously advances and is then arrested, with its tip at a new location. (The advance is considered instantaneous in comparison to the time scale of the crack growth process under consideration). Material in front of the crack is viewed as a sequence of randomly distributed obstacles. Once initiated, the crack jumps through all of the obstacles it can overcome and is arrested at the first 'insurmountable obstacle' (see below). Thus the length of each jump is random. What happens after the arrest is discussed in Sec. 4.

It is technically convenient to begin by introducing the probability that a crack with its tip at x will overcome all of the obstacles between x and an arbitrary $X \geq x$ (and, therefore, the crack arrest would occur to the right of X). This probability will be denoted $[X|x]$. This is a special case of the Crack Propagator that was introduced in [2] as the probability of instantaneous crack formation between two arbitrary points of a loaded two-dimensional solid. We now proceed to define the needed version of crack propagator.

Consider a solid with a straight crack (Figure 1). We will refer to the x-

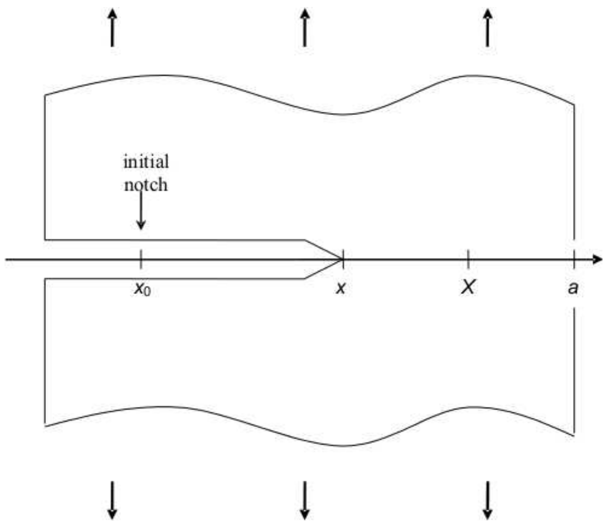

Figure 1: Specimen-loading configuration to be considered

coordinate of the crack tip as *crack depth*. Assume that the applied loading is symmetrical relative to the crack line to justify rectilinear crack growth along the x-axis. We assume that, in the course of the instantaneous phase, the crack, its tip currently at x, will advance farther by dx if the elastic energy release resulting from crack advancing from x to $x + dx$ would be greater than the energy required for breaking the material between x and $x + dx$. The (linear) "elastic energy release" is $G(x)dx$, where $G(x)$ is considered a known deterministic function, the so-called *energy release rate* (ERR). The "energy required for breaking" is taken to be $2\gamma(x)dx$, where $\gamma(x)$ is the value of a random field γ at x (when γ is a constant they refer to it as specific fracture energy). Thus, "no arrest at x" means $2\gamma(x) \leq G(x)$.

We make the following assumption about the random field γ:[1]

(a) γ is homogeneous in x,

(b) values of γ at every point are governed by a Weibull distribution,

$$F(\gamma) = \begin{cases} 1 - \exp\left(- \left[\Gamma\left(1 + \frac{1}{\alpha}\right) \frac{\gamma - \gamma_{\min}}{\gamma^* - \gamma_{\min}} \right]^{\alpha} \right) & \text{if } \gamma \geq \gamma_{\min} \\ 0 & \text{if } \gamma < \gamma_{\min}. \end{cases} \tag{1}$$

[1]One can find a discussion of the assumed properties of γ and consequences, 'in physical terms', in [3], [4].

Here the parameters γ^* and $\gamma_{\min}$ are the average and minimal values of γ, respectively, and $\alpha > 0$ is known as a 'shape parameter' – they characterize the scatter of material's strength on a microscale; $\Gamma(\cdot)$ is the Γ-function,

(c) properties of γ on disjoint intervals are independent, and

(d) for any coordinate $x \geq x_0$ and for any number $g \geq \gamma_{\min}$, the probability of γ exceeding g somewhere between x and $x + dx$ is proportional to dx, specifically,

$$\text{Prob}\{\,\gamma > g \text{ somewhere on } (x,\, x + dx]\,\} = (1 - F(g))\frac{dx}{r}, \tag{2}$$

where $r > 0$ is a parameter. (When the solid under consideration is finite, one assumes that r is small in comparison to the solid's size.)[2]

Therefore, the probability of the crack being arrested on $(x,\, x + dx]$ is

$$\text{Prob}\{\,2\gamma > G(x) \text{ somewhere on } (x,\, x + dx]\,\} = \frac{U(x)}{r}dx, \tag{3}$$

where $U(x) = 1 - F\left(\frac{G(x)}{2}\right)$. (In physical terms, one is describing the probability conditional on that either the crack has already passed through x or the initiation at x has occurred.)

For two arbitrary points x and X, $x \leq X$, on the x-axis let us define *crack propagator* (CP), $[X|x]$, as the probability that the crack can advance instantaneously to at least X conditional on that either the crack has already passed through x or the initiation at x has occurred. (One can formally define $[X|x] = 0$ for $x > X$, which would express the assumption that cracks do not cure.) The definition is a one-dimensional case of the notion introduced in [2] and is a cumulative version of the one that was used in [4]. Notice that if x were representing "time", then the above assumptions would mean that the random event of crack arrest is governed by a non-homogeneous Poisson process with "time"-dependent intensity $U(x)/r$. The absence of crack arrest between x and X (conditional, as described above), corresponds, in the Poisson process terminology, to having "zero number of events in "time" interval $(x, X]$". Its probability is therefore expressed by a well known formula (the last equality in equation (4) below):

$$[X\,|\,x] \qquad = \text{Prob}\{\,\text{no arrest on } (x,\, X]\,\}$$

[2]For examples of experimental evaluation of the parameters γ^*, $\gamma_{\min}$, α , and r of the γ-field, see [5], [6].

$$= \mathrm{Prob}\{2\gamma \ \le G_1(x), \text{everywhere on } (x,\, X]\,\}$$
$$= \exp\left\{ -\int_x^X \frac{U(\xi)}{r} d\xi \right\}. \tag{4}$$

Thus, the explicit expression for CP becomes (for $x \le X$)

$$[X\,|\,x] = \exp\left\{ -\int_x^X \exp\left(-\left[\Gamma\left(1+\frac{1}{\alpha}\right) \frac{\frac{G(x)}{2} - \gamma_{\min}}{\gamma^* - \gamma_{\min}} \right]^{\alpha} \right) \frac{d\xi}{r} \right\}. \tag{5}$$

The following useful property of CP is obvious from its definition and the independence of properties of the field γ on disjoint intervals (as well, as from the last expression in equation (4)): for any $x \le x' \le X$,

$$[X\,|\,x']\,[x'\,|\,x] = [X\,|\,x]. \tag{6}$$

Remark. Limiting transition to the case of a homogeneous material with no microstructure (constant γ) may be understood physically in more than one way. One possibility is to suppose that $r \to \infty$. This is inconsistent with the assumptions of the model. Indeed, by construction, the model applies to scales large relative to r. A second approach involves letting both r and the variance of γ tend to zero simultaneously. This one is consistent with the model.

3. Stable and Unstable Configurations

In physical terms, one may call a specimen-loading configuration stable, if a crack, once initiated from the notch, will surely be arrested, as opposed to there being a possibility of it reaching the specimen's right edge and breaking the specimen in two. In the latter case, the configuration may be called unstable.

Let a denote the x-coordinate of the right edge of the specimen under consideration, see Figure 1 (we include the possibility of $a = \infty$.) Formally, we will call a specimen-loading configuration *stable*, if the probability of a crack reaching a is zero for any crack depth x, i.e. if $[a|x] = 0$ for all $x \ge x_0$; otherwise we will call the configuration *unstable*.[3]

In [4], stable configurations were considered. In this paper, we are interested in unstable configurations, which are more common.

[3]Stable configurations occur, for example, when a half-infinite solid is loaded through prescribed displacements. Unstable configurations commonly occur in finite specimens (unless one is compressed for large crack depths) and in semi-infinite solid loaded through applied forces.

Notice that, for a given $x \geq x_0$, $[s|x]$ as a function of s monotonically decreases from $[x|x] = 1$ to $[a|x] > 0$. The distribution function for crack arrest depths conditional on the crack initiation occurring at x is

$$F(s\,|x) = \begin{cases} 1 - [s\,|x], & \text{if } x < s < a \\ 1, & \text{if } a \leq x. \end{cases} \tag{7}$$

For an unstable configuration, $F(\cdot\,|x)$ has a jump at a whose size is $[a\,|x] > 0$. From now on, unless the opposite is stated, let us assume $a < \infty$ (finite width specimen). Under this assumption one can consider the first two moments of the crack arrest distribution.

The mean value $m_1(x)$ of the crack arrest depth conditional on the crack initiation occurring at x is

$$m_1(x) = \int_x^a s\,dF(s\,|x) = x + \int_x^a [s\,|x]ds \tag{8}$$

(the second equality follows from equation (7) and integration by parts). The second moment of the crack arrest depth is

$$m_2(x) = \int_x^a s^2\,dF(s\,|x) = x^2 + 2\int_x^a s\,[s\,|x]ds \tag{9}$$

(again, the second equality follows from equation (7) and integration by parts).

4. Crack Arrest Duration

Let us make the following assumption about the (random) duration of crack arrest.

$$\text{Prob}\{\text{initiation from } x \text{ during } dt\} = \Lambda(\gamma^*,\, G(x))\frac{dt}{\tau} \tag{10}$$

Here τ (a characteristic time scale for the submicroscopic processes) is introduced explicitly to render the function $\Lambda(\gamma^*, G)$ dimensionless. In the absence of other relevant dimensional constants, $\Lambda(\gamma^*, G)$ has to be a function of γ^*/G. One might expect that Λ depend on the average relative energy barrier $(2\gamma^* - G)/G$. The dependence should be then such that a small barrier results in large values of Λ (short arrest) and a large barrier results in small values of Λ (long arrest), e.g. $\Lambda = G/(2\gamma^* - G)$. Actual choice of the function Λ is to be dictated by physical considerations outside of the scope of this paper. In Sec. 6, we make a choice for the sake of illustrative numerical examples.

Evidently, we have assumed that, for a crack arrested at x, its subsequent initiation is governed by a Poisson process in time with the time-independent intensity $\Lambda(x)/\tau$. Thus, in particular, the probability that initiation does not happen between times t_1 and $t_2 \geq t_1$ is

$$\text{Prob}\left\{ \begin{array}{c|c} \text{no crack initiation} & \text{the crack was arrested} \\ \text{between } t_1 \text{ and } t_2 \geq t_1 & \text{at } x \text{ prior to } t_1 \end{array} \right\}$$
$$= \exp\left(-\tfrac{\Lambda(x)}{\tau}(t_2 - t_1) \right). \tag{11}$$

5. Distribution of Crack Arrest Locations as a Function of Time

Let $\xi(t)$ denote the position of the crack tip at a time t. Since we model crack jumps as instantaneous, there is ambiguity in this description of $\xi(t)$. Namely, if t is an instant of a jump, then the crack tip position at t is not uniquely defined. Let us define $\xi(t)$ then as the *rightmost* position of the crack tip at t. Thus, the random process $\xi(t)$ has monotonically growing piecewise constant realizations. Also, notice that the statement $\xi(t) = x$ implies that, at t, the crack tip is at x *and* the crack is in the state of arrest.

Denote by $P(x,t)$ the probability that, at a time t, the crack depth is less than or equal x, assuming that, at $t = 0$, the crack tip was at x_0 in the state of arrest,
$P(x,\, t) = \text{Prob}\{\xi(t) \leq x \,|\, \xi(0) = x_0\}$.

We proceed to derive a PDE for $P(x,t)$. Let $x \geq x_0$ and $t \geq 0$. Assuming that $\xi(t) \leq x$, there are two and only two mutually exclusive possibilities to have $\xi(t + dt) \leq x$:

(a) $\xi(t) = x_0$ (in the state of arrest) and, during $[t, t+dt]$, it did *not* happen that crack initiation occurred and was followed by an instantaneous crack jump to a depth greater than x; the probability of this is

$$P(x_0, t)\left(1 - \frac{\Lambda(x_0)dt}{\tau}[x \,|\, x_0]\right);$$

(b) $\xi(t) \in (x',\, x' + dx']$ for some $x' \in (x_0,\, x]$ and, during $[t,\, t + dt]$, it did *not* happen that crack initiation occurred and was followed by an instantaneous crack jump to a depth greater than x; the probability of

$\xi(t) \in (x',\, x'+dx']$ is $P(x'+dx',\, t) - P(x',\, t) = P_x(x',\, t)dx'$,[4] the probability of crack initiation that is followed by an instantaneous crack advance beyond x is $\frac{\Lambda(x')dt}{\tau}[x\,|\,x']$; since, for non-intersecting intervals $(x',\, x'+dx']$, the events $\xi(t) \in (x',\, x'+dx']$ are mutually exclusive, then the probability of the event under consideration is

$$\int_{x_0}^{x} P_x\,(x',\, t)\left(1 - \frac{\Lambda(x')dt}{\tau}[x\,|\,x']\right)dx'.$$

Thus

$$\begin{aligned}
P(x,\, t+dt) =& P(x_0,\, t)\left(1 - \frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0]\right)\\
&+ \int_{x_0}^{x} P_x\,(x',\, t)\left(1 - \frac{\Lambda(x')dt}{\tau}[x\,|\,x']\right)dx'\\
=& -P(x_0,\, t)\,\frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0] + P(x_0,\, t) + \int_{x_0}^{x} P_x\,(x',\, t)dx'\\
&- \int_{x_0}^{x} P_x\,(x',\, t)\frac{\Lambda(x')dt}{\tau}[x\,|\,x']dx'\\
=& -P(x_0,\, t)\,\frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0] + P(x,\, t)\\
&- \int_{x_0}^{x} P_x\,(x',\, t)\frac{\Lambda(x')dt}{\tau}[x\,|\,x']dx'.
\end{aligned} \tag{12}$$

From equation (6), we have $[x\,|\,x'\,] = [x\,|\,x_0\,]/[x'\,|x_0]$. Substituting this into the integral above we get

$$\begin{aligned}
P(x,\, t+dt) = -P(x_0,\, t)\,\frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0\,] + P(x,\, t)\\
- [x\,|\,x_0\,]\int_{x_0}^{x} \frac{P_x\,(x',t)}{[x'\,|\,x_0\,]}\,\frac{\Lambda(x'\,)dt}{\tau}\,[x\,|\,x'\,]dx'
\end{aligned}$$

Move $P(x,t)$ to the left hand side, divide by dt, and factor out $[x\,|\,x_0\,]$ to get

$$P_t\,(x\,,t) = -[x\,|\,x_0\,]\left(P(x_0,\, t)\frac{\Lambda(x_0)}{\tau} + \int_{x_0}^{x} \frac{P_x\,(x',t)}{[x'\,|x_0]}\,\frac{\Lambda(x'\,)}{\tau}[x\,|\,x'\,]dx'\right). \tag{13}$$

Take $\partial/\partial x$ of both sides, use that $\frac{\partial}{\partial x}[x\,|\,x_0\,] = -[x\,|\,x_0\,]\,\frac{U(x)}{r}$ (see equation (4)), and, finally, use equation (13) to substitute the expression in parentheses by

[4] Subscripts stand for partial derivatives. Here, $P_x \equiv \frac{\partial P}{\partial x}$

$-P_t\,(x\,,t)/[x\,|\,x_0\,]$ to obtain

$$P_{tx}\,(x\,,t) + \frac{U(x)}{r} P_t\,(x\,,t) + \frac{\Lambda(x)}{\tau}\,P_x\,(x\,,t) = 0. \tag{14}$$

In addition, we have the initial condition $P(x,0) = 1$, $x \geq x_0$ (at $t = 0$ the crack tip is at x_0) and the boundary condition

$$P(x_0\,,t\,) = \exp\left(-\frac{\Lambda(x_0)}{\tau}\,t\right), \quad t \geq 0 \tag{15}$$

(the probability that, by the time t, crack growth has not yet begun, i.e. no initiation at x_0 has occurred; c.f. equation (11)).

6. Illustrative Numerical Examples

In this section, we consider a (vertical) strip in uniform tension applied far from the crack path. If the remote tension stress is σ and the strip width is a, then [7], [8]

$$\begin{aligned} G(x) =& \frac{2\sigma^2 a}{E}\,\beta \sec^2\beta\,\frac{\tan\beta}{\beta}\left[0.752 + 2.02\,\frac{2}{\pi}\,\beta + 0.37(1-\sin\beta)^3\right]^2, \\ \beta =& \frac{\pi x}{2a}, \end{aligned} \tag{16}$$

where E is the Young's modulus. We also use

$$\Lambda(x) = 2.0\left(\frac{G(x)}{2\gamma^*}\right)^{0.2}, \tag{17}$$

$\gamma^* = 4,000, \quad \gamma_{\min} = 0, \quad \alpha = 2, \quad r/a = 0.005, \quad \tau = 1, \quad 2\sigma^2 a/E = 32.$

6.1. Evolution of Crack Depth Distribution

Figure 2 below shows evolution of crack depth distribution with time.

6.2. Distribution of Time to Failure

In this sub-section, we illustrate the 'effect of microstructure', namely of the value of r, on a macroscopically observable distribution, namely that of time

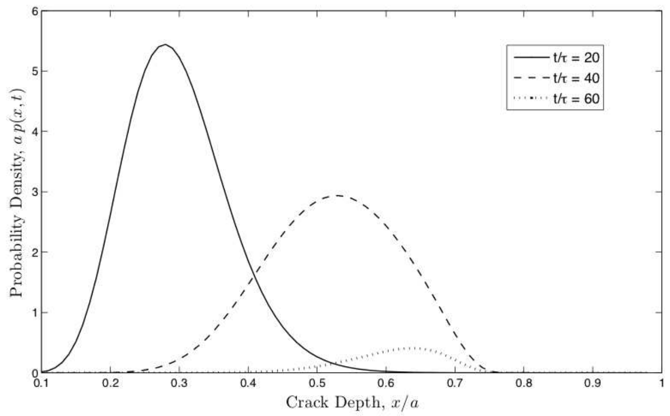

Figure 2: Probability densities of crack depth at various times. Not shown are the two atomic probabilities, one at $x = x_0$ and another at $x = a$, $P(x_0, t)$ and $1 - P(a, t)$, respectively

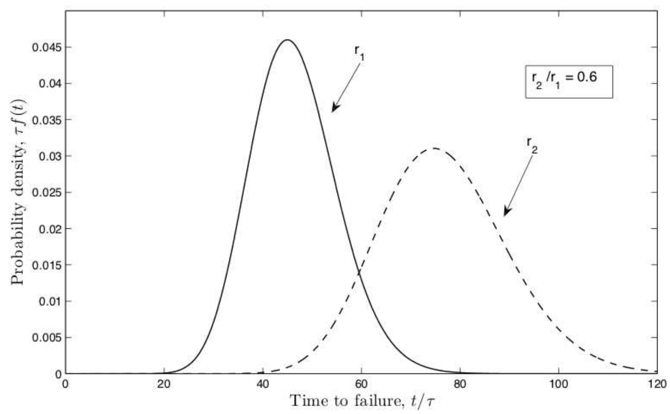

Figure 3: Probability density $f(t)$ of time to failure for two values of r: $r_1/a = 0.05$ and $r_2/a = 0.03$

to failure. By *time to failure* we mean the (random) time it takes for the crack to reach the end of the specimen, i.e. $x = a$. Evidently, the probability that failure occurs before time t is $P(a, t)$. Figure 3 shows the probability density function, $f(t) = P_t(a, t)$, for two different values of r.

6.3. Average Crack Speed vs. ERR

In this sub-section, we examine whether the model is capable of imitating the relation, known as Paris law. This relation – between the average crack speed and the amplitude of the 'stress intensity factor' (see below) – is commonly observed under fatigue conditions.

Notice that, in the scope of the model, if one considers any realization of a (random) growing crack, then the speed of the crack tip takes one of two values, zero or infinity. Thus, the notion of instantaneous crack speed, $v(x)$, is meaningless 'realization-wise', and so the average crack speed, $\hat{v}(x)$, cannot be defined as a straightforward average of $v(x)$.

Let us define *average crack propagation speed*, $\hat{v}(x)$, $x_0 < x < a$, as follows.[5] Let $T(x)$ denote the (random) time at which the crack tip either arrives at or flies by x, $x_0 < x < a$ and let $\hat{T}(x)$ denote the average value of $T(x)$. Evidently, the distribution function for $T(x)$ is given by

$$\begin{aligned} F_{T(x)}(t) &= \mathrm{Prob}\{T(x) \leq t\} = \mathrm{Prob}\{\text{the crack tip at } x \text{ at } t \text{ or earlier}\} \\ &= \mathrm{Prob}\{\text{at } t, \text{ the crack tip is at some } x' > x\} = 1 - P(x,t). \end{aligned} \tag{18}$$

Therefore,

$$\hat{T}(x) = \int_0^\infty t\, dF_{\hat{T}(x)}(t) = -\int_0^\infty t\, P_t\,(x,t)dt. \tag{19}$$

Finally, define

$$\hat{v}(x) = \frac{1}{\hat{T}'(x)}. \tag{20}$$

The second main character in the Paris law, the first one being $\hat{v}(x)$, is "stress intensity factor", K_I. This is one of the three coefficients that enter asymptotic expression for the stress field around a crack tip (the other two, K_{II} and K_{III}, equal zero in the case under consideration). The definition(s) can be found, for example, in [7]. In our situation $K_I = \sqrt{EG}$, where $G = G(x)$ is given by equation (16). For cyclic loading, let $K_{\max}$ and $K_{\min}$ denote the K_I-values that correspond to the highest and lowest values of the applied cyclic load, respectively (both $K_{\max}$ and $K_{\min}$ depend on the crack length). Denote $\Delta K_I = K_{\max} - K_{\min}$.

Paris law is the following observation, statistical in nature. Under cycling loading, crack speed v is proportional to a power of the amplitude of the stress

[5] As a motivation, consider the case of a deterministic particle moving along the x-axis with its coordinate $x = x(t)$ smoothly increasing with t. Let $t(x)$ denote the time, at which the particle reaches x, and let $v(x)$ denote the velocity of the particle when the particle is at x. Then, $t(x)$ being the inverse of $x(t)$, one has $v(x) = x'(t(x)) = 1/t'(x)$.

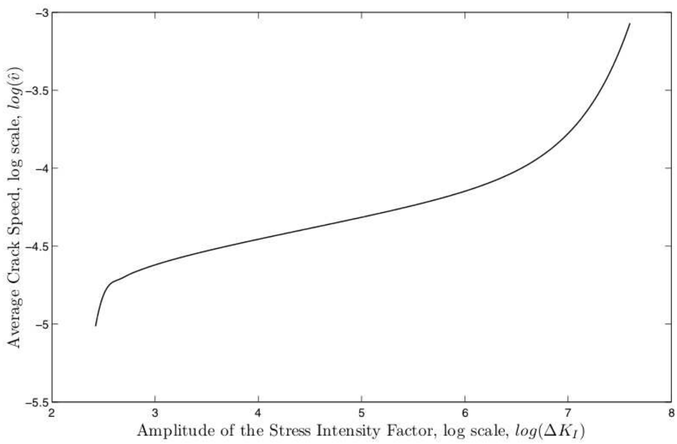

Figure 4: Dependence of $\ln \hat{v}$ on $\ln \Delta K_I$ predicted by the model

intensity factor ΔK_I over a wide range of the values of ΔK_I.[6] It is common to refer to the v-vs.-ΔK_I relation as 'linear in log-log scale'. Also of interest is that the commonly observed dependence of $\ln \hat{v}$ on $\ln \Delta K_I$ is 'steeper' outside of the (approximately) linear range.

In Figure 4, we exhibit the relation between $\ln \hat{v}$ and $\ln \Delta K_I$ predicted by the model (after finding $\hat{v} = \hat{v}(x)$ and excluding x from $\hat{v}(x)$ and $\Delta K_I(x)$; the expression for $\Delta K_I(x)$ is based on equation (16)).

We find it encouraging that Figure 4 is qualitatively similar to a typically observed (sigmoidal) relationship between the crack speed and the amplitude of the stress intensity factor.

7. Discussion

The proposed model has two aspects: (i) instantaneous formation of a random crack and (ii) slow crack growth consisting of a sequence of jumps of random lengths and arrests of random durations. The first aspect is developed for 'almost rectilinear' cracks, and, under simplest assumptions, it leads to a parabolic PDE [2]. The second aspect is developed above for a rectilinear

[6] In certain situations, it was found that restating Paris Law in terms of ERR is actually beneficial [9].

crack growth and, under simplest assumptions, leads to a hyperbolic PDE, as outlined in Sec. 5.

It would be desirable to leave behind the simplifying, highly restrictive, assumptions that allowed one to derive pretty equations (for CP and $P(x,t)$), so as to make accessible more realistic problems. To do so one needs to decide on the nature of the random crack trajectories, i.e. choosing the space Ω of random paths that a crack may potentially follow. In one of such choices, Ω consists of partially smoothed Brownian paths (their fractional integrals) [10]. Another possibility is to modify the approach to random crack formation found in [11] (from fragmentation to a single crack scenario). One may then proceed numerically by generating a finite ensemble of crack paths, Ω_N, find and tabulate G-values along each path ω from Ω_N, and evaluate and tabulate CP (given by a formula of the type of equation (5)). Finally, one would compute $P(x,t)$ (more generally, $P(\vec{r},t)$) by solving an integral-differential equation similar to equation (13). Finally, one may employ another minimal value distribution in place of Weibull, equation (1). Since the range of values of γ is bounded, the distribution introduced in [12] may be more adequate.

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