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COMPUTING PERIODS OF POWERS

J.S. Cánovas[1 §], A. Linero Bas[2]

[1]Department of Applied Mathematics and Statistics
Technical University of Cartagena
C/ Dr. Fleming sn, 30202 Cartagena, SPAIN
[2]Department of Mathematics
University of Murcia
30100, Espinardo, Murcia, SPAIN

Abstract: The aim of this note is to describe the set of periods of any iterate f^n, $n \geq 2$, of a continuous (interval or circle) map f with a known set of periods $\mathrm{Per}(f)$.

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1. Introduction

Let X be a topological space and denote by $C(X)$ the set of continuous selfmaps on X. For $f \in C(X)$, let $f^0 = \mathrm{Id}$ be the identity map on X, and $f^n = f \circ f^{n-1}$, $n \geq 1$. We say that $x \in X$ is *periodic* if there is $n \in \mathbb{N}$ such that $f^n(x) = x$. The *period* or *order* of x, which will be denoted by $\mathrm{ord}_f(x)$, is the smallest positive integer satisfying the above condition. Denote by $\mathrm{Per}(f) \subset \mathbb{N}$ and $\mathrm{P}(f) \subset X$ the sets of periods and periodic points of f, respectively.

The computation of sets of periods of continuous selfmaps on one dimen-

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§Correspondence author

sional spaces as the interval [7, 5], the circle, trees and graphs (see [2] and the references therein), and on surfaces (see [3], [4], [6] or [1]), has had an strongly development in the last 40 years.

The main aim of this paper is to characterize and compute the set of periods of the power f^n of a map $f \in C(X)$ whose set of periods $\mathrm{Per}(f)$ is known, when X is either the interval $I := [0,1]$ or the circle $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$. To this end, the following result from [2, Lemma 2.1.10] will be useful.

Lemma 1.1. *Let $f : X \to X$ be a map from a topological space X into itself. Assume that $x \in X$ is periodic with $\mathrm{ord}_f(x) = n$ and for some $p \in \mathbb{N}$, $\mathrm{ord}_{f^p}(x) = m$. Then $n = m \cdot \gcd(n,p)$, where $\gcd(n,p)$ denotes the great common divisor of n and p.*

As a consequence, we obtain the following combinatorial property (we will use it through the following sections) of the order of a periodic point: if $k \in \mathbb{N}$, then $f^k(x) = x$ if and only if $\mathrm{ord}_f(x)|k$, where $m|k$ means that m divides to k, that is, there is $r \in \mathbb{N}$ such that $k = m \cdot r$.

Moreover, according to Lemma 1.1, we can establish the following general characterization of the set of periods of a power of f.

Lemma 1.2. *Let $f \in C(X)$ and $p \in \mathbb{N}$. Then*

$$\mathrm{Per}(f^p) = \Big\{\frac{k}{\gcd(k,p)} : k \in \mathrm{Per}(f)\Big\}.$$

In the cases $X = I$ and $X = \mathbb{S}^1$ it is possible to compute exactly the set $\mathrm{Per}(f^p)$ in terms of initial segments $\mathcal{S}(n)$ of Sharkovsky's ordering (see Section 2) and of sets $M(\rho_l, \rho_l)$ (see Section 3).

The next two sections are devoted to computing the set of periods for continuous interval and circle maps, respectively.

2. The interval case

The sets of periods of interval maps are characterized by the well-known Sharkovsky's Theorem ([7]). We order the natural numbers in this form

$$\begin{aligned}&3 >_s 5 >_s 7 >_s \ldots >_s 2\cdot 3 >_s 2\cdot 5 >_s 2\cdot 7 >_s \ldots\\ &2^n\cdot 3 >_s 2^n\cdot 5 >_s 2^n\cdot 7 >_s \ldots >_s 2^{n+1} >_s 2^n >_s \ldots >_s 2 >_s 1.\end{aligned}$$

For $n \in \mathbb{N} \cup \{2^\infty\}$ define $\mathcal{S}(n) = \{m \in \mathbb{N} : n >_s m\} \cup \{n\}$ and $\mathcal{S}(2^\infty) = \{2^n : n \in \mathbb{N} \cup \{0\}\}$. Sharkovsky's Theorem states that if f has a periodic orbit of period n, then it has periodic points of period $m \in \mathcal{S}(n)$. Moreover, for any $n \in \mathbb{N} \cup \{2^\infty\}$ there is $f_n \in C(I)$ such that $\mathrm{Per}(f_n) = \mathcal{S}(n)$. (See [5] for a simple proof of Sharkovsky's Theorem).

Let $2 \cdot \mathbb{N} + 1$ be the set of odd numbers greater than 1. Let p and q be positive integers and write $\mathcal{A}(p,q) = \{m \in 2 \cdot \mathbb{N} + 1 : q \leq m \cdot p\}$. Let

$$t(p,q) = \min \mathcal{A}(p,q).$$

For instance, $t(1,5) = 5$, $t(3,3) = 3$ and $t(5,39) = 9$. Now we are in a position to state our main result.

Theorem 2.1. *Let $f \in C(I)$ be such that $\mathrm{Per}(f) = \mathcal{S}(q \cdot 2^r)$, with $q \geq 1$ odd and $r \in \mathbb{N} \cup \{0, \infty\}$. Let $p \in \mathbb{N}$, $p \neq 1$.*

(a) If $p \geq 3$ is odd, then

- *(a1) If $q = 1$, $\mathrm{Per}(f^p) = \mathcal{S}(2^r)$.*
- *(a2) If $q > 1$, $\mathrm{Per}(f^p) = \mathcal{S}(t(p,q) \cdot 2^r)$.*

(b) If $p = l \cdot 2^k$, $l \geq 1$ odd, $k \geq 1$, then

- *(b1) If $q = 1$,*

$$\mathrm{Per}(f^p) = \begin{cases} \mathcal{S}(1) & \text{if } k \geq r, \\ \mathcal{S}(2^{r-k}) & \text{if } k < r. \end{cases}$$

(here if $2^k = 2^\infty$, we consider that $2^{k-r} = 2^\infty$).

- *(b2) If $q > 1$,*

$$\mathrm{Per}(f^p) = \begin{cases} \mathcal{S}(3) & \text{if } k > r, \\ \mathcal{S}(t(l,q)) & \text{if } k = r, \\ \mathcal{S}(t(l,q) \cdot 2^{r-k}) & \text{if } k < r. \end{cases}$$

Proof of Theorem 2.1 (a1). By Lemma 1.1, for any $x \in \mathrm{P}(f^p) = \mathrm{P}(f)$, we have that $\mathrm{ord}_{f^p}(x) = 2^s / \gcd(2^s, p) = 2^s$, $s \leq r$, because $\gcd(p, 2^s) = 1$. This proves the result. □

Proof of Theorem 2.1 (a2). Assume that $p \geq 3$ and $q \geq 3$ are odd. We will prove the result in several steps.

1-st step: We claim that $\mathrm{Per}(f^p) \supseteq \mathcal{S}(q \cdot 2^r) = \mathrm{Per}(f)$. It suffices to show that $q \cdot 2^r \in \mathrm{Per}(f^p)$. Since $p \cdot q$ is odd and $p \cdot q > q$, by the Sharkovsky's ordering, there exists a periodic point x of f of period $p \cdot q \cdot 2^r$. Hence, by Lemma 1.1,

$$\mathrm{ord}_{f^p}(x) = \frac{p \cdot q \cdot 2^r}{\gcd(p \cdot q \cdot 2^r, p)} = q \cdot 2^r,$$

which proves the claim.

2-nd step: Next, we will prove that $\mathcal{S}(3 \cdot 2^r) \supseteq \mathrm{Per}(f^p)$. For $r = 0$ this is clear, and hence assume that $r \geq 1$. Let $x \in \mathrm{P}(f^p) = \mathrm{P}(f)$. Then $\mathrm{ord}_f(x) = 2^s$, $s \geq 0$, or $\mathrm{ord}_f(x) = w \cdot 2^s$, $w > 1$ odd and $s \leq r$. In the first case, since $\gcd(p, 2^s) = 1$, by Lemma 1.1 we have $\mathrm{ord}_{f^p}(x) = \frac{\mathrm{ord}_f(x)}{\gcd(p,2^s)} = 2^s$. In the second case, as $\gcd(w \cdot 2^s, p)$ is odd, we have that $\mathrm{ord}_{f^p}(x) = w' \cdot 2^s$, with $w' \geq 1$ odd and $s \leq r$. Hence, the second step is proved.

Summarizing the first two steps, we have

$$\mathcal{S}(3 \cdot 2^r) \supseteq \mathrm{Per}(f^p) \supseteq \mathcal{S}(q \cdot 2^r).$$

3-rd step: We are going to prove that if $q \leq 3 \cdot p$, then $\mathrm{Per}(f^p) = \mathcal{S}(3 \cdot 2^r)$. Note that it is enough to prove that $3 \cdot 2^r \in \mathrm{Per}(f^p)$. Since $q \leq 3 \cdot p$, there is $x \in \mathrm{P}(f)$ with period $3 \cdot p \cdot 2^r$. By Lemma 1.1, $\mathrm{ord}_{f^p}(x) = 3 \cdot p \cdot 2^r / \gcd(3 \cdot p \cdot 2^r, p) = 3 \cdot 2^r$, as desired.

4-th step: Finally, we suppose that $q > 3 \cdot p$ and prove that $\mathrm{Per}(f^p) = \mathcal{S}(t(p,q) \cdot 2^r)$. Notice that $5 \leq t(p,q) < q$. The inequality $5 \leq t(p,q)$ follows from the definition of $t(p,q)$ and the fact that $q > 3 \cdot p$. To check that $t(p,q) < q$, realize that $(t(p,q) - 2) \cdot p < q \leq t(p,q) \cdot p$. So, if $t(p,q)$ were greater than or equal to q, we would obtain that $(t(p,q) - 2)p < t(p,q)$, that is, $p < \frac{t(p,q)}{t(p,q)-2} < 2$, which is a contradiction. Now, since $q \leq p \cdot t(p,q)$ and $\mathrm{Per}(f) = \mathcal{S}(q \cdot 2^r)$, we have $p \cdot t(p,q) \cdot 2^r \in \mathrm{Per}(f)$. Reasoning as in the previous step, we see that $t(p,q) \cdot 2^r \in \mathrm{Per}(f^p)$ and hence,

$$\mathcal{S}(3 \cdot 2^r) \supseteq \mathrm{Per}(f^p) \supseteq \mathcal{S}(t(p,q) \cdot 2^r).$$

We finish the proof if we show that $(t(p,q) - 2) \cdot 2^r \notin \mathrm{Per}(f^p)$. Suppose that $x \in \mathrm{P}(f^p)$ has period $(t(p,q) - 2) \cdot 2^r$ and get a contradiction. Let $s := \mathrm{ord}_f(x)$. Clearly, $s | p \cdot 2^r \cdot (t(p,q) - 2)$, and so $s \leq p \cdot 2^r \cdot (t(p,q) - 2)$. From the definition of $t(p,q)$ we have that $p \cdot (t(p,q) - 2) < q \leq p \cdot t(p,q)$ but at the same time $p(t(p,q) - 2)$ is an odd positive integer greater than one. Then, by force $s = 2^m$ for some $0 \leq m \leq r$. Then, we would obtain that $f^s(x) = x = f^{p \cdot s}(x)$, and therefore $s \geq (t(p,q) - 2) \cdot 2^r = \mathrm{ord}_{f^p}(x)$, which is a contradiction because $t(p,q) - 2$ would be smaller than 1. □

Proof of Theorem 2.1 (b1). Notice that either $\mathrm{Per}(f^{2^k}) = \mathcal{S}(1)$ if $r \leq k$ or $\mathrm{Per}(f^{2^k}) = \mathcal{S}(2^{r-k})$ if $r > k$, and then apply Theorem 2.1 (a1). □

Proof of Theorem 2.1 (b2). By Theorem 2.1 (a), $\mathrm{Per}(f^l) = \mathcal{S}(t(l,q) \cdot 2^r)$. We distinguish several cases:

i): $r < k$. Since $3 \cdot 2^{r+1} \in \operatorname{Per}(f^l)$, there is $x \in \mathrm{P}(f)$ with $\operatorname{ord}_{f^l}(x) = 3 \cdot 2^{r+1}$. By Lemma 1.1, taking into account f^l instead of f, $\operatorname{ord}_{f^p}(x) = \operatorname{ord}_{(f^l)^{2^k}}(x) = \frac{3 \cdot 2^{r+1}}{\gcd(3 \cdot 2^{r+1}, 2^k)} = 3$, and so $\operatorname{Per}(f^{2^k \cdot l}) = \mathcal{S}(3)$.

ii): $r = k$. We claim that $\operatorname{Per}(f^{2^k}) = \mathcal{S}(q)$. Obviously, $q \in \operatorname{Per}(f^{2^k})$. If $q = 3$, the claim is obtained. If $q \geq 5$, suppose that $q - 2 \in \operatorname{Per}(f^{2^k})$. Then there exists $x \in \mathrm{P}(f) = \mathrm{P}(f^{2^k})$ with $\operatorname{ord}_{f^{2^k}}(x) = q - 2$. This implies that $s := \operatorname{ord}_f(x) | (q-2) \cdot 2^k$, that is, $s = w \cdot 2^t$, with $w \geq 1$ odd and $t \leq k$. By Lemma 1.1, $q - 2 = \operatorname{ord}_{f^{2^k}}(x) = \frac{w \cdot 2^t}{\gcd(w \cdot 2^t, 2^k)} = w$. Consequently, $s = (q-2) \cdot 2^t$, which contradicts that $\operatorname{Per}(f) = S(q \cdot 2^k)$. Therefore, $q - 2$ does not belong to the set of periods of f^{2^k}, which ends the claim.
Finally, by applying Theorem 2.1 (a2) to f^{2^k} and the odd value l, we conclude $\operatorname{Per}(f^{2^k \cdot l}) = \mathcal{S}(t(l, q))$.

iii): $r > k$. It is easy to deduce that $\operatorname{Per}(f^{2^k}) = \mathcal{S}(2^{r-k} \cdot q)$. Then, by Theorem 2.1 (a2), $\operatorname{Per}((f^{2^k})^l) = \operatorname{Per}(f^p) = \mathcal{S}(2^{k-s} \cdot t(l, q))$, and the proof ends. $\square$

Example. Let $f \in C(I)$ be such that $\operatorname{Per}(f) = \mathcal{S}(2^5 \cdot 59)$. Since $t(3, 59) = 21$, $t(9, 59) = 7$, $t(41, 59) = 3$, we have

$$\operatorname{Per}(f^3) = \mathcal{S}(2^5 \cdot 21), \quad \operatorname{Per}(f^9) = \mathcal{S}(2^5 \cdot 7), \quad \operatorname{Per}(f^{41}) = \mathcal{S}(2^5 \cdot 3).$$

On the other hand, since $t(1, 59) = 59$, $t(5, 59) = 13$, $t(7, 59) = 9$, we have

$$\begin{array}{lll} \operatorname{Per}(f^2) = \mathcal{S}(2^4 \cdot 59), & \operatorname{Per}(f^{2^2 \cdot 5}) = \mathcal{S}(2^3 \cdot 13), & \operatorname{Per}(f^{2^5}) = \mathcal{S}(59), \\ \operatorname{Per}(f^{2^5 \cdot 9}) = \mathcal{S}(7), & \operatorname{Per}(f^{2^6}) = \mathcal{S}(3), & \operatorname{Per}(f^{2^7 \cdot 11}) = \mathcal{S}(3). \end{array}$$

Example. Let $f \in C(I)$ hold $\operatorname{Per}(f) = \mathcal{S}(2^{12})$. Then $\operatorname{Per}(f^{2m+1}) = \mathcal{S}(2^{12})$ for all $m \geq 1$. For some even values we have:

$$\begin{array}{ll} \operatorname{Per}(f^2) = \mathcal{S}(2^{11}), & \operatorname{Per}(f^{2^3 \cdot 7}) = \mathcal{S}(2^9), \\ \operatorname{Per}(f^{2^{12}}) = \mathcal{S}(1), & \operatorname{Per}(f^{2^{19} \cdot 11}) = \mathcal{S}(1). \end{array}$$

3. The circle case

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map on the circle and let $F : \mathbb{R} \to \mathbb{R}$ be its lifting, that is, $e^{2\pi i F(x)} = f(e^{2\pi i x})$ for all $x \in \mathbb{R}$. Recall that $\deg(f) = F(1) - F(0) \in \mathbb{Z}$ is called the degree of f. Moreover, $\deg(f^k) = (\deg f)^k$ for any positive integer k (see [2, Prop. 3.1.8]). For maps of degree one, it is defined the rotation number of $x \in \mathbb{R}$ by

$$\rho(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

when such limit exists. When F is nondecreasing, the rotation number exists for any $x \in \mathbb{R}$ and, moreover, all points $x \in \mathbb{R}$ have the same rotation number. Hence, it is defined the lower and upper maps

$$F_l(x) = \inf\{F(y) : y \geq x\}$$

and

$$F_u(x) = \sup\{F(y) : y \leq x\},$$

respectively. These maps are nondecreasing and have rotation numbers ρ_l and ρ_u, $\rho_l \leq \rho_u$. Let $\mathrm{Rot}(F)$ be the set of rotation numbers of F. Then, it is proved (see [2, Theorem 3.7.20]) that $\mathrm{Rot}(F) = [\rho_l, \rho_u]$. For $c, d \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{2^\infty\}$, define

$$S(c,n) = \begin{cases} \emptyset & \text{if } c \notin \mathbb{Q}, \\ \{m \cdot q : q \in S(n)\} & \text{if } c = l/m,\ l \text{ and } m \text{ coprime}, \end{cases}$$

and if $c \leq d$,

$$M(c,d) = \{m \in \mathbb{N} : c < l/m < d \text{ for some integer } l\}.$$

The degree and, in the case of degree one maps, the rotation intervals are the keys for characterizing the set of periods of a continuous circle map (see [2, Chapter 3]).

Theorem 3.1. *Let $f \in C(\mathbb{S}^1)$. Then:*

(a) If $\deg(f) \in \mathbb{Z} \setminus \{-2,-1,0,1\}$, *then* $\mathrm{Per}(f) = \mathbb{N}$.

(b) If $\deg(f) = -2$, *then* $\mathrm{Per}(f)$ *is either* $\mathbb{N}$ *or* $\mathbb{N} \setminus \{2\}$, *and both possibilities may occur.*

(c) If $\deg(f) \in \{-1, 0\}$, *then* $\mathrm{Per}(f) = S(n)$ *for some* $n \in \mathbb{N} \cup \{2^\infty\}$.

(d) If $\deg(f) = 1$ *and* $\mathrm{Rot}(F) = [\rho_l, \rho_u]$, *then* $\mathrm{Per}(f) = S(\rho_l, n_l) \cup M(\rho_l, \rho_u) \cup S(\rho_u, n_u)$ *for some* $n_l, n_u \in \mathbb{N} \cup \{2^\infty\}$.

Before establishing our main result for circle maps, we present a pair of previous results which aid us to clarify the general statement of Theorem 3.4 and its corresponding proof.

Lemma 3.2. *Let $\rho_l, \rho_u \in \mathbb{R}$, $\rho_l < \rho_u$. For any $p \in \mathbb{N}$ we have*

$$\{\frac{m}{\gcd(m,p)} : m \in M(\rho_l, \rho_u)\} = M(p \cdot \rho_l, p \cdot \rho_u).$$

Proof. Let $q \in M(\rho_l, \rho_u)$. Then $\rho_l < \frac{k}{q} < \rho_u$ for some integer k. From here, we have

$$p\rho_l < \frac{\frac{p}{\gcd(p,q)}k}{\frac{q}{\gcd(p,q)}} < p\rho_u,$$

and according to definition of sets $M(c,d)$ we deduce that $\frac{q}{\gcd(p,q)} \in M(p \cdot \rho_l, p \cdot \rho_u)$. This proves the inclusion

$$\{\frac{m}{\gcd(m,p)} : m \in M(\rho_l, \rho_u)\} \subseteq M(p \cdot \rho_l, p \cdot \rho_u).$$

For the other part notice that if $t \in M(p \cdot \rho_l, p \cdot \rho_u)$, then $p \cdot \rho_l < \frac{k}{t} < p \cdot \rho_u$ for some $k \in \mathbb{Z}$, or equivalently $\rho_l < \frac{k}{pt} < \rho_u$, which implies $pt \in M(\rho_l, \rho_u)$, and we can write $t = \frac{pt}{\gcd(pt,p)}$. □

Lemma 3.3. *Let $n_\rho \in \mathbb{N}$ and let $m_\rho \in \big(\mathbb{N} \cup \{2^\infty\}\big)$. Then, for any $p \in \mathbb{N}$,*

$$\{\frac{t}{\gcd(t,p)} : t \in n_\rho \mathcal{S}(m_\rho)\} = \frac{n_\rho}{\gcd(n_\rho, p)} \cdot \{\frac{q}{\gcd(q, \frac{p}{\gcd(p,n_\rho)})} : q \in \mathcal{S}(m_\rho)\}.$$

Proof. We use the property

$$\gcd(n_\rho q, k) = \gcd(n_\rho, k) \cdot \gcd(q, \frac{k}{\gcd(n_\rho, k)}),$$

whose proof is immediate and will be omitted. According to it, for each $t \in n_\rho \mathcal{S}(m_\rho)$, to write $\frac{t}{\gcd(t,p)}$ is equivalent to put $\frac{n_\rho q}{\gcd(n_\rho q, p)}$, that is, $\frac{n_\rho}{\gcd(n_\rho,p)} \cdot \frac{q}{\gcd(q, \frac{p}{\gcd(p,n_\rho)})}$ for some $q \in \mathcal{S}(m_\rho)$. □

Now, we can state our main result for circle maps.

Theorem 3.4. *Let $f \in C(\mathbb{S}^1)$ and $p \in \mathbb{N}$, $p \neq 1$. Then*

(a) If $\deg(f) \in \mathbb{Z} \setminus \{-1, 0, 1\}$, then $\mathrm{Per}(f^p) = \mathbb{N}$.

(b) If $\deg(f) \in \{-1, 0\}$, then $\mathrm{Per}(f^p)$ is characterized by Theorem 2.1.

(c) If $\deg(f) = 1$, with $\mathrm{Rot}(F) = [\rho_l, \rho_u]$, and $\mathrm{Per}(f) = S(\rho_l, n_l) \cup M(\rho_l, \rho_u) \cup S(\rho_u, n_u)$ for some $n_l, n_u \in \mathbb{N} \cup \{2^\infty\}$, then for any $p \in \mathbb{N}$ we have:

(c1) If $\rho_l, \rho_u \notin \mathbb{Q}$*, then* $\mathrm{Per}(f^p) = M(p \cdot \rho_l, p \cdot \rho_u)$.

(c2) If $\rho_i \in \mathbb{Q}$ *and* $\rho_j \notin \mathbb{Q}$*, with* $i, j \in \{l, u\}, i \neq j$*, then*

$$\mathrm{Per}(f^p) = \frac{m_i}{\gcd(m_i, p)} \mathrm{Per}\big((f^{m_i})^{\frac{p}{\gcd(m_i,p)}}\big) \cup M(p \cdot \rho_j, p \cdot \rho_j),$$

where $\rho_i = k_i/m_i$*, with* k_i *and* m_i *coprime. Moreover,* $\mathrm{Per}(f^{m_i}) = \mathcal{S}(n_i)$*, so the set* $\mathrm{Per}\big((f^{m_i})^{\frac{p}{\gcd(m_i,p)}}\big)$ *is given by Theorem 2.1 with* $\widetilde{p} = \frac{p}{\gcd(m_i,p)}$ *and* $\widetilde{f} = f^{m_i}$.

(c3) If $\rho_l, \rho_u \in \mathbb{Q}$*, then*

$$\begin{aligned} \mathrm{Per}(f^p) &= \frac{m_l}{\gcd(m_l, p)} \mathrm{Per}\big((f^{m_l})^{\frac{p}{\gcd(m_l,p)}}\big) \\ &\cup M(p \cdot \rho_l, p \cdot \rho_u) \\ &\cup \frac{m_u}{\gcd(m_u, p)} \mathrm{Per}\big((f^{m_u})^{\frac{p}{\gcd(m_u,p)}}\big), \end{aligned}$$

where $\rho_i = k_i/m_i$*, with* k_i *and* m_i *coprime,* $i \in \{l, u\}$*. Moreover, the sets* $\mathrm{Per}\big((f^{m_i})^{\frac{p}{\gcd(m_i,p)}}\big)$*,* $i \in \{l, u\}$*, are given by Theorem 2.1 with* $\widetilde{p_i} = \frac{p}{\gcd(m_i,p)}$ *and by considering an interval map* $\widetilde{f_i} \in C(I)$ *with* $(\widetilde{f_i}) = \mathcal{S}(m_i)$.

Proof. (a) If $\deg(f) \in \mathbb{Z}\backslash\{-1, 0, 1\}$, then $\deg(f^p) = (\deg(f))^p \notin \{-2, -1, 0, 1\}$, and hence by Theorem 3.1, $\mathrm{Per}(f^p) = \mathbb{N}$.

(b) If $\deg(f) = 0$, then $\deg(f^p) = (\deg(f))^p = 0$ and $\mathrm{Per}(f^p) = S(n_p)$ for some $n_p \in \mathbb{N}$. Hence, the set of periods of f^p is characterized by Theorem 2.1. The same argument applies for the case $\deg(f) = -1$ and $p \geq 3$ odd since $\deg(f^{2m+1}) = -1$ and the corresponding set of periods for f^{2m+1} is an initial segment of the Sharkovsky's ordering. Any case, for an even value $2m$ by Lemma 1.2 we know that $\mathrm{Per}(f^{2m}) = \big\{\frac{k}{\gcd(k,2m)} : k \in \mathrm{Per}(f)\big\}$, and attending to the combinatorial nature of this result we deduce that the set of periods obtained in this way must be also a new initial segment of the Sharkovsky's ordering and, therefore, characterized by Theorem 2.1.

(c) The conclusions of this paragraph are based on Lemma 1.2 and the definitions of the sets $S(c, n), M(c, d)$, and on Lemmas 3.2-3.3. □

Remark 3.5. With the notation of Theorem 3.4, if $m \in M(\rho_l, \rho_u)$, there is $t \in \mathbb{Z}$ such that $\rho_l < t/m < \rho_u$. Then $p \cdot \rho_l < p \cdot t/m < p \cdot \rho_u$ and so $m \in M(p \cdot \rho_l, p \cdot \rho_u)$. So, $M(\rho_l, \rho_u) \subset M(p \cdot \rho_l, p \cdot \rho_u)$. Then, by the proof of Proposition 3.9.8 from [2] we have that if $(\rho_l, \rho_u) \cap \mathbb{Z} \neq \emptyset$, then $\mathrm{Per}(f) = \mathbb{N}$ and if $\rho_l \neq \rho_u$ and $\rho_l \in \mathbb{Z}$ (respectively $\rho_u \in \mathbb{Z}$), then $M(\rho_l, \rho_u) = B(t)$, $t \in \mathbb{N} \cup \{\infty\}$, where

$$B(t) = \{t, t+1, t+2, ...\}$$

if $t \in \mathbb{N}$ and $B(\infty) = \emptyset$, and $\mathrm{Per}(f) = \mathcal{S}(n) \cup B(t)$ for some $n \in \mathbb{N} \cup \{2^\infty\}$ and $t \in \mathbb{N} \cup \{\infty\}$. Then, $\mathrm{Per}(f^p)$ has a similar form using by Theorem 2.1. In addition, if $0 \in \mathrm{Rot}(F) \subset \mathrm{Rot}(F^n)$, then $\mathrm{Per}(f^n) = \mathbb{N}$ for all $n \in \mathbb{N}$ by Lemma 3.9.1 from [2].

Remark 3.6. It could be interesting to check whether similar results hold for maps defined in spaces whose structure of periods are known, for instance tree or graph maps (see [2]) or some two dimensional maps (see, again [2] or [3, 4]). Note that the proof of Theorem 2.1 is purely combinatory, that is, the topological structure of the interval is not used. So Theorem 2.1 holds for triangular maps, which are maps on $[0,1]^n$, $n \in \mathbb{N}$, that have the same periodic structure as interval maps (see [6]).

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