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# CONVERGENCE OF ITERATIVE ALGORITHMS FOR NONLINEAR VARIATIONAL-LIKE INEQUALITIES

Shin Min Kang[1] §, Juhe Sun[2], Zeqing Liu[3]

[1]Department of Mathematics and RINS
Gyeongsang National University
Jinju, 660-701, KOREA

[2]Department of Science
Shenyang Aerospace University
Shenyang, Liaoning 110034, P.R. CHINA

[3]Department of Mathematics
Liaoning Normal University
Dalian, Liaoning 116029, P.R. CHINA

**Abstract:** In this paper, we introduce and study a new class of nonlinear variational-like inequalities. By using the auxiliary principle technique, we suggest two iterative algorithms, establish the existence of solutions for the nonlinear variational-like inequality and give the convergence criteria of the sequences generated by the iterative algorithms.

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## 1. Introduction

One of the most interesting and important problems in variational inequality theory is the development of efficient and implementable iterative algorithms

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§Correspondence author

for solving various variational inequalities. It is well known that there are a lot of iterative type algorithms for finding the approximate solutions of various variational inequalities in Hilbert spaces, for example, see [3] and [9], [11]-[15]. Among the most effective numerical technique is the projection method and its variant forms. However, the standard projection technique can no longer be applied to suggest the iterative type algorithm for some nonlinear variational inequalities and variational-like inequalities. This fact motivated Gowinski, Lions and Tremoliers [8] to develop the auxiliary principle technique, which does not depend on the projection. By using the auxiliary principle technique, Ding [4]-[6] suggested several iterative algorithms to compute approximate solutions for some classes of general nonlinear mixed variational inequalities and variational-like inequalities in reflexive Banach spaces.

Motivated and inspired by the results in [4]-[9], [11]-[15], in this paper, we introduce and study a new class of nonlinear variational-like inequalities. By applying a result due to Chang [1] and the auxiliary principle technique, we suggest and analyze two new iterative algorithms, establish the existence of solutions for the nonlinear variational-like inequality. The convergence criteria of the sequence generated by the iterative algorithms are given.

## 2. Preliminaries

Let $H$ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let $K$ be a nonempty closed convex subset of $H$, $A, B, C : K \to H$, $N : H \times H \times H \to H$ and $\eta : K \times K \to H$ be mappings. Suppose that $b : H \times H \to (-\infty, +\infty)$ is nondifferential and satisfies the following conditions:

(c1) $b$ is linear in the first argument;

(c2) $b$ is convex in the second argument;

(c3) $b$ is bounded, that is, there exists a constant $l > 0$ satisfying

$$|b(u, v)| \leq l\|u\|\|v\|, \quad \forall u, v \in H;$$

(c4) $b(u, v) - b(u, w) \leq b(u, v - w)$, $\forall u, v, w \in H$.

Now we consider the following nonlinear variational-like inequality problem: Find $u \in K$ such that

$$\langle N(Au, Bu, Cu), \eta(v, u) \rangle \geq b(u, u) - b(u, v), \quad \forall v \in K. \tag{2.1}$$

Special Cases (A) If $N(Au, Bu, Cu) = Au - Bu$ and $b(u, v) = f(v)$ for all

$u, v \in K$, then the nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au - Bu, \eta(v, u)\rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.2}$$

which was introduced and studied by Ding [4].

(B) If $N(Au, Bu, Cu) = Au - Bu$, $\eta(u, v) = gu - gv$ and $b(u, v) = f(v)$ for all $u, v \in K$, then the nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au - Bu, gv - gu\rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.3}$$

which was studied by Yao [15].

**Definition 2.1.** Let $A : K \to H$, $N : H \times H \times H \to H$ and $\eta : K \times K \to H$ be mappings.

(1) $A$ is said to be *Lipschitz continuous* with constant $\alpha$ if there exists a constant $\alpha > 0$ such that

$$\|Au - Av\| \leq \alpha \|u - v\|, \quad \forall u, v \in K;$$

(2) $N$ is said to be *Lipschitz continuous* with constant $\beta$ in the third argument if there exists a constant $\beta > 0$ such that

$$\|N(w, z, u) - N(w, z, v)\| \leq \beta \|u - v\|, \quad \forall u, v, w, z \in H;$$

(3) $N$ is said to be *strongly monotone* with constant $\gamma$ with respect to $A$ in the third argument if there exists a constant $\gamma > 0$ such that

$$\langle N(w, z, Au) - N(w, z, Av), u - v\rangle \geq \gamma \|u - v\|^2, \quad \forall u, v \in K,\ w, z \in H;$$

(4) $N$ is said to be *$\eta$-strongly monotone* with constant $\xi$ with respect to $A$ in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(Au, w, z) - N(Av, w, z), \eta(u, v)\rangle \geq \xi \|u - v\|^2, \quad \forall u, v \in K,\ w, z \in H;$$

(5) $N$ is said to be *$\eta$-monotone* with respect to $A$ in the second argument if

$$\langle N(w, Au, z) - N(w, Av, z), \eta(u, v)\rangle \geq 0, \quad \forall u, v \in K,\ w, z \in H;$$

(6) $\eta$ is said to be *Lipschitz continuous* with constant $\delta$ if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta \|u - v\|, \quad \forall u, v \in K;$$

(7) $\eta$ is said to be *strongly monotone* with constant $\omega$ if there exists a constant $\omega > 0$ such that

$$\langle u - v, \eta(u, v)\rangle \geq \omega \|u - v\|^2, \quad \forall u, v \in K.$$

**Lemma 2.1.** ([1,2]) *Let $X$ be a nonempty closed convex subset of a Hausdorff linear topological space $E$, and $\phi, \psi : X \times X \to R$ be mappings satisfying the following conditions:*

(a) $\psi(x,y) \le \phi(x,y)$, $\forall x,y \in X$, *and* $\psi(x,x) \ge 0$, $\forall x \in X$;

(b) *for each* $x \in X$, $\phi(x,\cdot)$ *is upper semicontinuous on* $X$;

(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x,y) < 0\}$ *is a convex set;*

(d) *there exists a nonempty compact set* $K \subset X$ *and* $x_0 \in K$ *such that* $\psi(x_0,y) < 0$, $\forall y \in X \setminus K$.

*Then there exists* $\hat{y} \in K$ *such that* $\phi(x,\hat{y}) \ge 0$, $\forall x \in X$.

**Lemma 2.2.** ([10]) *Let* $\{\alpha_n\}_{n\ge 0}$, $\{\beta_n\}_{n\ge 0}$ *and* $\{\gamma_n\}_{n\ge 0}$ *be nonnegative sequences satisfying*

$$\alpha_{n+1} \le (1-\lambda_n)\alpha_n + \beta_n\lambda_n + \gamma_n, \quad \forall n \ge 0,$$

*where* $\{\lambda_n\}_{n\ge 0} \subset [0,1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\lim_{n\to\infty} \alpha_n = 0$.

## 3. Auxiliary Problem and Algorithms

In this section, we give an existence theorem of a solution for auxiliary problem of the nonlinear variational-like inequality (2.1). Based on this existence theorem, we construct two iterative algorithms for the nonlinear variational-like inequality (2.1).

Now, given $u \in K$, we consider the following auxiliary problem: Find $w \in K$ such that

$$\begin{aligned} \langle w, \eta(v,w)\rangle \ge \langle u, \eta(v,w)\rangle - \rho\langle N(Aw, Bw, Cu), \eta(v,w)\rangle \\ + \rho b(u,w) - \rho b(u,v), \quad \forall v \in K, \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant.

**Theorem 3.1.** *Suppose that* $b : H \times H \to (-\infty,\infty)$ *satisfies* (c1)-(c4), $A, B, C : K \to H$ *and* $N : H \times H \times H \to H$ *are continuous mappings. Let* $\eta : K \times K \to H$ *be Lipschitz continuous with constant* $\delta$ *and strongly monotone with constant* $\omega$, *for each* $v \in K$, $\eta(\cdot,v)$ *be continuous and* $\eta(v,u) = -\eta(u,v)$ *for all* $u, v \in K$. *Assume that* $N$ *is* $\eta$*-strongly monotone with constant* $\alpha$ *with respect to* $A$ *in the first argument and* $\eta$*-monotone with respect to* $B$ *in the second argument. If for given* $x, y, z \in H$ *and* $v \in K$, *the mapping* $u \mapsto \langle N(x,y,z), \eta(v,u)\rangle$ *is concave and upper semicontinuous, then the auxiliary problem* (3.1) *has a solution in* $K$.

*Proof.* Define the functionals $\phi$ and $\psi : K \times K \to R$ by

$$\phi(v,w) = \langle v, \eta(v,w)\rangle - \langle u, \eta(v,w)\rangle + \rho\langle N(Av,Bv,Cu), \eta(v,w)\rangle \\ + \rho b(u,v) - \rho b(u,w)$$

and

$$\psi(v,w) = \langle w, \eta(v,w)\rangle - \langle u, \eta(v,w)\rangle + \rho\langle N(Aw,Bw,Cu), \eta(v,w)\rangle \\ + \rho b(u,v) - \rho b(u,w)$$

for all $v, w \in K$.

We check that the functionals $\phi$ and $\psi$ satisfy all the conditions of Lemma 2.1 in the weak topology. It is easy to see for all $v, w \in K$,

$$\begin{aligned} &\phi(v,w) - \psi(v,w) \\ &= \langle v - w, \eta(v,w)\rangle + \rho\langle N(Av,Bv,Cu) - N(Aw,Bv,Cu), \eta(v,w)\rangle \\ &\quad + \rho\langle N(Aw,Bv,Cu) - N(Aw,Bw,Cu), \eta(v,w)\rangle \\ &\geq (\omega + \rho\alpha)\|v - w\|^2 \geq 0, \end{aligned}$$

which yields that $\phi$ and $\psi$ satisfy the condition (a) of Lemma 2.1. Note that $b$ is convex and lower semicontinuous with respect to the second argument and for given $x, y, z \in H$ and $v \in K$, the mapping $u \mapsto \langle N(x,y,z), \eta(v,u)\rangle$ is concave and upper semicontinuous. It follows that $\phi(v,w)$ is weakly upper semicontinuous with respect to $w$ and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. Therefore the conditions (b) and (c) of Lemma 2.1 hold. Let $v^*$ be in $K$. Put

$$D = (\omega + \rho\alpha)^{-1}\big(\delta\rho\|N(Av^*,Bv^*,Cu)\| + \delta\|v^* - u\| + \rho l\|u\|\big)$$

and

$$T = \{w \in K : \|w - v^*\| \leq D\}.$$

Clearly, $T$ is a weakly compact subset of $K$ and for any $w \in K \setminus T$

$$\begin{aligned} \psi(v^*,w) &= -\langle w - v^*, \eta(w,v^*)\rangle + \langle u - v^*, \eta(w,v^*)\rangle \\ &\quad - \rho\langle N(Aw,Bw,Cu) - N(Av^*,Bw,Cu), \eta(w,v^*)\rangle \\ &\quad - \rho\langle N(Av^*,Bw,Cu) - N(Av^*,Bv^*,Cu), \eta(w,v^*)\rangle \\ &\quad - \rho\langle N(Av^*,Bv^*,Cu), \eta(w,v^*)\rangle + \rho b(u,v^*) - \rho b(u,w) \\ &\leq -\|w - v^*\|\big((\omega + \rho\alpha)\|w - v^*\| - \delta\rho\|N(Av^*,CBv^*,Cu)\| \\ &\qquad - \delta\|v^* - u\| - \rho l\|u\|\big) \\ &< 0, \end{aligned}$$

which means that the condition (d) of Lemma 2.1 holds. Thus Lemma 2.1 ensures that there exists a $\hat{w} \in K$ such that $\phi(v, \hat{w}) \geq 0$ for all $v \in K$, that is,

$$\begin{aligned}\langle v, \eta(v, \hat{w})\rangle \geq \langle u, \eta(v, \hat{w})\rangle - \rho\langle N(Av, Bv, Cu), \eta(v, \hat{w})\rangle \\ - \rho b(u, v) + \rho b(u, \hat{w}).\end{aligned} \tag{3.2}$$

Let $t$ be in $(0, 1]$ and $v$ be in $K$. Replacing $v$ by $v_t = tv + (1 - t)\hat{w}$ in (3.2), we know that

$$\begin{aligned}\langle v_t, \eta(v_t, \hat{w})\rangle \geq \langle u, \eta(v_t, \hat{w})\rangle - \rho\langle N(Av_t, Bv_t, Cu), \eta(v_t, \hat{w})\rangle \\ - \rho b(u, v_t) + \rho b(u, \hat{w}).\end{aligned} \tag{3.3}$$

Notice that $b$ is convex with respect to the second argument. From (3.3) we deduce that

$$\begin{aligned}t\langle v_t, \eta(v, \hat{w})\rangle \geq t\big(\langle u, \eta(v, \hat{w})\rangle - \rho\langle N(Av_t, Bv_t, Cu), \eta(v, \hat{w})\rangle \\ - \rho b(u, v) + \rho b(u, \hat{w})\big),\end{aligned}$$

which implies that

$$\begin{aligned}\langle v_t, \eta(v, \hat{w})\rangle \geq \langle u, \eta(v, \hat{w})\rangle - \rho\langle N(Av_t, Bv_t, Cu), \eta(v, \hat{w})\rangle \\ - \rho b(u, v) + \rho b(u, \hat{w}).\end{aligned}$$

Letting $t \to 0^+$ in the above inequality, we conclude that

$$\begin{aligned}\langle \hat{w}, \eta(v, \hat{w})\rangle \geq \langle u, \eta(v, \hat{w})\rangle - \rho\langle N(A\hat{w}, B\hat{w}, Cu), \eta(v, \hat{w})\rangle \\ - \rho b(u, v) + \rho b(u, \hat{w}), \quad \forall v \in K.\end{aligned}$$

That is, $\hat{w}$ is a solution of (3.1). This completes the proof. □

By using Theorem 3.1, we now suggest the following iterative algorithms for solving the nonlinear variational-like inequality (2.1).

**Algorithm 3.1.** *Suppose that $b : H \times H \to (-\infty, \infty)$ satisfies* (c1)-(c4), *$A, B, C : K \to H$, $N : H \times H \times H \to H$ and $\eta : K \times K \to H$ are mappings. For any given $u_0 \in K$, compute sequence $\{u_n\}_{n \geq 0}$ by the iterative scheme*

$$\begin{aligned}&\langle u_{n+1}, \eta(v, u_{n+1})\rangle \\ &\geq \langle u_n, \eta(v, u_{n+1})\rangle - \rho\langle N(Au_{n+1}, Bu_{n+1}, Cu_n), \eta(v, u_{n+1})\rangle \\ &\quad - \rho b(u_n, v) + \rho b(u_n, u_{n+1})\end{aligned} \tag{3.4}$$

*for all $v \in K$ and $n \geq 0$, where $\rho > 0$ is a constant.*

**Algorithm 3.2.** *Suppose that $b : H \times H \to (-\infty, \infty)$ satisfies* (c1)-(c4), *$A, B, C : K \to H$, $N : H \times H \times H \to H$ and $\eta : K \times K \to H$ are mappings. For any given $u_0 \in K$, compute sequences $\{u_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ by the iterative schemes*

$$\begin{aligned}&\langle w_n, \eta(v, w_n)\rangle \\ &\geq (1-\alpha_n)\langle u_n, \eta(v, w_n)\rangle \\ &\quad + \alpha_n\langle u_n - \rho N(Aw_n, Bw_n, Cu_n), \eta(v, w_n)\rangle \\ &\quad - \alpha_n\rho b(u_n, v) + \alpha_n\rho b(u_n, w_n) + \langle q_n, \eta(v, w_n)\rangle,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\langle z_n, \eta(v, z_n)\rangle \\ &\geq (1-\beta_n)\langle u_n, \eta(v, z_n)\rangle \\ &\quad + \beta_n\langle w_n - \rho N(Az_n, Bz_n, Cw_n), \eta(v, z_n)\rangle \\ &\quad - \beta_n\rho b(w_n, v) + \beta_n\rho b(w_n, z_n) + \langle r_n, \eta(v, z_n)\rangle\end{aligned} \tag{3.6}$$

*and*

$$\begin{aligned}&\langle u_{n+1}, \eta(v, u_{n+1})\rangle \\ &\geq (1-\gamma_n)\langle u_n, \eta(v, u_{n+1})\rangle \\ &\quad + \gamma_n\langle z_n - \rho N(Au_{n+1}, Bu_{n+1}, Cz_n), \eta(v, u_{n+1})\rangle \\ &\quad - \gamma_n\rho b(z_n, v) + \gamma_n\rho b(z_n, u_{n+1}) + \langle s_n, \eta(v, u_{n+1})\rangle\end{aligned} \tag{3.7}$$

*for all $v \in K$ and $n \geq 0$, where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0} \subset [0, 1]$, and $\{q_n\}_{n\geq 0}$, $\{r_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0} \subset H$ and $\rho > 0$ is a constant.*

## 4. Existence and Convergence Theorems

In this section, we prove the existence of solution of the nonlinear variational-like inequality (2.1) and discuss the convergence of the iterative sequences generated by the Algorithms 3.1 and 3.2.

**Theorem 4.1.** *Let $b, A, B, N$ and $\eta$ be as in Theorem 3.1 and $C : K \to H$ be Lipschitz continuous with constant $\xi$. Assume that $N$ is Lipschitz continuous with constant $\varepsilon$ in the third argument and strongly monotone with constant $\beta$ with respect to $C$ in the third argument. Let $k = \varepsilon\xi$, $j = \frac{l-\alpha}{\delta}$ and $p = \frac{\omega}{\delta}$. If there exists a constant $\rho > 0$ satisfying*

$$\rho j < p, \tag{4.1}$$

*and one of the following conditions:*

$$\begin{aligned}&\left|\rho-\frac{jp-\beta}{j^2-k^2}\right|<\frac{\sqrt{(jp-\beta)^2-(k^2-j^2)(1-p^2)}}{k^2-j^2},\\&k>|j|,\quad |\beta-jp|>\sqrt{(k^2-j^2)(1-p^2)};\end{aligned}\tag{4.2}$$

$$\left|\rho-\frac{jp-\beta}{j^2-k^2}\right|>\frac{\sqrt{(jp-\beta)^2+(j^2-k^2)(1-p^2)}}{j^2-k^2},\quad |j|>k,\tag{4.3}$$

*then the iterative sequence* $\{u_n\}_{n\ge 0}$ *generated by Algorithm 3.1 converges strongly to some solution of the nonlinear variational-like inequality* (2.1) *in* $K$ .

*Proof.* Using Algorithm 3.1, we obtain that

$$\begin{aligned}&\langle u_n,\eta(v,u_n)\rangle\\&\ge\langle u_{n-1},\eta(v,u_n)\rangle-\rho\langle N(Au_n,Bu_n,Cu_{n-1}),\eta(v,u_n)\rangle\\&\quad-\rho b(u_{n-1},v)+\rho b(u_{n-1},u_n)\end{aligned}\tag{4.4}$$

for all $n\ge 1$. Taking $v=u_n$ in (3.4) and $v=u_{n+1}$ in (4.4), respectively, and adding the two inequalities, we conclude that

$$\begin{aligned}&\omega\|u_{n+1}-u_n\|^2\\&\le\langle u_{n+1}-u_n,\eta(u_{n+1},u_n)\rangle\\&\le\langle u_n-u_{n-1}-\rho(N(Au_n,Bu_n,Cu_n)\\&\qquad-N(Au_n,Bu_n,Cu_{n-1})),\eta(u_{n+1},u_n)\rangle\\&\quad-\rho\langle N(Au_{n+1},Bu_{n+1},Cu_n)-N(Au_n,Bu_{n+1},Cu_n),\eta(u_{n+1},u_n)\rangle\\&\quad-\rho\langle N(Au_n,Bu_{n+1},Cu_n)-N(Au_n,Bu_n,Cu_n),\eta(u_{n+1},u_n)\rangle\\&\quad+\rho b(u_n-u_{n-1},u_n-u_{n+1})\\&\le\left(\delta\sqrt{1-2\rho\beta+(\rho k)^2}+\rho l\right)\|u_n-u_{n-1}\|\|u_{n+1}-u_n\|\\&-\rho\alpha\|u_{n+1}-u_n\|^2\end{aligned}$$

for all $n\ge 1$. That is,

$$\|u_{n+1}-u_n\|\le\theta\|u_n-u_{n-1}\|,\tag{4.5}$$

where

$$\theta=\frac{\delta\sqrt{1-2\rho\beta+(\rho k)^2}+\rho l}{\omega+\rho\alpha}<1\tag{4.6}$$

by (4.1) and one of (4.2) and (4.3). It follows from (4.5) and (4.6) that $\{u_n\}_{n\geq 0}$ is a Cauchy sequence. Consequently, $u_n \to u$ as $n \to \infty$. Letting $n \to \infty$ in (3.4), we infer that

$$\langle N(Au, Bu, Cu), \eta(v, u)\rangle \geq b(u, u) - b(u, v), \quad \forall v \in K.$$

Hence $u$ is a solution of the nonlinear variational-like inequality (2.1). This completes the proof. □

**Theorem 4.2.** *Let $b, A, B, C, N, \eta, k, j$ and $p$ be as in Theorem 4.1. Assume that*

$$\lim_{n\to\infty} \|q_n\| = \lim_{n\to\infty} \|r_n\| = \lim_{n\to\infty} \|s_n\| = 0 \tag{4.7}$$

*and*

$$\inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\} > 0. \tag{4.8}$$

*If there exists a constant $\mu > 0$ satisfying* (4.1) *and*

$$\frac{\delta - \omega}{\alpha \inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\}} \leq \rho < \min\left\{\frac{\delta}{l}, \frac{2\delta(\delta\beta - l)}{(\delta k)^2 - (l)^2}\right\}, \tag{4.9}$$

*then the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.2 converges strongly to some solution of the nonlinear variational-like inequality* (2.1) *in $K$ .*

*Proof.* Put

$$\theta_1 = \frac{\delta}{\omega + \rho\alpha \inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\}}$$

and

$$\theta_2 = \frac{\rho l}{\delta} + \sqrt{1 - 2\rho\beta + (\rho k)^2}.$$

In view of (4.1), (4.6) and (4.9), we conclude easily that

$$\begin{aligned} \theta &= \frac{\delta\sqrt{1 - 2\rho\beta + (\rho k)^2} + \rho l}{\omega + \rho\alpha} = \frac{\delta\theta_2}{\omega + \rho\alpha} \\ &\leq \frac{\delta\theta_2}{\omega + \rho\alpha \inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\}} \leq \theta_2 < 1, \end{aligned}$$

which implies that one of (4.2) and (4.3) holds. It follows from Theorem 4.1 that the nonlinear variational-like inequality (2.1) has a solution $u \in K$ such that

$$\begin{aligned} &\langle u, \eta(v, u)\rangle \\ &\geq (1 - \alpha_n)\langle u, \eta(v, u)\rangle + \alpha_n\langle u - \rho N(Au, Bu, Cu), \eta(v, u)\rangle \\ &\quad - \alpha_n\rho b(u, v) + \alpha_n\rho b(u, u), \end{aligned} \tag{4.10}$$

$$\begin{aligned}&\langle u,\eta(v,u)\rangle\\&\geq(1-\beta_n)\langle u,\eta(v,u)\rangle+\beta_n\langle u-\rho N(Au,Bu,Cu),\eta(v,u)\rangle\\&\quad-\beta_n\rho b(u,v)+\beta_n\rho b(u,u)\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&\langle u,\eta(v,u)\rangle\\&\geq(1-\gamma_n)\langle u,\eta(v,u)\rangle+\gamma_n\langle u-\rho N(Au,Bu,Cu),\eta(v,u)\rangle\\&\quad-\gamma_n\rho b(u,v)+\gamma_n\rho b(u,u)\end{aligned}\tag{4.12}$$

for all $v\in K$ and $n\geq 0$. Taking $v=u$ in (3.5), $v=w_n$ in (4.10) and adding these inequalities, we know that

$$\begin{aligned}&\omega\|w_n-u\|^2\\&\leq(1-\alpha_n)\langle u_n-u,\eta(w_n,u)\rangle\\&\quad-\alpha_n\rho\langle N(Aw_n,Bw_n,Cu_n)-N(Au,Bw_n,Cu_n),\eta(w_n,u)\rangle\\&\quad-\alpha_n\rho\langle N(Au,Bw_n,Cu_n)-N(Au,Bu,Cu_n),\eta(w_n,u)\rangle\\&\quad+\alpha_n\langle u_n-u-\rho(N(Au,Bu,Cu_n)-N(Au,Bu,Cu)),\eta(w_n,u)\rangle\\&\quad+\alpha_n\rho b(u_n-u,u-w_n)+\langle q_n,\eta(w_n,u)\rangle\\&\leq\big((1-\alpha_n)\delta+\alpha_n\delta\sqrt{1-2\rho\beta+\rho^2k^2}+\alpha_n\rho l\big)\|u_n-u\|\|w_n-u\|\\&\quad-\alpha_n\alpha\rho\|w_n-u\|^2+\delta\|q_n\|\|w_n-u\|,\quad\forall n\geq 0,\end{aligned}$$

that is,

$$\begin{aligned}\|w_n-u\|&\leq\theta_1\big(1-\alpha_n(1-\theta_2)\big)\|u_n-u\|+\theta_1\|q_n\|\\&\leq\|u_n-u\|+\|q_n\|,\quad\forall n\geq 0.\end{aligned}\tag{4.13}$$

Similarly we infer that that

$$\begin{aligned}\|z_n-u\|&\leq\big(1-\beta_n(1-\theta_2)\big)\|w_n-u\|+\|r_n\|\\&\leq\big(1-\beta_n(1-\theta_2)\big)\|u_n-u\|+\|q_n\|+\|r_n\|\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1}-u\|&\leq\big(1-\gamma_n(1-\theta_2)\big)\|z_n-u\|+\|s_n\|\\&\leq\big(1-\gamma_n(1-\theta_2)\big)\|u_n-u\|+\|s_n\|\\&\quad+\|q_n\|+\|r_n\|\end{aligned}\tag{4.14}$$

for all $n\geq 0$. It follows from Lemma 2.2, (4.7)-(4.9) and (4.14) that $\lim_{n\to\infty}\|u_{n+1}-u\|=0$. This completes the proof. □

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