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AN UNCOUNTABLE FAMILY OF REGULAR BOREL MEASURES ON CERTAIN PATH SPACES OF LIPSCHITZ FUNCTIONS WITH APPLICATIONS TO FEYNMAN-TYPE PATH INTEGRALS

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Abstract: Let $c > 0$ be a fixed constant. Let $0 \le r < s$ be an arbitrary pair of real numbers. Let a, b be any pair of real numbers such that $|\, b - a \,| \le c(s - r)$. Define C_r^s to be the set of continuous real-valued functions on $[r, s]$, and define C_r to be the set of continuous real-valued functions on $[\, r, +\infty)$. Finally, consider the following sets of Lipschitz functions:

$$\Lambda_r^s = \{\, x \in C_r^s \mid |x(v) - x(u)| \le c|v - u|, \quad \textit{for all} \quad u, v \in [\, r, s\,] \,\}, \tag{1}$$

$$\Lambda_r = \{\, x \in C_r \mid |x(v) - x(u)| \le c|v - u|, \quad \textit{for all} \quad u, v \in [\, r, +\infty) \,\}, \tag{2}$$

$$\Lambda_{r,a}^{s,b} = \{\, x \in \Lambda_r^s \mid x(r) = a, \quad x(s) = b \,\}, \tag{3}$$

$$\Lambda_{r,a}^{s} = \{\, x \in \Lambda_r^s \mid x(r) = a \,\}, \tag{4}$$

$$\Lambda_{r}^{s,b} = \{\, x \in \Lambda_r^s \mid x(s) = b \,\}, \tag{5}$$

$$\Lambda_{r,a} = \{\, x \in \Lambda_r \mid x(r) = a \,\}. \tag{6}$$

We present a general method of constructing an uncountable family of regular Borel measures on each of the sets (1), (2), and an uncountable family of regular Borel probability measures on each of the sets (3)-(6). Using this method, we give a definition of **Lebesgue measure** on the sets (1) and (2), and a definition of **the uniform probability** measure on each of the sets (3)-(6). By interpreting c as the speed of light, we then use Lebesgue measure on the sets (1), (2) and the uniform probability measure on the sets (3)-(6) to *rigorously define* versions of **the relativistic Feynman integral** and the **relativistic Wiener integral** on the sets of relativistic paths (1)-(6).

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1. Introduction

If $\mathcal{X}$ is any infinite dimensional locally convex topological vector space, then it is known that there does not exist a non-trivial translation invariant σ-finite Borel measure on $\mathcal{X}$ (see [13], p. 143; [6], pp. 354-362). Because of this impossibility theorem, it is part of the folklore of infinite dimensional analysis that there is no satisfactory analogue of Lebesgue measure on the space $\underline{\mathrm{R}}^\infty = \prod_{i=1}^\infty \underline{\mathrm{R}}$. In the papers [1] and [2], we demonstrated, without upsetting this folklore that there exists a non-trivial translation invariant Borel measure λ on $\underline{\mathrm{R}}^\infty$ which is analogous to Lebesgue measure in the sense that if $R = \prod_{i=1}^\infty (a_i, b_i)$ is any infinite dimensional rectangle such that the "volume" $\prod_{i=1}^\infty (b_i - a_i)$ of R is a nonnegative real number, then

$$\lambda(R) = \prod_{i=1}^{\infty} (b_i - a_i).$$

Therefore, it does not seem to be inappropriate to call λ an **infinite dimensional Lebesgue measure on** $\underline{\mathrm{R}}^\infty$. Of course, according to the above no-go result, the measure λ can not be σ-finite. For another example, let $t > 0$ and define the **Wiener space** $C_0[\,0, t\,]$ to be the vector space of all continuous functions x on $[\,0, t\,]$ such that $x(0) = 0$. The above no-go result implies that $C_0[\,0, t\,]$ can not be equipped with a satisfactory analogue of Lebesgue measure. Given this no-go result, it seems natural to ask the following question: Does there exist a topological subspace $\mathcal{X} \neq \emptyset$ of $C_0[\,0, t\,]$ such that (a) $\mathcal{X}$ has *infinite topological dimension* (see [11], p. 302) and (b) there exists on $\mathcal{X}$ a non-trivial σ-finite Borel measure that may appropriately be called **Lebesgue measure** on $\mathcal{X}$? In the present paper, we give an affirmative answer to this question: we present a method of constructing analogues of Lebesgue measure on certain spaces of Lipschitz functions. To define these spaces of functions, let $c > 0$ be a fixed constant. For each interval $I \subseteq \underline{\mathrm{R}}$, let $\Lambda(I)$ be the set of functions $x : I \to \underline{\mathrm{R}}$ such that x satisfies the Lipschitz condition

$$|x(t) - x(s)| \leq c|t - s|, \quad \text{for all} \quad s, t \in I.$$

Let r, s be any pair of real numbers such that $0 \leq r < s$. Define

$$\Lambda_r^s = \Lambda([r,s]), \quad \Lambda_r = \Lambda([r,+\infty)).$$

Finally, let a, b be any pair of real numbers such that $|b-a| \leq c(s-r)$. Then define

$$\begin{aligned} \Lambda_{r,a}^{s,b} &= \{\, x \in \Lambda_r^s \mid x(r) = a, \quad x(s) = b \,\}; \\ \Lambda_{r,a}^{s} &= \{\, x \in \Lambda_r^s \mid x(r) = a \,\}; \\ \Lambda_{r}^{s,b} &= \{\, x \in \Lambda_r^s \mid x(s) = b \,\}; \\ \Lambda_{r,a} &= \{\, x \in \Lambda_r \mid x(r) = a \,\}. \end{aligned} \tag{1}$$

The problem of finding an analogue of Lebesgue measure on the spaces in (1) arises naturally in quantum mechanics in the context of searching for rigorous versions of the so-called *Feynman integral*. Let $L(x, \dot{x})$ be the **Lagrangian** of a one-dimensional quantum mechanical article and let $S(x)$ be the **action** corresponding to $L(x, \dot{x})$, i.e.,

$$L(x,\dot{x}) = \frac{1}{2} m\dot{x}^2 - V(x,\dot{x}), \quad S(x) = \int_0^t L(x,\dot{x})\, ds.$$

Feynman postulated (see Section 5 for more details) that, formally,

$$(e^{-\frac{i}{\hbar}tH}\psi)(b) = \int_{x(t)=b} e^{\frac{i}{\hbar}S(x)}\, \psi(x(0))\, \mathcal{D}(x), \tag{2}$$

where H is the operator

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,\dot{x}).$$

In (2) the "integration" is supposed to be over the space $C_0^{t,b}$ of all continuous paths $x : [0,t] \to \mathbb{R}$ such that $x(t) = b$. $\mathcal{D}(x)$ is supposed to be "Lebesgue measure" on $C_0^{t,b}$. Formally,

$$\mathcal{D}(x) = \mathcal{N} \prod_{0 \leq s \leq t} dx(s), \tag{3}$$

where $\mathcal{N}$ is a formal normalization factor. The main obstruction to making rigorous mathematical sense of the heuristic statement (2) is that the purely formal definition (3) can not be directly replaced by a mathematically well-defined σ-finite translation invariant Borel measure on $C_0^{t,b}$—simply because, according to the no-go result stated above, no such measure exists. Ever since the 1950's mathematicians have attacked the problem of how to make rigorous mathematical sense of the formal "integral" in (2) (see Section 5 for more detail). Now, if we take c to be the speed of light then the **relativistic Lagrangian** and **relativistic action** of a spinless relativistic particle with rest mass $m_0 > 0$ can be written as

$$L_{rel}(x, \dot{x}) = -m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - V(x, \dot{x}),$$

$$S_{rel}(x) = \int_0^t \left(- m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - V(x, \dot{x}) \right) ds.$$

Therefore, following Feynman's prescription in the relativistic case, if in (2) we replace $L(x, \dot{x})$ by $L_{rel}(x, \dot{x})$, then we are naturally led to consider "relativistic Feynman integral" of the form

$$\int_{x(t)=b} \exp\left\{ \frac{i}{\hbar} \int_0^t \left(- m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - V(x, \dot{x}) \right) ds \right\} \psi(x(0))\, \mathcal{D}(x). \tag{4}$$

In (4), $|b| \leq c$ and the "integration" is over a suitable space **relativistic paths** x such that x is continuous on $[\,0, t\,]$, $x(t) = b$, and the average velocity of x on $[\,0, t\,]$ does not exceed c in absolute value, i.e., for any $r, s \in [\,0, t\,]$, if $r \neq s$, then $\dfrac{|x(s) - x(r)|}{|s - r|} \leq c$, i.e, $|x(s) - x(r)| \leq c|s - r|$. Therefore the formal integration in (4) is over a suitable subspace of paths $\mathcal{X} \subseteq \Lambda_0^{t,b}$. Thus, in the quest to make sense of the formal measure $\mathcal{D}(x)$, we are naturally led to ask if there are analogues of Lebesgue measure on the function space $\Lambda_0^{t,b}$. In the present paper we show how to construct an analogue of Lebesgue measure on $\Lambda_0^{t,b}$ and (see Section 5) we show how to use this measure to give a rigorous *version* of the measure $\mathcal{D}(x)$ in (4).

As stated above, the main result of the present paper (Sections 2 and 3) is a general method of constructing an uncountable family of regular Borel measures

$$\lambda_{r,a}^{s,b}, \quad \lambda_{r,a}^{s}, \quad \lambda_{r}^{s,b}, \quad \lambda_{r,a}, \tag{5}$$

$$\lambda_{r}^{s}, \quad \lambda_{r}, \tag{6}$$

respectively, on each of the following sets:

$$\Lambda_{r,a}^{s,b}, \quad \Lambda_{r,a}^{s}, \quad \Lambda_{r}^{s,b}, \quad \Lambda_{r,a}; \tag{7}$$

$$\Lambda_{r}^{s}, \quad \Lambda_{r}. \tag{8}$$

The method of constructing the measures in (2) and (3) was inspired by Lévy's bisection algorithm for constructing Brownian motion (see [4], p. 56, [10], and [12], pp. 142-145). Each Borel measure in (2) is a probability measure, constructed as the continuous image of Lebesgue measure under a certain family of continuous surjective mappings

$$\varphi_{r,a}^{s,b} : \Omega_r^s \to \Lambda_{r,a}^{s,b}, \quad \varphi_{r,a}^{s} : \overset{\wedge}{\Omega}{}_r^s \to \Lambda_{r,a}^{s}, \quad \varphi_{r}^{s,b} : \overset{\vee}{\Omega}{}_r^s \to \Lambda_{r}^{s,b}, \quad \overset{\wedge}{\Omega}_r \to \Lambda_{r,a}. \tag{9}$$

Each of the spaces

$$\Omega_r^s, \quad \overset{\wedge}{\Omega}{}_r^s, \quad \overset{\vee}{\Omega}{}_r^s, \quad \overset{\wedge}{\Omega}_r \tag{10}$$

is endowed with Lebesgue measure, and has the form $[0,1]^A$, where A is some indexing set. Likewise, each Borel measure in (6) is constructed as the continuous image of Lebesgue measure under a certain family of continuous surjective mappings

$$\varphi_r^s : \overset{\sim}{\Omega}{}_r^s \to \Lambda_r^s, \quad \varphi_r : \overset{\sim}{\Omega}_r \to \Lambda_r, \tag{11}$$

where each of the spaces

$$\overset{\sim}{\Omega}{}_r^s, \quad \overset{\sim}{\Omega}_r \tag{12}$$

has the form $[0,1]^B$ for some indexing set B, and is endowed with Lebesgue measure. In Section 4, certain members

$$\lambda_{r,a}^{s,b}, \quad \lambda_{r,a}^{s}, \quad \lambda_{r}^{s,b}, \quad \lambda_{r,a}$$

of the uncountable family (5) are singled out and defined to be **the uniform probability measure** on the spaces

$$\Lambda_{r,a}^{s,b}, \quad \Lambda_{r,a}^{s}, \quad \Lambda_{r}^{s,b}, \quad \Lambda_{r,a}, \tag{13}$$

and certain members

$$\lambda_r^s, \quad \lambda_r$$

of the uncountable family (6) are singled out and defined to be **Lebesgue measure** on the spaces

$$\Lambda_r^s, \quad \Lambda_r. \tag{14}$$

Finally, in Section 5, we use the uniform probability measures on the spaces (13) and Lebesgue measure on the spaces (14) to *rigorously define* versions of **the relativistic Feynman integral** and **the relativistic Wiener integral** on these spaces.

2. Construction of the Functions $\varphi_{\square}^{\square}$

In this section we construct the families of continuous surjective mappings mentioned in (6) and (1) of the introduction.

Definition 2.1. Let $0 \leq r < s$ be given. Let $(r, a), (s, b)$ be two given points in the plane. Define

$$\begin{aligned} F_{r,a} &= \{ (t,x) \,|\, r \leq t \quad \text{and} \quad a - c(t-r) \leq x \leq a + c(t-r) \}, \\ B_{s,b} &= \{ (t,x) \,|\, t \leq s \quad \text{and} \quad b - c(s-t) \leq x \leq b + c(s-t) \}, \\ P_{r,a}^{s,b} &= F_{r,a} \cap B_{s,b}. \end{aligned}$$

Proposition 2.1. *For arbitrary pairs $(r, a), (s, b)$, with $0 \leq r < s$, we have*

$$P_{r,a}^{s,b} \neq \emptyset \quad \text{if and only if} \quad |b - a| \leq c(s - r).$$

If $P_{r,a}^{s,b} \neq \emptyset$, then $P_{r,a}^{s,b}$ is either the line segment connecting (r, a) to (s, b), or $P_{r,a}^{s,b}$ is a nondegenerate parallelogram containing this line segment.

Proof. The proof of this proposition is routine. □

Definition 2.2. Assume that $P_{r,a}^{s,b} \neq \emptyset$. Let $u = \dfrac{1}{2}(r + s)$. Define $I_{r,a}^{s,b}$ to be the projection of the following set onto the x-axis.

$$\{ (u, x) \,|\, -\infty < x < +\infty \} \cap P_{r,a}^{s,b}.$$

Because $P_{r,a}^{s,b}$ is a parallelogram containing the line segment joining (r, a) to (s, b), and because $r < u < s$, we see that $I_{r,a}^{s,b}$ is either a point or a nondegenerate closed interval.

Proposition 2.2. *The intervals $I_{r,a}^{s,b}$ are given by*

$$I_{r,a}^{s,b} = \begin{cases} [\, b - \frac{1}{2}c(s-r), a + \frac{1}{2}c(s-r) \,], & \text{if } a \leq b; \\ [\, a - \frac{1}{2}c(s-r), b + \frac{1}{2}c(s-r) \,], & \text{if } b \leq a. \end{cases}$$

Proof. The proof of this proposition is clear. □

Definition 2.3. We shall assume that for each pair (r,a), (s,b) of points in the plane such that $0 \le r < s$ and $P^{s,b}_{r,a} \neq \emptyset$, we are given a continuous function

$$\lambda^{s,b}_{r,a} : [0,1] \to I^{s,b}_{r,a}$$

mapping $[0,1]$ onto $I^{s,b}_{r,a}$. We shall also assume that the mapping $(a,b,r,s,\xi) \mapsto \lambda^{s,b}_{r,a}(\xi)$ is continuous on the set D_λ, where

$$D_\lambda = \{\,(a,b,r,s,\xi) \in \mathbb{R}^5 \,|\, 0 \le r < s, \quad |b-a| \le c(s-r), \quad \textit{and} \quad \xi \in [\,0,1\,]\,\}.$$

Proposition 2.3. *Assume that* $P^{s,b}_{r,a} \neq \emptyset$. *Let* $u = \frac{1}{2}(r+s)$. *If* $d = \lambda^{s,b}_{r,a}(\xi)$, *where* $\xi \in [0,1]$ *is arbitrary, then*

$$\emptyset \neq P^{u,d}_{r,a}, \quad P^{s,b}_{u,d} \subseteq P^{s,b}_{r,a}.$$

Proof. We will only prove that $\emptyset \neq P^{u,d}_{r,a} \subseteq P^{s,b}_{r,a}$ and $P^{u,d}_{r,a} \subseteq P^{s,b}_{r,a}$. The rest of the proof is similar. Also, we only give the proof for the case where $a \le b$, the proof for the case $b \le a$ is similar. We then have

$$I^{s,b}_{r,a} = [\,b - \frac{1}{2}c(s-r), a + \frac{1}{2}c(s-r)\,].$$

Because $d = \lambda^{s,b}_{r,a}(\xi)$, it follows from Definition 2.3 that $d \in I^{s,b}_{r,a}$, therefore

$$b - \frac{1}{2}c(s-r) \le d \le a + \frac{1}{2}c(s-r). \tag{1}$$

By Proposition 2.1, to prove that $P^{u,d}_{r,a} \neq \emptyset$, it suffices to prove that

$$|d-a| \le c(u-r). \tag{2}$$

Note that $u - r = \frac{1}{2}(s-r)$. Hence, from (1),

$$d \le a + \frac{1}{2}c(s-r) = a + c(u-r).$$

It also follows from (1) that

$$d \ge b - \frac{1}{2}c(s-r) = b - c(u-r) \ge a - c(u-r).$$

We conclude that

$$a - c(u-r) \le d \le a + c(u-r).$$

Therefore, $|d-a| \le c(u-r)$, hence (2) holds. To prove that $P_{r,a}^{u,d} \subseteq P_{r,a}^{s,b}$, let $(t,x) \in P_{r,a}^{u,d}$ be arbitrary. Then by definition, $r \le t \le s$ and

$$\begin{aligned} a - c(t-r) &\le x \le a + c(t-r), \\ d - c(u-t) &\le x \le d + c(u-t). \end{aligned} \tag{3}$$

We have $u-r = s-u$, and by (2), $d \le a + c(u-r)$, consequently, (3) implies that

$$\begin{aligned} x &\le d + c(u-t) \\ &\le [\, a + c(u-r)\,] + c(u-t) \\ &= a + c(s-u) + c(u-t) \\ &\le b + c(s-u) + c(u-r) \\ &= b + c(s-r). \end{aligned}$$

Thus, $x \le b + c(s-r)$. Similarly, we see that $b - c(s-r) \le x$, therefore,

$$b - c(s-r) \le x \le b + c(s-r).$$

Hence, by definition, $(t,x) \in B_{s,b}$. The same type of argument shows that $(t,x) \in F_{r,a}$, consequently, $(t,x) \in F_{r,a} \cap B_{s,b} = P_{r,a}^{s,b}$. This proves that $P_{r,a}^{u,d} \subseteq P_{r,a}^{s,b}$. □

Definition 2.4. Let $0 \le r < s$ be arbitrary, and let a, b be real numbers such that $P_{r,a}^{s,b} \ne \emptyset$. Define

$$t_{nj} = r + \frac{j}{2^n}(s-r), \quad 0 \le j \le 2^n, \quad n = 0, 1, 2, \ldots .$$

For $n \ge 1$, define

$$\begin{aligned} V_{r,n}^s &= \{\, t_{nj} \mid 0 < j < 2^n \,\}, \\ V_r^s &= \bigcup_{n=1}^{\infty} V_{r,n}^s, \\ \Omega_r^s &= [0,1]^{V_r^s}. \end{aligned}$$

Theorem 2.1. *Assume that $P_{r,a}^{s,b} \ne \emptyset$. Then there exists a function*

$$\varphi_{r,a}^{s,b} : \Omega_r^s \to \underset{\sim}{R}^{V_r^s}$$

such that for each $\omega \in \Omega_r^s$, $\varphi_{r,a}^{s,b}(\omega)$ *satisfies conditions (a)–(c) below. For each* $n \geq 1$, *we define*

$$a_{mj}(\omega) = \varphi_{r,a}^{s,b}(\omega)(t_{mj}), \quad 0 \leq j \leq 2^m, \quad 0 \leq m \leq n.$$

Write

$$a_{mj} = a_{mj}(\omega), \quad \varphi(\omega) = \varphi_{r,a}^{s,b}(\omega).$$

The properties that $\varphi(\omega)$ *has are as follows.*

(a) *For all* $1 \leq m \leq n$ *and* $1 \leq j \leq 2^{m-1}$,

$$\emptyset \neq P_{t_{m-1,j-1},a_{m-1,j-1}}^{t_{m,2j-1},a_{m,2j-1}}, \quad P_{t_{m,2j-1},a_{m,2j-1}}^{t_{m-1,j},a_{m-1,j}} \subseteq P_{t_{m-1,j-1},a_{m-1,j-1}}^{t_{m-1,j},a_{m-1,j}}.$$

(b) *For all* $1 \leq m \leq n$ *and* $1 \leq j \leq 2^{m-1}$,

$$\varphi(\omega)(t_{m,2j-1}) = \lambda_{t_{m-1,j-1},a_{m-1,j-1}}^{t_{m-1,j},a_{m-1,j}}(\omega_{t_{m,2j-1}}).$$

(c) $P_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}} \neq \emptyset, \quad 1 \leq j \leq 2^n.$

Proof. Fix $\omega \in \Omega_r^s$. We will use induction on n to define $\varphi(\omega) = \varphi_{r,a}^{s,b}(\omega)$ consistently on each $V_n = V_{r,n}^s$.

Define $\varphi(\omega)$ on $V_1 = t_{11}$ as follows. First, define

$$\varphi(\omega)(r) = a, \quad \varphi(\omega)(s) = b.$$

Because $P_{r,a}^{s,b} \neq \emptyset$, Definition 2.3 gives a function

$$\lambda_{r,a}^{s,b} : [0,1] \to I_{r,a}^{s,b}.$$

Define $\varphi(\omega)$ on V_1 by

$$\varphi(\omega)(t_{11}) = \lambda_{r,a}^{s,b}(\omega_{t_{11}}).$$

Define $u = t_{11} = \dfrac{1}{2}(r+s)$. Then Proposition 2.3 implies that, with $d = \lambda_{r,a}^{s,b}(\omega_{t_{11}})$,

$$\emptyset \neq P_{r,a}^{u,d}, \quad P_{u,d}^{s,b} \subseteq P_{r,a}^{s,b}.$$

It is now easy to see that (a)–(c) hold for $n = 1$.

Now assume that $\varphi(\omega)$ has been defined on V_n in such a way that (a)–(c) hold, where $n \geq 1$ is given. We will then define $\varphi(\omega)$ on V_{n+1} in such a way that (a)–(c) hold when n is replaced by $n+1$. That is, we want to define $\varphi(\omega)$ on V_{n+1} so that the following conditions hold.

(a') For all $1 \le m \le n+1$ and $1 \le j \le 2^{m-1}$,

$$\emptyset \neq P_{t_{m-1,j-1},a_{m-1,j-1}}^{t_{m,2j-1},a_{m,2j-1}}, \quad P_{t_{m,2j-1},a_{m,2j-1}}^{t_{m-1,j},a_{m-1,j}} \subseteq P_{t_{m-1,j-1},a_{m-1,j-1}}^{t_{m-1,j},a_{m-1,j}}.$$

(b') For all $1 \le m \le n+1$ and $1 \le j \le 2^{m-1}$,

$$\varphi(\omega)(t_{m,2j-1}) = \lambda_{t_{m-1,j-1},a_{m-1,j-1}}^{t_{m-1,j},a_{m-1,j}}(\omega_{t_{m,2j-1}}).$$

(c') $P_{t_{n+1,j-1},a_{n+1,j-1}}^{t_{n+1,j},a_{n+1,j}} \neq \emptyset, \quad 1 \le j \le 2^{n+1}.$

To define $\varphi(\omega)$ on V_{n+1}, let $0 < k < 2^{n+1}$ be arbitrary. Then k has one of the following forms.

$$k = \begin{cases} 2j, & 1 \le j < 2^n; \\ 2j-1, & 1 \le j \le 2^n. \end{cases}$$

In case, $k = 2j$, we have $t_{n+1,k} = t_{n+1,2j} = t_{nj}$, and hence we define

$$a_{n+1,k} = \varphi(\omega)(t_{n+1,k}) = \varphi(\omega)(t_{nj}).$$

Suppose that $k = 2j-1$. By (c) of the inductive hypothesis, we have

$$P_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}} \neq \emptyset.$$

Hence, by Definition 2.3, we are given a function

$$\lambda_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}} : [\,0,1\,] \to I_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}}.$$

Now define

$$a_{n+1,k} = \varphi(\omega)(t_{n+1,k}) = \lambda_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}}(\omega_{t_{n+1,k}}).$$

This defines $\varphi(\omega)$ on V_{n+1}.

To prove (a'), let $1 \le j \le 2^{m-1}$, where $1 \le m \le m+1$. Suppose that $m \le n$, then by the inductive hypothesis, (a) holds for n. Because $m \le n$, (a') reduces to (a). On the other hand, suppose that $m = n+1$, and let $1 \le j \le 2^{m-1} = 2^n$. Define

$$u = t_{n+1,2j-1} = \frac{1}{2}(t_{n,j-1} + t_{nj}).$$

Then Proposition 2.3 implies that, with

$$\begin{aligned} d &= \lambda_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}}(\omega_{t_{n+1,k}}) \\ &= \varphi(\omega)(t_{n+1,2j-1}) \end{aligned}$$

$$= a_{n+1,2j-1},$$

we have

$$\emptyset \neq P^{u,d}_{tn,j-1,a_{n,j-1}}, \quad P^{t_{nj},a_{nj}}_{u,d} \subseteq P^{t_{nj},a_{nj}}_{tn,j-1,a_{n,j-1}}.$$

This statement is equivalent to (a') for the case where $m = n+1$. We conclude that (a') holds for all $1 \leq m \leq n+1$.

To prove (b'), let $1 \leq m \leq n+1$, and let $1 \leq j \leq 2^{m-1}$. If $m \leq n$, then by the induction hypothesis, (b) holds, and hence (b') holds because (b') reduces to (b). On the other hand, suppose that $m = n+1$. Then by definition,

$$\begin{aligned}\varphi(\omega)(t_{m,2j-1}) &= \varphi(\omega)(t_{n+1,2j-1}) \\ &= \lambda^{t_{nj},a_{nj}}_{t_{n,j-1},a_{n,j-1}}(\omega_{t_{n+1,2j-1}}) \\ &= \lambda^{t_{m-1,j},a_{m-1,j}}_{t_{m-1,j-1},a_{m-1,j-1}}(\omega_{t_{m,2j-1}}),\end{aligned}$$

which is (b') for the case $m = n+1$. We conclude that (b') holds for all $1 \leq m \leq n+1$.

To prove (c'), let $1 \leq k \leq 2^{n+1}$. Then k has one of the following forms.

$$k = \begin{cases} 2j, & 1 \leq j < 2^n; \\ 2j-1, & 1 \leq j \leq 2^n. \end{cases}$$

Suppose that $k = 2j$. We proved above that (a') holds, hence, with $m = n+1$ in (a'), we have

$$P^{t_{n+1,k},a_{n+1,k}}_{t_{n+1,k-1},a_{n+1,k-1}} = P^{t_{nj},a_{nj}}_{t_{n+1,2j-1},a_{n+1,2j-1}} \neq \emptyset.$$

On the other hand, suppose that $k = 2j-1$. Then by (a'), with $m = n+1$, we have

$$P^{t_{n+1,k},a_{n+1,k}}_{t_{n+1,k-1},a_{n+1,k-1}} = P^{t_{n+1,2j-1},a_{n+1,2j-1}}_{t_{n,j-1},a_{n,j-1}} \neq \emptyset.$$

Because $1 \leq k \leq 2^{n+1}$ is arbitrary, we conclude that (c') holds. Therefore, $(a') - (c')$ hold, and hence we have completed the inductive definitions of $\varphi(\omega)$ on each V_n in such a way that (a)–(c) hold for each n.

To complete the definition of $\varphi(\omega)$ as a function on

$$V^s_r = \bigcup_{n=1}^{\infty} V^s_{r,n},$$

we show that $\varphi(\omega)$ is consistently defined on V^s_r. To this end, note first that by the above construction, $\varphi(\omega)$ has the property that for all $n \geq 1$, $\varphi(\omega)(t_{n+1,2j}) =$

$\varphi(\omega)(t_{nj})$, for $0 < j < 2^n$. Now let $1 \leq m \leq n$, and let $t_{mj} \in V_m$, that is, let $0 < j < 2^m$. Then $t_{mj} = t_{n,j2^{n-m}} \in V_n$. Hence we have

$$\varphi(\omega)(t_{n,j2^{n-m}}) = \varphi(\omega)(t_{n-1,j2^{n-1-m}}) = \cdots = \varphi(\omega)(t_{mj}).$$

This shows that $\varphi(\omega)$ is a well defined function on V_r^s. This completes the proof of the theorem. □

Theorem 2.2. *Assume that* $P_{r,a}^{s,b} \neq \emptyset$, *where* $0 \leq r < s$. *For any* $\omega \in \Omega_r^s$, *the function*

$$\varphi_{r,a}^{s,b}(\omega) : V_r^s \to \underline{R}^{V_r^s}$$

satisfies

$$\left|\varphi_{r,a}^{s,b}(\omega)(v) - \varphi_{r,a}^{s,b}(\omega)(u)\right| \leq c|v-u|, \tag{1}$$

for all $u, v \in V_r^s$.

Proof. Fix $\omega \in \Omega_r^s$, and let $x = \varphi_{r,a}^{s,b}(\omega) = \varphi(\omega)$. Since $V_r^s = \bigcup_{n=1}^{\infty} V_{r,n}^s$, we prove (1) by induction on n. For $n = 1$, $V_{r,1}^s = t_{11}$, and hence (1) holds for $n = 1$. Now assume that (1) holds for $u, v \in V_{r,n}^s$, where $n \geq 1$. Then we want to prove (1) for $u, v \in V_{r,n+1}^s$. To this end, let $u = t_{n+1,p}$ and $v = t_{n+1,q}$ be arbitrary members of $V_{r,n+1}^s$, where $0 < p < q < 2^{n+1}$. Then p has the form

$$p = \begin{cases} 2j, & 1 \leq j < 2^n; \\ 2j-1, & 1 \leq j \leq 2^n, \end{cases}$$

and q has the form

$$q = \begin{cases} 2k, & 1 \leq k < 2^n; \\ 2k-1, & 1 \leq k \leq 2^n. \end{cases}$$

Therefore we must consider the following cases.

(a) $p = 2j$ and $q = 2k$, where $0 < j, k < 2^n$, and $j \leq k$.

(b) $p = 2j$ and $q = 2k-1$, where $0 < j, k < 2^n$, $0 < k \leq 2^n$, and $2j \leq 2k-1$.

(c) $p = 2j-1$ and $q = 2k$, where $0 < j \leq 2^n$, $0 < k < 2^n$, and $2j-1 \leq 2k$.

(d) $p = 2j-1$ and $q = 2k-1$, $0 < j \leq 2^n$, $0 < k \leq 2^n$, and $2j-1 \leq 2k-1$.

Assume that (a) holds. We have $u = t_{n+1,p} = t_{n+1,2j} = t_{nj}$ and $v = t_{n+1,q} = t_{n+1,2k} = t_{nk}$, and $u, v \in V_{r,n}^s$, therefore (1) holds by the induction hypothesis.

Now suppose that (b) holds. Then because $1 \leq k \leq 2^n$, Theorem 2.1(a) implies that

$$P_{t_{n,k-1},a_{n,k-1}}^{t_{n+1,2k-1},a_{n+1,2k-1}} \neq \emptyset.$$

Hence by Proposition 2.1, we have

$$|x(v) - x(t_{n,k-1})| \leq c(v - t_{n,k-1}). \tag{2}$$

Condition (b) implies that $j \leq k-1$. If $j = k-1$, then by (2), we have

$$\begin{aligned} |x(v) - x(u)| &= |x(v) - x(t_{n,k-1})| \\ &\leq c(v - t_{n,k-1}) \\ &= c|v - u|. \end{aligned}$$

On the other hand, suppose that $j < k-1$. By the induction hypothesis, we have

$$|x(t_{nm}) - x(t_{n,m-1})| \leq c(t_{nm} - t_{n,m-1}),$$

for $j \leq m \leq k-1$. Therefore, (2) implies that

$$\begin{aligned} |x(v) - x(u)| &= |[x(v) - x(t_{n,k-1})] + \cdots + [x(t_{n,j+1}) - x(t_{nj})]| \\ &\leq |x(v) - x(t_{n,k-1})| + \cdots + |x(t_{n,j+1}) - x(t_{nj})| \\ &\leq c(v - t_{n,k-1}) + \cdots + c(t_{n,j+1} - t_{nj}) \\ &= c(v - u). \end{aligned}$$

Hence, (b) implies (1).

Similar arguments show that (1) holds if either (c) or (d) is true. Hence, (1) holds for all $u, v \in V_{r,n+1}^s$. It follows by induction that (1) is true for all $u, v \in V_r^s$. □

Corollary 2.1. *Under the hypothesis of Theorem 2.2, for any $\omega \in \Omega_r^s$, we have*

$$\left|\varphi_{r,a}^{s,b}(\omega)(v) - \varphi_{r,a}^{s,b}(\omega)(u)\right| \leq c|v - u|, \tag{1}$$

for all $u, v \in V_r^s \cup \{r, s\}$.

Proof. Set $\varphi(\omega) = \varphi_{r,a}^{s,b}(\omega)$. Let $u, v \in V_r^s \cup \{r, s\}$. We consider the following cases.

(a) $u = r$, $v = s$.

(b) $u, v \in V_r^s$.

(c) $u = r$, $v \in V_r^s$.

(*d*) $u \in V_r^s$, $v = s$.

Assume that (*a*) holds. Because $P_{r,a}^{s,b} \neq \emptyset$, we have $|b-a| \leq c(s-r)$, i.e.,

$$|\varphi(\omega)(s) - \varphi(\omega)(r)| \leq c|v-u|,$$

which is (1) for $u = r$, $v = s$. If (*b*) holds, then (1) follows from Theorem 2.2. Suppose that (*c*) holds. Let $v \in V_{r,n}^s$, say $v = t_{nj}$, $0 < j < 2^n$. Then $u = r = t_{n,0}$. By Theorem 2.1(c), we have $P_{t_{n,0},a_{n,0}}^{t_{n,1},a_{n,1}} \neq \emptyset$, hence Proposition 2.1 implies that

$$|\varphi(\omega)(t_{n,1}) - \varphi(\omega)(t_{n,0})| \leq c(t_{n,1} - t_{n,0}),$$

that is,

$$|\varphi(\omega)(t_{n,1}) - \varphi(\omega)(r)| \leq c(t_{n,1} - r). \tag{2}$$

Then by Theorem 2.2 and (2), we get

$$\begin{aligned}
|\varphi(\omega)(v) - \varphi(\omega)(u)| &= |\varphi(\omega)(t_{nj}) - \varphi(\omega)(r)| \\
&\leq |\varphi(\omega)(t_{nj}) - \varphi(\omega)(t_{n,j-1})| + \cdots + |\varphi(\omega)(t_{n,1}) - \varphi(\omega)(r)| \\
&\leq c(t_{nj} - t_{n,j-1}) + \cdots + c(t_{n,1} - r) \\
&= c(t_{nj} - r) \\
&= c|v-u|,
\end{aligned}$$

which is (1) for the case where $u = r$, $v \in V_r^s$. The proof of (1) for the case (*d*) is similar to the proof of (1) for the case (*c*). □

Definition 2.5. For $P_{r,a}^{s,b} \neq \emptyset$, define

$$\begin{aligned}
C_r^s &= \{\, x : [r,s] \to \underline{\mathrm{R}} \,|\, x \quad \text{is continuous on} \quad [\, r, s\,] \,\}, \\
C_{r,a}^{s,b} &= \{\, x \in C_r^s \,|\, x(r) = a, x(s) = b \,\}, \\
\Lambda_r^s &= \{\, x \in C_r^s \,|\, |x(v) - x(u)| \leq c|v-u|, \quad \text{for all} \quad u, v \in [\, r, s\,] \,\}, \\
\Lambda_{r,a}^{s,b} &= C_{r,a}^{s,b} \cap \Lambda_r^s, \\
L_r^s &= \{\, (a,b) \in \underline{\mathrm{R}}^2 \,|\, |b-a| \leq c(s-r) \,\}.
\end{aligned}$$

Proposition 2.4. *Assume that* $P_{r,a}^{s,b} \neq \emptyset$. *Then for any* $\omega \in \Omega_r^s$, *the function*

$$\varphi_{r,a}^{s,b}(\omega) : V_r^s \to \underline{\mathrm{R}}^{V_r^s}$$

can be uniquely extended to a function $\varphi_{r,a}^{s,b}(\omega) \in \Lambda_{r,a}^{s,b}$.

Proof. Fix $\omega \in \Omega$, and set $\varphi_{r,a}^{s,b}(\omega) = \varphi(\omega)$. According to Corollary 2.1, we have

$$|\varphi(\omega)(v) - \varphi(\omega)(u)| \leq c|v-u|, \tag{1}$$

for all $u, v \in V_r^s \cup \{r, s\}$. To define $\varphi(\omega)$ on all of $[r, s]$, let $u \in [r, s]$ be arbitrary. The set $V_r^s \cup \{r, s\}$ is dense in $[r, s]$, hence there exists a sequence (u_n) in $[r, s]$ such that $\lim_{n\to\infty} u_n = u$. Then (1) implies that the sequence $\big(\varphi(\omega)(u_n)\big)$ is a Cauchy sequence in $\underline{\mathrm{R}}$, therefore we may define

$$\varphi(\omega)(u) = \lim_{n\to\infty} \varphi(\omega)(u_n).$$

It is easy to see that (1) then implies that

$$|\varphi(\omega)(v) - \varphi(\omega)(u)| \leq c|v-u|,$$

for all $u, v \in [r, s]$. Thus, the extended function $\varphi(\omega) : V_r^s \to \underline{\mathrm{R}}^{V_r^s}$ is in $\Lambda_{r,a}^{s,b}$. It is clear from the definition of $\varphi(\omega)$ on $[r, s]$ that this extension is unique. $\square$

Lemma 2.1. *Assume that* $P_{r,a}^{s,b} \neq \emptyset$. *Let* $u = \frac{1}{2}(r+s)$. *Assume that* $d \in \underline{\mathrm{R}}$ *satisfies the following conditions.*

$$\begin{aligned} |d-a| &\leq c(u-r); \\ |d-b| &\leq c(s-u). \end{aligned} \tag{1}$$

Then $d \in I_{r,a}^{s,b}$.

Proof. We give a proof for the case where $a \leq b$. The proof for the case $a \geq b$ is similar. By Proposition 2.2,

$$I_{r,a}^{s,b} = [\, b - \frac{1}{2}c(s-r), a + \frac{1}{2}c(s-r)\,]. \tag{2}$$

Hence (1) implies that the following conditions hold.

$$\begin{aligned} a - \frac{c}{2}(s-r) \leq d \leq a + \frac{c}{2}(s-r); \\ b - \frac{c}{2}(s-r) \leq d \leq b + \frac{c}{2}(s-r). \end{aligned} \tag{3}$$

Then by (2) and (3), we get that $d \in I_{r,a}^{s,b}$. $\square$

Theorem 2.3. *For* $P_{r,a}^{s,b} \neq \emptyset$, *the function*

$$\varphi_{r,a}^{s,b} : \Omega_r^s \to \Lambda_{r,a}^{s,b}$$

is onto.

Proof. Let $x \in \Lambda_{r,a}^{s,b}$. For $n \geq 0$, define $a_{nj} = x(t_{nj})$, where $0 \leq j \leq 2^n$. Because $x \in \Lambda_{r,a}^{s,b}$, for any $n \geq 0$, and for any $1 \leq j \leq 2^n$, we have

$$\begin{aligned} |a_{nj} - a_{n,j-1}| &= |x(t_{nj}) - x(t_{n,j-1})| \\ &\leq c(t_{nj} - t_{n,j-1}). \end{aligned}$$

Therefore, for any $n \geq 0$, and for any $1 \leq j \leq 2^n$, Proposition 2.1 implies that

$$P_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}} \neq \emptyset. \tag{1}$$

Condition (1) and Definition 2.3 then imply that for any $n \geq 0$, and for any $1 \leq j \leq 2^n$, we are given the surjective function

$$\lambda_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}} : [\,0,1\,] \to I_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}}.$$

We will construct by induction an $\omega \in \Omega_r^s$ such that for all $n \geq 1$, the following equation holds:

$$x(t_{n,2j-1}) = \lambda_{t_{n-1,j-1},a_{n-1,j-1}}^{t_{n-1,j},a_{n-1,j}}(\omega_{t_{n,2j-1}}), \quad 1 \leq j \leq 2^{n-1}. \tag{2}$$

To prove (2) for $n = 1$, note first that because $P_{r,a}^{s,b} \neq \emptyset$, Definition 2.3 gives the surjective function

$$\lambda_{r,a}^{s,b} : [\,0,1\,] \to I_{r,a}^{s,b}.$$

Because $x \in \Lambda_{r,a}^{s,b}$ and $t_{11} = \dfrac{1}{2}(s + r)$, we have

$$\begin{aligned} |x(t_{11}) - a| &= |x(t_{11}) - x(r)| \leq c(t_{11} - r); \\ |b - x(t_{11})| &= |x(s) - x(t_{11})| \leq c(s - t_{11}). \end{aligned} \tag{3}$$

Lemma 2.1 and (3) then imply that $x(t_{11}) \in I_{r,a}^{s,b}$. Hence there exists $\omega_{t_{11}} \in [\,0,1\,]$ such that

$$\begin{aligned} x(t_{11}) &= \lambda_{r,a}^{s,b}(\omega_{t_{11}}) \\ &= \lambda_{t_{00},a_{00}}^{t_{01},a_{01}}(\omega_{t_{11}}). \end{aligned} \tag{4}$$

Statement (4) gives that (2) holds for $n = 1$. Therefore, as the inductive hypothesis, we assume the $n \geq 1$ is given, and that the numbers $\omega_{t_{nj}}$, $0 \leq j \leq 2^n$ have been constructed in such a way that (2) holds for n. We then want to prove (2) for the case where n is replaced by $n+1$, that is, we want to construct numbers $\omega_{t_{n+1,k}}$, $0 \le k \le 2^{n+1}$ in $[0,1]$ such that the following statement holds:

$$x(t_{n+1,2j-1}) = \lambda^{t_{nj},a_{nj}}_{t_{n,j-1},a_{n,j-1}}(\omega_{t_{n+1,2j-1}}), \quad 1 \le j \le 2^n. \tag{5}$$

To construct $\omega_{t_{n+1,k}}$, $0 \le k \le 2^{n+1}$, let $1 \le k \le 2^{n+1}$ be arbitrary. Then k has one of the following forms.

$$k = \begin{cases} 2j, & 1 \le j \le 2^n; \\ 2j-1, & 1 \le j \le 2^n. \end{cases}$$

If $k = 2j$, $1 \le j \le 2^n$, define

$$\omega_{t_{n+1,k}} = \omega_{t_{nj}}.$$

Assume that $k = 2j-1$, $1 \le j \le 2^n$. Because $x \in \Lambda^{s,b}_{r,a}$ and

$$t_{n+1,2j-1} = \frac{1}{2}(t_{n,j-1} + t_{nj}),$$

we have

$$\begin{aligned} |x(t_{n+1,2j-1}) - a_{n,j-1}| &= |x(t_{n+1,2j-1}) - x(t_{n,j-1})| \le c(t_{n+1,2j-1} - t_{n,j-1}); \\ |a_{nj} - x(t_{n+1,2j-1})| &= |x(t_{nj}) - x(t_{n+1,2j-1})| \le c(t_{nj} - t_{n+1,2j-1}). \end{aligned} \tag{6}$$

It follows from (6) and Lemma 2.1 that

$$x(t_{n+1,2j-1}) \in I^{t_{nj},a_{nj}}_{t_{n,j-1},a_{n,j-1}}.$$

Consequently, there exists $\omega_{t_{n+1,2j-1}} \in [0,1]$ such that

$$x(t_{n+1,2j-1}) = \lambda^{t_{nj},a_{nj}}_{t_{n,j-1},a_{n,j-1}}(\omega_{t_{n+1,2j-1}}). \tag{7}$$

This defines $\omega_{t_{n+1,k}} = \omega_{t_{n+1,2j-1}}$. It follows from (7) that the numbers $\omega_{t_{n+1,k}}$, $0 \le k \le 2^{n+1}$ satisfy (5). This completes the inductive construction of $\omega \in \Omega^s_r$ such that (2) holds for all $n \ge 1$. We claim that for $n \ge 1$,

$$x(t_{nj}) = \varphi^{s,b}_{r,a}(\omega)(t_{nj}), \quad 1 \le j < 2^n. \tag{8}$$

We prove (8) by induction. To this end, write $\varphi(\omega) = \varphi^{s,b}_{r,a}(\omega)$. For $n \ge 0$, define

$$b_{nj} = \varphi(\omega)(t_{nj}), \quad 0 \le j \le 2^n.$$

According to Theorem 2.1(b), for $n \geq 1$, we have

$$\varphi(\omega)(t_{n,2j-1}) = \lambda_{t_{n-1,j-1},b_{n-1,j-1}}^{t_{n-1,j},b_{n-1,j}}(\omega)(t_{n,2j-1}), \quad 1 \leq j \leq 2^{n-1}. \tag{9}$$

A simple computation shows that

$$x(t_{11}) = \varphi(\omega)(t_{11}) = \lambda_{r,a}^{s,b}(\omega_{t_{11}}),$$

and hence (8) holds for $n = 1$. Now assume that (8) holds for $n \geq 1$. We want to then prove that (8) holds when n is replaced by $n+1$, i.e.,

$$x(t_{n+1,j}) = \varphi(\omega)(t_{n+1,j}), \quad 1 \leq j < 2^{n+1}. \tag{10}$$

Let $1 \leq k < 2^{n+1}$. Then k has one of the following forms:

$$k = \begin{cases} 2j, & 1 \leq j < 2^n; \\ 2j-1, & 1 \leq j \leq 2^n. \end{cases}$$

If $k = 2j$, $1 \leq j < 2^n$, then by the inductive hypothesis, (8) holds, and hence

$$\begin{aligned} x(t_{n+1,k}) = x(t_{n+1,2j}) = x(t_{nj}) = \varphi(\omega)(t_{nj}) &= \varphi(\omega)(t_{n+1,2j}) \\ &= \varphi(\omega)(t_{n+1,k}). \end{aligned} \tag{11}$$

On the other hand, suppose that $k = 2j-1$, $1 \leq j \leq 2^n$. By the induction hypothesis, (8) holds, hence we have

$$\begin{aligned} a_{n,j-1} &= x(t_{n,j-1}) = \varphi(\omega)(t_{n,j-1}) = b_{n,j-1}, \\ a_{nj} &= x(t_{nj}) = \varphi(\omega)(t_{nj}) = b_{nj}. \end{aligned} \tag{12}$$

By (2), (9), and (12), we get

$$\begin{aligned} x(t_{n+1,k}) &= x(t_{n+1,2j-1}) = \lambda_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}}(\omega_{t_{n+1,2j-1}}) \\ &= \lambda_{t_{n,j-1},b_{n,j-1}}^{t_{nj},b_{nj}}(\omega_{t_{n+1,2j-1}}) \\ &= \varphi(\omega)(t_{n+1,2j-1}) \\ &= \varphi(\omega)(t_{n+1,k}). \end{aligned} \tag{13}$$

Statements (11) and (13) prove (10). Hence, by induction, the claim (8) holds for all $n \geq 1$. Statement (8) is equivalent to the following:

$$x(t) = \varphi(\omega)(t), \quad t \in V_r^s.$$

Because $x, \varphi(\omega) \in \Lambda_{r,a}^{s,b}$, we see that (8) implies

$$x(t) = \varphi(\omega)(t), \quad t \in \{ r,\ s\} \cup V_r^s.$$

The set $\{ r, s \} \cup V_r^s$ is dense in $[r, s]$, therefore we have

$$x(t) = \varphi(\omega)(t), \quad t \in [r, s],$$

i.e.,

$$\varphi(\omega) = x.$$

It follows that the function

$$\varphi_{r,a}^{s,b} : \Omega_r^s \to \Lambda_{r,a}^{s,b}$$

is onto. □

Lemma 2.2. *Let U be any open subset of C_r^s such that*

$$\Lambda_r^s \cap U \neq \emptyset.$$

Let $x_0 \in \Lambda_r^s \cap U$. Then there exists a $\delta > 0$ such that if

$$\tau : \quad r = t_0 < t_1 < \cdots < t_{n-1} < t_n = s$$

is any partition of $[r, s]$ with

$$\max_{1 \leq j \leq n} \triangle t_j < \delta, \tag{1}$$

then there exist open intervals I_{t_j}, $0 \leq j \leq n$, such that

$$x_0 \in \Lambda_r^s \cap U_\tau \subseteq U, \tag{2}$$

where

$$U_\tau = \{ x \in C_r^s \,|\, x(t_j) \in I_{t_j}, \quad 0 \leq j \leq n \}.$$

Proof. Because U is open in C_r^s and $x_0 \in U$, there exists an $\epsilon > 0$ such that if $x \in C_r^s$ with $||x - x_0|| < \epsilon$, then $x \in U$. Define $\delta = \dfrac{1}{6c}\epsilon$, and assume that

$$\tau : \quad r = t_0 < t_1 < \cdots < t_{n-1} < t_n = s$$

is a partition of $[r, s]$ satisfying condition (1). Define

$$I_{t_j} = \left(x_0(t_j) - \frac{1}{2}\epsilon, x_0(t_j) + \frac{1}{2}\epsilon \right), \quad 0 \leq j \leq n.$$

Let $x \in \Lambda_r^s \cap U_\tau$. Let $t \in [\,r, s\,]$ be arbitrary, say $t_{j-1} \le t \le t_j$, for some $1 \le j \le n$. Because $x \in U_\tau$, we have $|x(t_{j-1}) - x_0(t_{j-1})| < \frac{1}{2}\epsilon$, and consequently (1) implies that

$$\begin{aligned}|x(t) - x_0(t)| &\le |x(t_j) - x(t)| + |x(t_j) - x(t_{j-1})| + |x(t_{j-1}) - x_0(t_{j-1})| \\ &\qquad + |x_0(t) - x_0(t_{j-1})| \\ &\le c(t_j - t) + c(t_j - t_{j-1}) + \frac{1}{2}\epsilon + c(t - t_{j-1}) \\ &\le 3c \triangle t_j + \frac{1}{2}\epsilon \\ &\le 3c \max_{1\le i\le n} \triangle t_j + \frac{1}{2}\epsilon \\ &< 3c\delta + \frac{1}{2}\epsilon = \epsilon.\end{aligned}$$

This shows that $||x - x_0|| < \epsilon$, i.e., $x \in U$. Therefore (2) holds. □

Definition 2.6. *Let $n \ge 1$ be an arbitrary positive integer, and define*

$$\Omega_{r,n}^s = [0,1]^{V_{r,n}^s}.$$

Let $\omega \in \Omega_r^s$ be arbitrary. For $1 \le j < 2^n$, define

$$\omega_{t_{nj}} = (\omega)_{t_{nj}}.$$

Define the function $\pi_n : \Omega_r^s \to \Omega_{r,n}^s$ by

$$\pi_n(\omega) = (\omega_{t_{n1}}, \ldots, \omega_{t_{n,2^n-1}}).$$

Theorem 2.4. *Let $0 \le r < s$ be fixed real numbers. For each $n \ge 1$ and for all $0 \le j \le 2^n$, there exists continuous functions*

$$\theta_{nj} : L_r^s \times \Omega_{r,n}^s \to \underline{R},$$

such that for any $(a, b, \omega) \in L_r^s \times \Omega_r^s$,

$$\varphi_{r,a}^{s,b}(\omega)(t_{nj}) = \theta_{nj}(a, b, \pi_n(\omega)), \quad 0 \le j \le 2^n. \tag{1}$$

Proof. Write $\Omega_r^s = \Omega$, $L_r^s = L$ and $\Omega_{r,n}^s = \Omega_n$. We will prove (1) by induction on $n \geq 1$. To prove (1) for $n = 1$, let $(a, b, \omega_{t_{11}}) \in L \times \Omega_1$ be arbitrary, and define

$$\theta_{1j} : L \times \Omega_1 \to \underline{\mathrm{R}}, \quad 0 \leq j \leq 2$$

by

$$\begin{aligned} \theta_{10}(a, b, \omega_{t_{11}}) &= a, \\ \theta_{11}(a, b, \omega_{t_{11}}) &= \lambda_{r,a}^{s,b}(\omega_{t_{11}}), \\ \theta_{12}(a, b, \omega_{t_{11}}) &= b. \end{aligned} \tag{2}$$

Then for any $(a, b, \omega) \in L \times \Omega$, Theorem 2.1(b) implies that

$$\varphi_{r,a}^{s,b}(\omega)(t_{11}) = \lambda_{r,a}^{s,b}(\omega_{t_{11}}) = \theta_{11}(a, b, \omega_{t_{11}}) = \theta_{11}(a, b, \pi_1(\omega)). \tag{3}$$

Also, by definition, we have

$$\begin{aligned} \varphi(\omega)(t_{10}) &= a = \theta_{10}(a, b, \omega_{t_{11}}) = \theta_{10}(a, b, \pi_1(\omega)), \\ \varphi(\omega)(t_{12}) &= b = \theta_{12}(a, b, \omega_{t_{11}}) = \theta_{12}(a, b, \pi_1(\omega)). \end{aligned} \tag{4}$$

The functions θ_{10} and θ_{12} are clearly continuous on $L \times \Omega_1$. It follows from Definition 2.3 that the function

$$(a, b, \xi) \mapsto \lambda_{r,a}^{s,b}(\xi)$$

is a continuous function on $L \times \Omega_1$. Hence (3) and (4) together give (1) for the case $n = 1$. Now assume that $n \geq 1$ is given and that there exist continuous functions

$$\theta_{nj} : L \times \Omega_n \to \underline{\mathrm{R}}, \quad 0 \leq j \leq 2^n,$$

such that (1) holds. We then want to construct continuous functions

$$\theta_{n+1,k} : L \times \Omega_{n+1} \to \underline{\mathrm{R}}, \quad 0 \leq j \leq 2^{n+1},$$

such that for all $(a, b, \omega) \in L \times \Omega$, the following statement holds:

$$\varphi_{r,a}^{s,b}(\omega)(t_{n+1,k}) = \theta_{n+1,k}(a, b, \pi_{n+1}(\omega)), \quad 0 \leq k \leq 2^{n+1}. \tag{5}$$

To this end, let $(a, b, \omega_{t_{n+1,1}}, \ldots, \omega_{t_{n+1,2^{n+1}-1}}) \in L \times \Omega_{n+1}$, and let $0 \leq k \leq 2^{n+1}$. Then k has one of the following forms:

$$k = \begin{cases} 2j, & 0 \leq j \leq 2^n; \\ 2j - 1, & 1 \leq j \leq 2^n. \end{cases}$$

If $k = 2j$, $0 \le j \le 2^n$, then define

$$\theta_{n+1,k}(a, b, \omega_{t_{n+1,1}}, \dots, \omega_{t_{n+1,2^{n+1}i-1}}) = \theta_{nj}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}). \quad (6)$$

On the other hand, assume that $k = 2j - 1$, $1 \le j \le 2^n$, and let $\omega^{n+1} \in \Omega$ be any member of Ω such that

$$\pi_{n+1}(\omega^{n+1}) = (\omega_{t_{n+1,1}}, \dots, \omega_{t_{n+1,2^{n+1}-1}}).$$

Then we have

$$\pi_n(\omega^{n+1}) = (\omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}).$$

By the induction hypothesis, (1) holds for n, hence we have

$$\theta_{nj}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}) = \varphi_{r,a}^{s,b}(\omega^{n+1})(t_{nj}),$$
$$\theta_{n,j-1}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}) = \varphi_{r,a}^{s,b}(\omega^{n+1})(t_{n,j-1}).$$

Therefore, we have

$$\begin{aligned} |\theta_{nj}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}) - \theta_{n,j-1}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}})| \\ = |\varphi_{r,a}^{s,b}(\omega^{n+1})(t_{nj}) - \varphi_{r,a}^{s,b}(\omega^{n+1})(t_{n,j-1})| \\ \le c(t_{nj} - t_{n,j-1}). \end{aligned}$$

Consequently, we have

$$\big(\, \theta_{n,j-1}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}), \theta_{nj}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}), t_{n,j-1}, t_{nj}, \omega_{t_{2j-1}} \,\big) \in D_\lambda. \quad (7)$$

Therefore, by Definition 2.3, if we write

$$\begin{aligned} b_{n,j-1} &= \theta_{n,j-1}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}), \quad (8) \\ b_{nj} &= \theta_{nj}(a, b, \omega_{t_{n1}}, \dots, \omega_{t_{n,2^n-1}}), \end{aligned}$$

then (7) implies that we may define

$$\theta_{n+1,k}(a, b, \omega_{t_{n+1,1}}, \dots, \omega_{t_{n+1,2^{n+1}-1}}) = \lambda_{t_{n,j-1},b_{n,j-1}}^{t_{nj},b_{nj}}(\omega_{t_{n+1,2j-1}}). \quad (9)$$

Definitions (6) and (9) together give the definition of $\theta_{n+1,k}$ on Ω_{n+1}, for $0 \le k \le 2^{n+1}$. By Definition 2.3, the functions

$$(p, q, u, v, \xi) \to \lambda_{u,p}^{v,q}(\xi)$$

are continuous on D_λ, and by the induction hypothesis, the functions θ_{nj}, $\theta_{n,j-1}$ are continuous on $L \times \Omega_n$, hence (6), (7), and (9) together imply that for $0 \leq k \leq 2^{n+1}$, $\theta_{n+1,k}$ is continuous on $L \times \Omega_{n+1}$. To show that (5) holds, let $(a, b, \omega) \in \Omega$ and let $0 \leq k \leq 2^{n+1}$. Set

$$\pi_{n+1}(\omega) = (\omega_{t_{n+1,1}}, \ldots, \omega_{t_{n+1,2^{n+1}-1}}).$$

Then k has one of the following forms:

$$k = \begin{cases} 2j, & 0 \leq j \leq 2^n; \\ 2j-1, & 1 \leq j \leq 2^n. \end{cases}$$

If $k = 2j$, $0 \leq j \leq 2^n$, then by the induction hypothesis and (9) together imply that

$$\begin{aligned} \varphi_{r,a}^{s,b}(\omega)(t_{n+1,k}) &= \varphi_{r,a}^{s,b}(\omega)(t_{n+1,2j}) \\ &= \varphi_{r,a}^{s,b}(\omega)(t_{nj}) \\ &= \theta_{nj}(a, b, \omega_{t_{n1}}, \ldots, \omega_{t_{n,2^n-1}}) \\ &= \theta_{n+1,2j}(a, b, \pi_{n+1}(\omega)) \\ &= \theta_{n+1,k}(a, b, \pi_{n+1}(\omega)). \end{aligned} \tag{10}$$

On the other hand, if $k = 2j-1$, $1 \leq j \leq 2^n$, then by Theorem 2.1(b) and (9) together imply that

$$\begin{aligned} \varphi_{r,a}^{s,b}(\omega)(t_{n+1,k}) &= \varphi_{r,a}^{s,b}(\omega)(t_{n+1,2j-1}) \\ &= \lambda_{t_{n,j-1},a_{n,j-1}}^{t_{nj},a_{nj}}(\omega)(t_{n+1,2j-1}) \\ &= \lambda_{t_{n,j-1},b_{n,j-1}}^{t_{nj},b_{nj}}(\omega)(t_{n+1,2j-1}) \\ &= \theta_{n+1,2j-1}(a, b, \omega_{t_{n+1,1}}, \ldots, \omega_{t_{n+1,2^{n+1}-1}}) \\ &= \theta_{n+1,k}(a, b, \pi_{n+1}(\omega)). \end{aligned} \tag{11}$$

Statements (10) and (11) together prove (5). Therefore the inductive proof of (1) is complete. □

Theorem 2.5. *Fix $0 \leq r < s$. Define the function*

$$\phi_r^s : L_r^s \times \Omega_r^s \to \Lambda_r^s$$

by

$$\phi_r^s(a, b, \omega) = \varphi_{r,a}^{s,b}(\omega), \quad (a, b, \omega) \in L_r^s \times \Omega_r^s.$$

Then ϕ_r^s is continuous. In particular, for any $(a,b) \in L_r^s$, the function

$$\varphi_{r,a}^{s,b} : \Omega_r^s \to \Lambda_{r,a}^{s,b}$$

is continuous.

Proof. Set $\Omega = \Omega_r^s$ and $\phi_r^s = \phi$. For $n \geq 1$ and $0 \leq j \leq n$, set $\Omega_{r,n}^s = \Omega_n$. Let $(a_0, b_0, \omega^0) \in L \times \Omega$ be arbitrary, and define

$$x_0 = \phi(a_0, b_0, \omega^0).$$

Let U be any open set in C_r^s such that $x_0 \in U$. We want to find an open set X in $L \times \Omega$ such that

$$(a_0, b_0, \omega_0) \in X \text{ and } \phi(X) \subseteq U. \tag{1}$$

To prove (1) let $\delta > 0$ (with respect to U) be a positive number given in the hypothesis of Lemma 2.2. Select $n \geq 1$ so large that

$$\frac{1}{2^n}(s-r) < \delta. \tag{2}$$

Let τ_n be the partition of $[\,r,s\,]$ defined by

$$r = t_{n,0}, \ldots, t_{n,2^n} = s.$$

By Lemma 2.2, (2) implies that there exists open intervals

$$I_{t_{n,0}}, \ldots, I_{t_{n,2^n}}$$

such that

$$x_0 \in \Lambda_r^s \cap U_{\tau_n} \subseteq \Lambda_r^s \cap U, \tag{3}$$

where

$$U_{\tau_n} = \{\, x \in C_r^s \,|\, x(t_{nj}) \in I_{t_{nj}}, \quad 0 \leq j \leq 2^n \,\}.$$

By (3), $x_0 \in U_{\tau_n}$, and hence Theorem 2.4 implies that

$$x_0(t_{nj}) = \phi(a_0, b_0, \omega^0)(t_{nj}) = \varphi_{r,a}^{s,b}(\omega^0)(t_{nj}) = \theta_{nj}(a, b, \pi_n(\omega^0)) \in I_{t_{nj}}, \quad 0 \leq j \leq 2^n. \tag{4}$$

By Theorem 2.4, the function $(a,b,\omega) \mapsto \theta_{nj}(a,b,\pi_n(\omega))$ is continuous on $L \times \Omega$, hence (4) implies that for each $0 \leq j \leq 2^n$, there exists an open sets V_{nj} in L and W_{nj} in Ω such that

$$(a_0, b_0, \omega^0) \in V_{nj} \times W_{nj} \quad \text{and} \quad \theta_{nj}(a, b, \pi_n(\omega)) \subseteq I_{t_{nj}}, \qquad (a, b, \omega) \in V_{nj} \times W_{nj}. \tag{5}$$

Define

$$X = \bigcap_{j=0}^{2^n} V_{nj} \times W_{nj}.$$

Then X is open in $L \times \Omega$ and $(a_0, b_0, \omega^0) \in X$. Let $(a, b, \omega) \in X$ and $0 \le j \le 2^n$. Then (5) implies that

$$\phi(a, b, \omega)(t_{nj}) = \varphi^{s,b}_{r,s}(\omega)(t_{nj}) = \theta_{nj}(a, b, \pi_n(\omega)) \in I_{t_{nj}}.$$

Consequently, because $0 \le j \le 2^n$ is arbitrary, (3) implies that

$$\phi(a, b, \omega) = \varphi^{s,b}_{r,s}(\omega) \in \Lambda^{s,b}_{r,a} \cap U_{\tau_n} \subseteq \Lambda^s_r \cap U_{\tau_n} \subseteq U.$$

Because $\omega \in W$ is arbitrary, we see that statement (1) holds. Therefore ϕ is continuous on $L \times \Omega$. □

Definition 2.7. Fix $0 \le r < s$. For $n \ge 1$, define

$$\begin{aligned}
\hat{V}{}^s_{r,n} &= V^s_{r,n} \cup \{s\},\\
\overset{\vee}{V}{}^s_{r,n} &= V^s_{r,n} \cup \{r\},\\
\hat{V}{}^s_{r} &= \bigcup_{n=1}^{\infty} \hat{V}{}^s_{r,n} = V^s_r \cup \{ s \},\\
\overset{\vee}{V}{}^s_{r} &= \bigcup_{n=1}^{\infty} \overset{\vee}{V}{}^s_{r,n} = V^s_r \cup \{ r \},\\
\overset{\wedge}{\Omega}{}^s_r &= [0,1]^{\hat{V}{}^s_r},\\
\overset{\vee}{\Omega}{}^s_r &= [0,1]^{\overset{\vee}{V}{}^s_r}.
\end{aligned}$$

Now define the functions

$$\overset{\wedge}{\pi}\colon \overset{\wedge}{\Omega}{}^s_r \to \Omega^s_r,$$

$$\overset{\vee}{\pi}\colon \overset{\vee}{\Omega}{}^s_r \to \Omega^s_r$$

as follows. For $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}{}^s_r$, define $\omega = \overset{\wedge}{\pi}(\overset{\wedge}{\omega}) \in \Omega^s_r$ by

$$\omega_t = \overset{\wedge}{\omega}_t,$$

where $t \in V_r^s$. For $\overset{\vee}{\omega} \in \overset{\vee}{\Omega}{}_r^s$, define $\omega = \overset{\vee}{\pi}(\overset{\vee}{\omega}) \in \Omega_r^s$ by

$$\omega_t = \overset{\vee}{\omega}_t,$$

where $t \in V_r^s$. Finally, for arbitrary $a, b \in \underline{\mathrm{R}}$, define

$$\Lambda_{r,a}^s = \{\, x \in \Lambda_r^s \,|\, x(r) = a \,\},$$
$$\Lambda_a^{s,b} = \{\, x \in \Lambda_r^s \,|\, x(s) = b \,\}.$$

Definition 2.8. Let $0 \le r < s$ be arbitrary. For arbitrary $a \in \underline{\mathrm{R}}$ define

$$I_{r,a}^s = [\, a - c(s-r), a + c(s-r) \,].$$

We shall assume that for every $a \in \underline{\mathrm{R}}$, we are given a continuous function

$$\lambda_{r,a}^s : [0,1] \to I_{r,a}^s$$

onto $I_{r,a}^s$. Moreover, we will assume that the mapping

$$(a, r, s, \xi) \to \lambda_{r,a}^s(\xi)$$

is continuous on the set

$$\{\, (a, r, s, \xi) \,|\, a \in \underline{\mathrm{R}}, 0 \le r < s, \xi \in [0,1] \,\}.$$

For $a \in \underline{\mathrm{R}}$, define

$$\varphi_{r,a}^s : \overset{\wedge}{\Omega}{}_r^s \to \Lambda_{r,a}^s$$

by

$$\varphi_{r,a}^s(\overset{\wedge}{\omega}) = \varphi_{r,a}^{s,b}(\overset{\wedge}{\pi}(\overset{\wedge}{\omega})),$$

where $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}{}_r^s$, and $b = \lambda_{r,a}^s(\overset{\wedge}{\omega}_s)$. For $b \in \underline{\mathrm{R}}$, define

$$\varphi_r^{s,b} : \overset{\vee}{\Omega}{}_r^s \to \Lambda_r^{s,b}$$

by

$$\varphi_r^{s,b}(\overset{\vee}{\omega}) = \varphi_{r,a}^{s,b}(\overset{\vee}{\pi}(\overset{\vee}{\omega})),$$

where $\overset{\vee}{\omega} \in \overset{\vee}{\Omega}{}_r^s$, and $a = \lambda_{r,b}^s(\overset{\vee}{\omega}_r)$. For $a \in \underline{\mathrm{R}}$ and $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}{}_r^s$, if $b = \lambda_{r,a}^s(\overset{\wedge}{\omega}_s)$, then $b \in I_{r,a}^s$, and hence

$$a - c(s-r) \le b \le a + c(s-r),$$

that is

$$|b-a| \le c(s-r).$$

It follows from Proposition 2.1 that $P^{s,b}_{r,a} \neq \emptyset$. Hence by Proposition 2.4, we see that

$$\varphi^s_{r,a}(\overset{\wedge}{\omega}) = \varphi^{s,b}_{r,a}(\overset{\wedge}{\pi}(\overset{\wedge}{\omega})) \in \Lambda^{s,b}_{r,a} \subseteq \Lambda^s_{r,a}.$$

A similar argument shows that for $b \in \underline{\mathrm{R}}$ and $\overset{\vee}{\omega} \in \overset{\vee}{\Omega}{}^s_r$, if $a = \lambda^s_{r,b}(\overset{\wedge}{\omega}_s)$, then

$$\varphi^{s,b}_r(\overset{\vee}{\omega}) = \varphi^{s,b}_{r,a}(\overset{\vee}{\pi}(\overset{\vee}{\omega})) \in \Lambda^{s,b}_{r,a} \subseteq \Lambda^{s,b}_r.$$

Consequently, for $a, b \in \underline{\mathrm{R}}$, we have

$$\varphi^s_{r,a} : \overset{\wedge}{\Omega}{}^s_r \to \Lambda^s_{r,a},$$
$$\varphi^{s,b}_r : \overset{\vee}{\Omega}{}^s_r \to \Lambda^{s,b}_r.$$

Theorem 2.6. *Let $0 \le r < s$ be arbitrary. Then for any $a, b \in \underline{\mathrm{R}}$, the functions*

$$\varphi^s_{r,a} : \overset{\wedge}{\Omega}{}^s_r \to \Lambda^s_{r,a},$$
$$\varphi^{s,b}_r : \overset{\vee}{\Omega}{}^s_r \to \Lambda^{s,b}_r$$

are onto.

Proof. Fix $a, b \in \underline{\mathrm{R}}$. Let $x \in \Lambda^s_{r,a}$, and set $d = x(s)$. Then

$$|d-a| = |x(s) - x(a)| \le c(s-r),$$

hence we have $|d-a| \le c(s-r)$, i.e.,

$$a - c(s-r) \le d \le a + c(s-r).$$

It follows that $d \in I^s_{r,a}$. By Definition 2.8, the function $\lambda^s_{r,a} : [0,1] \to I^s_{r,a}$ is onto, hence there exists $\overset{\wedge}{\omega}_s \in [0,1]$ such that $d = \lambda^s_{r,a}(\overset{\wedge}{\omega}_s)$. Because $x \in \Lambda^s_{r,a}$, we have $x \in \Lambda^{s,d}_{r,a}$, therefore Theorem 2.3 implies that there exists $\omega \in \Omega^s_r$ such that $x = \varphi^{s,d}_{r,a}(\omega)$. Define $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}{}^s_r$ by

$$\overset{\wedge}{\omega}_t = \begin{cases} \omega_t, & \text{if } t \in V^s_r; \\ \overset{\wedge}{\omega}_s, & \text{if } t = s. \end{cases}$$

Then $\hat{\pi}(\hat{\omega}) = \omega$, and hence

$$x = \varphi_{r,a}^{s,d}(\omega) = \varphi_{r,a}^{s,d}(\hat{\pi}(\hat{\omega})) = \varphi_{r,a}^{s}(\hat{\omega}).$$

Because $x \in \Lambda_{r,a}^s$ is arbitrary, this proves that $\varphi_{r,a}^s$ is onto. A similar proof shows that $\varphi_r^{s,b}$ is onto. ☐

Theorem 2.7. *Let $0 \leq r < s$ be arbitrary. Define*

$$\begin{aligned}
\hat{\phi}_r^s &: \underline{R} \times \hat{\Omega}_r^s \to \Lambda_r^s, \\
\check{\phi}_r^s &: \underline{R} \times \check{\Omega}_r^s \to \Lambda_r^s
\end{aligned}$$

by

$$\begin{aligned}
\hat{\phi}_r^s(a, \hat{\omega}) &= \varphi_{r,a}^s(\hat{\omega}), \quad (a, \hat{\omega}) \in \underline{R} \times \hat{\Omega}_r^s, \\
\check{\phi}_r^s(b, \check{\omega}) &= \varphi_r^{s,b}(\check{\omega}), \quad (b, \check{\omega}) \in \underline{R} \times \check{\Omega}_r^s.
\end{aligned}$$

Then $\hat{\phi}_r^s$ and $\check{\phi}_r^s$ are continuous. In particular, for $a, b \in \underline{R}$ arbitrary, the functions

$$\begin{aligned}
\varphi_{r,a}^s &: \hat{\Omega}_r^s \to \Lambda_{r,a}^s, \\
\varphi_r^{s,b} &: \check{\Omega}_r^s \to \Lambda_r^{s,b}
\end{aligned}$$

are continuous.

Proof. For $(a, \hat{\omega}) \in \underline{R} \times \hat{\Omega}_r^s$, and define $b = \lambda_{r,a}^s(\hat{\omega}_s)$. Then

$$\begin{aligned}
\hat{\phi}_r^s(a, \hat{\omega}) &= \varphi_{r,a}^s(\hat{\omega}) \qquad (1) \\
&= \varphi_{r,a}^{s,b}(\hat{\pi}(\hat{\omega})) \\
&= \phi_r^s(a, b, \hat{\pi}(\hat{\omega})) \\
&= \phi_r^s(a, \lambda_{r,a}^s(\hat{\omega}_s), \hat{\pi}(\hat{\omega})).
\end{aligned}$$

By Theorem 2.5 the function $\phi_r^s : L_r^s \times \hat{\Omega}_r^s \to \Lambda_r^s$ is continuous, and by Definition 2.8 the function

$$\underline{R} \times [0, 1] \ni (d, \xi) \mapsto \lambda_{r,d}^s(\xi)$$

is continuous. Hence we see that the function

$$\underset{\sim}{R} \times \overset{\wedge}{\Omega}{}_r^s \ni (a, \overset{\wedge}{\omega}) \mapsto \phi_r^s(a, \lambda_{r,a}^s(\overset{\wedge}{\omega}_s), \overset{\wedge}{\pi}(\overset{\wedge}{\omega}))$$

is continuous. Therefore, (1) implies that the function $\overset{\wedge}{\phi}{}_r^s$ is continuous. A similar argument shows that the function $\overset{\vee}{\phi}{}_r^s$ is continuous. □

Definition 2.9. Let $r \geq 0$ be arbitrary, and let m be the smallest integer such that $m \geq r$. let $a \in \underset{\sim}{R}$ be arbitrary. Define

$$\begin{aligned}
C_r &= \{ x : [r, +\infty) \to \underset{\sim}{R} \,|\, x \quad \textit{is continuous on} \quad [r, +\infty) \}, \\
\Lambda_r &= \{ x \in C_r \,|\, |x(v) - x(u)| \leq c|v - u|, \quad \textit{for all} \quad u, v \in [r, +\infty) \}, \\
\Lambda_{r,a} &= \{ x \in \Lambda_r \,|\, x(r) = a \}, \\
\hat{V}_r &= \hat{V}{}_r^m \cup \bigcup_{j=m}^{\infty} \hat{V}{}_j^{j+1}, \\
\overset{\wedge}{\Omega}_r &= [0, 1]^{\hat{V}_r}.
\end{aligned}$$

For $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}_r$, define $\overset{\wedge}{\pi}_{r,m}(\overset{\wedge}{\omega})$ and $\overset{\wedge}{\pi}_{j,j+1}(\overset{\wedge}{\omega})$ by

$$\begin{aligned}
\overset{\wedge}{\pi}_{r,m}(\overset{\wedge}{\omega})_t &= \overset{\wedge}{\omega}_t, \ t \in \hat{V}{}_r^m, \\
\overset{\wedge}{\pi}_{j,j+1}(\overset{\wedge}{\omega})_t &= \overset{\wedge}{\omega}_t, \ j \geq m, \ t \in \hat{V}{}_j^{j+1}.
\end{aligned}$$

Let $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}_r$ be arbitrary. Define $\varphi_{r,a}(\overset{\wedge}{\omega})$ by induction as follows.

$$\begin{aligned}
\varphi_{r,a}(\overset{\wedge}{\omega})(t) &= \varphi_{r,a}^m(\overset{\wedge}{\pi}_{r,m}(\overset{\wedge}{\omega}))(t), \ t \in [r, m], \\
\varphi_{r,a}(\overset{\wedge}{\omega})(t) &= \varphi_{m,b}^{m+1}(\overset{\wedge}{\pi}_{m,m+1}(\overset{\wedge}{\omega}))(t), \ b = \varphi_{r,a}(\overset{\wedge}{\omega})(m), \ t \in [m, m+1].
\end{aligned}$$

Now assume that $\varphi_{r,a}(\overset{\wedge}{\omega})$ has been defined on $[j, j+1]$ as above, where $j \geq m$. Then define

$$\varphi_{r,a}(\overset{\wedge}{\omega})(t) = \varphi_{j+1,d}^{j+2}(\overset{\wedge}{\pi}_{j+1,j+2}(\overset{\wedge}{\omega}))(t), \ d = \varphi_{r,a}(\overset{\wedge}{\omega})(j+1), \ t \in [j+1, j+2].$$

Because the ranges of the functions $\varphi_{r,a}^m$, $\varphi_{m,b}^{m+1}$, and $\varphi_{j+1,d}^{j+2}$ $(j \geq m)$ are, respectively, $\Lambda_{r,a}^m$, $\Lambda_{m,b}^{m+1}$, and $\Lambda_{j+1,d}^{j+2}$, we see that

$$\varphi_{r,a} : \overset{\wedge}{\Omega}_r \to \Lambda_{r,a}.$$

Theorem 2.8. *Let $r \geq 0$ be arbitrary. Give the set C_r the compact-open topology, and give $\Lambda_r \subseteq C_r$ the induced subspace topology. Define the function*

$$\overset{\wedge}{\phi}_r\colon \underline{R} \times \overset{\wedge}{\Omega}_r \to \Lambda_r$$

by

$$\overset{\wedge}{\phi}_r\,(a, \overset{\wedge}{\omega}) = \varphi_{r,a}(\overset{\wedge}{\omega}), \quad (a, \overset{\wedge}{\omega}) \in \underline{R} \times \overset{\wedge}{\Omega}_r\ .$$

Then $\overset{\wedge}{\phi}_r$ is continuous. In particular, for fixed $a \in \underline{R}$, the function

$$\varphi_{r,a} : \overset{\wedge}{\Omega}_r \to \Lambda_{r,a}$$

is continuous.

Proof. Note first that the compact-open topology on C_r coincides with the topology of compact convergence on C_r (see [11], Theorem 5.1). Recall that a basis for the topology of compact convergence on C_r consists of all sets of the form

$$B_C(x_0, \epsilon) = \{\, x \in C_r \mid \sup_{t \in C} |x(t) - x_0(t)| < \epsilon\, \},$$

where $\epsilon > 0$ is arbitrary, and C is and arbitrary compact subset of $[r, \infty)$. Therefore, to show that $\overset{\wedge}{\phi}_r$ is continuous on $\underline{R} \times \overset{\wedge}{\Omega}_r$, it suffices to show that for arbitrary $(a_0, \overset{\wedge}{\omega}{}^0) \in \underline{R} \times \overset{\wedge}{\Omega}_r$, if $x_0 = \overset{\wedge}{\phi}_r\,(a_0, \overset{\wedge}{\omega}{}^0)$, then for every $\epsilon > 0$ and for every compact subset C of $[r, \infty)$, there exists an open subset W of $\underline{R} \times \overset{\wedge}{\Omega}_r$ such that

$$(a_0, \overset{\wedge}{\omega}{}^0) \in W \quad \text{and} \quad \overset{\wedge}{\phi}_r\,(W) \subseteq B_C(x_0, \epsilon). \tag{1}$$

To this end, let m be the smallest positive integer greater than r. Let $\epsilon > 0$ be arbitrary, and let C be any nonempty compact subset of $[r, \infty)$. Then there exists a positive integer $k \geq m$ such that $C \subseteq [r, k+1]$. We claim that there exist positive numbers

$$0 < \epsilon_r,\ \epsilon_m,\ \cdots \epsilon_k,\ \epsilon_{k+1} = \epsilon,$$

and open subsets

$$W_r \subseteq \overset{\wedge}{\Omega}{}_r^m, \quad W_m \subseteq \overset{\wedge}{\Omega}{}_m^{m+1}, \ldots, W_k \subseteq \overset{\wedge}{\Omega}{}_k^{k+1},$$

having the following properties, where

$I_r = (x_0(r)-\epsilon_r, x_0(r)+\epsilon_r)$ and $I_j = (x_0(j)-\epsilon_j, x_0(j)+\epsilon_j), \quad m \leq j \leq k+1:$

$$\epsilon_r < \epsilon_j, \quad m \leq j \leq k+1; \tag{2}$$

$$\epsilon_j < \epsilon_{j+1}, \quad m \leq j \leq k; \tag{3}$$

$$\hat{\pi}_{r,m}(\hat{\omega}^0) \in W_r \quad \text{and} \quad \hat{\phi}_r^m(I_r \times W_r) \subseteq B_{[r,m]}(x_0, \epsilon_m); \tag{4}$$

$$\hat{\pi}_{j,j+1}(\hat{\omega}^0) \in W_j \quad \text{and} \quad \hat{\phi}_j^{j+1}(I_j \times W_j) \subseteq B_{[j,j+1]}(x_0, \epsilon_{j+1}), \quad m \leq j \leq k. \tag{5}$$

To prove the claim, we first use "backward induction" on j to define ϵ_j and W_j for $m \leq j \leq k$, in such a way that (3) and (5) hold. By Theorem 2.7, the function

$$\hat{\phi}_k^{k+1} : \underset{\sim}{\mathrm{R}} \times \hat{\Omega}_k^{k+1} \to \Lambda_k^{k+1}$$

is continuous. Hence there exists an $0 < \epsilon_k < \epsilon_{k+1} = \epsilon$ and an open subset $W_k \subseteq \hat{\Omega}_k^{k+1}$ such that

$$\hat{\pi}_{k,k+1}(\hat{\omega}^0) \in W_k \quad \text{and} \quad \hat{\phi}_k^{k+1}(I_k \times W_k) \subseteq B_{[k,k+1]}(x_0, \epsilon_{k+1}). \tag{6}$$

This defines ϵ_k and W_k. Clearly, (6) implies (5) for the case where $j = k$. Also, (3) clearly holds for $j = k$.

Now assume that $m < j \leq k$, and that ϵ_j and W_j have been defined in such a way that (3) and (5) hold for j. By Theorem 2.7, the function

$$\hat{\phi}_{j-1}^j : \underset{\sim}{\mathrm{R}} \times \hat{\Omega}_{j-1}^j \to \Lambda_{j-1}^j$$

is continuous, hence there exists an $0 < \epsilon_{j-1} < \epsilon_j$ and an open subset W_{j-1} of $\hat{\Omega}_{j-1}^j$ such that

$$\hat{\pi}_{j,j-1}(\hat{\omega}^0) \in W_{j-1} \quad \text{and} \quad \hat{\phi}_{j-1}^j(I_{j-1} \times W_{j-1}) \subseteq B_{[j-1,j]}(x_0, \epsilon_j). \tag{7}$$

This defines ϵ_{j-1} and W_{j-1}. It is clear that (3) holds when j is replaced by $j-1$. Also, statement (7) implies that (5) holds when j is replaced by $j-1$. It follows from backward induction on j that (3) and (5) hold for all $m \leq j \leq k$. We now define ϵ_r and W_r in such a way that (2) and (4) are true. By Theorem 2.7, the function

$$\hat{\phi}_r^m : \underset{\sim}{\mathrm{R}} \times \hat{\Omega}_r^m \to \Lambda_r^m$$

is continuous at $(x_0(r), \hat{\pi}_{r,m}(\hat{\omega}^0))$. Hence there exist $0 < \epsilon_r < \epsilon_m$ and an open subset W_r of $\hat{\Omega}_r^m$ such that

$$\hat{\pi}_{r,m}(\hat{\omega}^0) \in W_r \quad \text{and} \quad \hat{\phi}_r^m(I_r \times W_r) \subseteq B_{[r,m]}(x_0, \epsilon_m). \tag{8}$$

This defines ϵ_r and W_r. Because $\epsilon_r < \epsilon_m$, (3) implies (2). Clearly, (8) implies (4).

To prove (1), define W as follows.

$$W = I_r \times (W_r \times \cdots \times W_k \times \hat{\Omega}_{k+1}).$$

Then W is open in $\underset{\sim}{\mathbb{R}} \times \hat{\Omega}_r$. By (4) and (5) we see that

$$\hat{\pi}_{r,m}(\hat{\omega}^0) \in W_r \quad \text{and} \quad \hat{\pi}_{j,j+1}(\hat{\omega}^0) \in W_j, \quad m \le j \le k.$$

It follows that

$$(a_0, \hat{\omega}^0) \in W. \tag{9}$$

Now let $x \in \hat{\phi}_r(W)$ be arbitrary, say

$$x = \hat{\phi}_r(a, \hat{\omega}), \quad (a, \hat{\omega}) \in W.$$

By definition of $\hat{\phi}_r$, we have

$$x|[r,m] = \hat{\phi}_r(a, \hat{\omega})|[r,m] = \varphi_{r,a}^m(\hat{\pi}_{r,m}(\hat{\omega})).$$

Because $\hat{\pi}_{r,m}(\hat{\omega}) \in W_r$ and $x(r) = a \in W_r$, (3) and (4) together imply that

$$\begin{aligned} x|[r,m] &= \varphi_{r,a}^m(\hat{\pi}_{r,m}(\hat{\omega})) \\ &= \hat{\phi}_r^m(a, \hat{\pi}_{r,m}(\hat{\omega})) \\ &\in \hat{\phi}_r^m(I_r \times W_r) \\ &\subseteq B_{[r,m]}(x_0, \epsilon_m) \\ &\subseteq B_{[r,m]}(x_0, \epsilon). \end{aligned}$$

Therefore, we have

$$x|[r,m] \in B_{[r,m]}(x_0, \epsilon_m) \subseteq B_{[r,m]}(x_0, \epsilon). \tag{10}$$

We claim that

$$x|[j,j+1] \in B_{[j,j+1]}(x_0,\epsilon_{j+1}) \subseteq B_{[j,j+1]}(x_0,\epsilon), \quad m \leq j \leq k. \tag{11}$$

We prove this claim by using induction on $m \leq j \leq k$. By (10), we have

$$|x(m) - x_0(m)| < \epsilon_m,$$

hence $x(m) \in I_m$. Therefore, by (9),

$$(x(m), \hat{\pi}_{m,m+1}(\hat{\omega})) \in I_m \times W_m.$$

It follows from (5) and Definition 2.9 that, with $b = \varphi_{r,a}(\hat{\omega})(m) = x(m)$, we have

$$\begin{aligned} x|[m,m+1] &= \varphi_{m,b}^{m+1}(\hat{\pi}_{m,m+1}(\hat{\omega})) \\ &= \hat{\phi}_m^{m+1}(b, \hat{\pi}_{m,m+1}(\hat{\omega})) \\ &\subseteq \hat{\phi}_m^{m+1}(I_m \times W_m) \\ &\subseteq B_{[m,m+1]}(x_0,\epsilon_{m+1}) \\ &\subseteq B_{[m,m+1]}(x_0,\epsilon). \end{aligned}$$

Therefore (11) holds for $j = m$. Assume that (11) holds for $m \leq j < k$. Then by (11), $|x(j+1) - x_0(j+1)| < \epsilon_{j+1}$, and hence $x(j+1) \in I_{j+1}$. Consequently, by (9), we have

$$(x(j+1), \hat{\pi}_{j+1,j+2}(\hat{\omega})) \in I_{j+1} \times W_{j+1}.$$

It follows from (5) and Definition 2.9 that, with $d = \varphi_{r,a}(\hat{\omega})(j+1) = x(j+1)$, we have

$$\begin{aligned} x|[j+1,j+2] &= \varphi_{j+1,d}^{j+2}(\hat{\pi}_{j+1,j+2}(\hat{\omega})) \\ &= \hat{\phi}_{j+1}^{j+2}(d, \hat{\pi}_{j+1,j+2}(\hat{\omega})) \\ &\subseteq \hat{\phi}_{j+1}^{j+2}(I_{j+1} \times W_{j+1}) \\ &\subseteq B_{[j+1,j+2]}(x_0,\epsilon_{j+2}) \\ &\subseteq B_{[j+1,j+2]}(x_0,\epsilon). \end{aligned}$$

That is,

$$x|[j+1,j+2] \in B_{[j+1,j+2]}(x_0,\epsilon_{j+2}) \subseteq B_{[j+1,j+2]}(x_0,\epsilon). \tag{12}$$

Statement (12) is just statement (11) with j replaced by $j+1$. Hence, by induction, (11) holds. Now, (10) and (11) together imply the following statements:

$$x|([r,m]\cap C) \in B_{[r,m]}(x_0,\epsilon); \tag{13}$$
$$x|([j,j+1]\cap C) \in B_{[j,j+1]}(x_0,\epsilon), \quad m \le j \le k. \tag{14}$$

Because $C \subseteq [r,m] \cup \bigcup_{j=m}^{k} [j,j+1] = [r,k+1]$, we see that (13) and (14) together imply that

$$x \in B_C(x_0,\epsilon).$$

Because $x \in W$ is arbitrary, we see that (1) holds. This completes the proof that

$$\hat{\phi}_r\colon \underset{\sim}{R} \times \hat{\Omega}_r \to \Lambda_r$$

is continuous in the compact-open topology on Λ_r. □

Theorem 2.9. *Let $r \ge 0$ be arbitrary. Then the function*

$$\hat{\phi}_r\colon \underset{\sim}{R} \times \hat{\Omega}_r \to \Lambda_r$$

is onto.

Proof. Let m be defined as in Theorem 2.8. Let $x \in \Lambda_r$. Then $x|[r,m] \in \Lambda^m_{r,x(r)}$, hence by Theorem 2.6, there exists $\hat{\omega}^{\,r} \in \hat{\Omega}^{\,m}_{r}$ such that, with $a = x(r)$, we have

$$x|[r,m] = \varphi^m_{r,a}(\hat{\omega}^{\,r}).$$

Now let $j \ge m$ be an arbitrary integer. By Theorem 2.6, there exists an $\hat{\omega}^{\,j} \in \hat{\Omega}^{\,j+1}_{j}$ such that

$$x|[j,j+1] = \varphi^{j+1}_{j,x(j)}(\hat{\omega}^{\,j}).$$

Define $\hat{\omega} \in \hat{\Omega}_r$ by

$$\hat{\omega} = (\hat{\omega}^{\,r}, \hat{\omega}^{\,m}, \ldots, \hat{\omega}^{\,j}, \ldots).$$

We claim that

$$x = \hat{\phi}_r\,(\hat{\omega}). \tag{1}$$

To prove this claim, note first that by Definition 2.9, for $t \in [r,m]$, we have

$$x(t) = (x|[r,m])(t) = \varphi^m_{r,a}(\hat{\omega}^{\,r})(t) \tag{2}$$

$$= \varphi_{r,m}^{m}(\overset{\wedge}{\pi}_{r,m}(\overset{\wedge}{\omega}))(t)$$
$$= \varphi_{r,a}(\overset{\wedge}{\omega})(t)$$
$$= \overset{\wedge}{\phi}_{r}(a,\overset{\wedge}{\omega})(t).$$

Define $d = \varphi_{r,x(r)}(\overset{\wedge}{\omega})(m)$. Then by (2), we have $d = x(r)$. Hence by Definition 2.9, for $t \in [m, m+1]$, we have

$$x(t) = (x|[m,m+1])(t) = \varphi_{m,x(m)}^{m+1}(\overset{\wedge}{\omega}^{m})(t) \tag{3}$$
$$= \varphi_{m,x(m)}^{m+1}(\overset{\wedge}{\pi}_{m,m+1}(\overset{\wedge}{\omega}))(t)$$
$$= \varphi_{r,d}(\overset{\wedge}{\omega})(t)$$
$$= \overset{\wedge}{\phi}_{r}(a,\overset{\wedge}{\omega})(t).$$

To finish the proof of the claim (1), we will prove by induction on $j \geq m$ that

$$x(t) = \overset{\wedge}{\phi}_{r}(a,\overset{\wedge}{\omega})(t), \quad t \in [j, j+1]. \tag{4.}$$

Statement (3) implies that (4) holds for $j = m$. Assume that (4) holds for some $j \geq m$. Define $h = \varphi_{r,a}(\overset{\wedge}{\omega})(j+1)$. Then, by the induction hypothesis, we have

$$x(t) = (x|[j+1,j+2])(t) = \varphi_{j+1,x(j+1)}^{j+2}(\overset{\wedge}{\omega}^{j+1})(t) \tag{5}$$
$$= \varphi_{j+1,h}^{j+2}(\overset{\wedge}{\pi}_{j+1,j+2}(\overset{\wedge}{\omega}))(t)$$
$$= \varphi_{r,a}(\overset{\wedge}{\omega})(t)$$
$$= \overset{\wedge}{\phi}_{r}(a,\overset{\wedge}{\omega})(t).$$

Statement (5) is obtained from statement (4) by replacing j in (4) by $j+1$. Hence (4) holds by induction on $j \geq m$. Now statements (2) and (4) together imply (1). Because $x \in \Lambda_r$ is arbitrary, we have proved that $\overset{\wedge}{\phi}_{r}$ is onto. □

Definition 2.10. Let $0 \leq r < s$ be arbitrary. Define

$$\overset{\sim}{\Omega}{}_{r}^{s} = \underset{\sim}{R} \times \overset{\wedge}{\Omega}{}_{r}^{s},$$
$$\overset{\sim}{\Omega}_{r} = \underset{\sim}{R} \times \overset{\wedge}{\Omega}_{r}.$$

Assume that we are given a function

$$\lambda_r : \underset{\sim}{R} \to \underset{\sim}{R}$$

such that λ_r is continuous and onto. Then define the functions

$$\varphi_r^s : \widetilde{\Omega}\,{}_r^s \to \Lambda_r^s,$$
$$\varphi_r : \widetilde{\Omega}\,{}_r \to \Lambda_r$$

as follows.

$$\varphi_r^s(a, \overset{\wedge}{\omega}) = \varphi_{r,b}^s(\overset{\wedge}{\omega}), \quad b = \lambda_r(a), \quad (a, \overset{\wedge}{\omega}) \in \widetilde{\Omega}\,{}_r^s,$$
$$\varphi_r(a, \overset{\wedge}{\omega}) = \varphi_{r,b}(\overset{\wedge}{\omega}), \quad b = \lambda_r(a), \quad (a, \overset{\wedge}{\omega}) \in \widetilde{\Omega}\,{}_r.$$

Theorem 2.10. *Let $0 \le r < s$ be arbitrary. Then the functions*

$$\varphi_r^s : \widetilde{\Omega}\,{}_r^s \to \Lambda_r^s,$$
$$\varphi_r : \widetilde{\Omega}\,{}_r \to \Lambda_r$$

are continuous and onto.

Proof. To prove that φ_r^s is onto, let $x \in \Lambda_r^s$. Because λ_r is onto, there exists an $a \in \underset{\sim}{\mathrm{R}}$ such that $\lambda_r(a) = x(r)$. Define $b = x(r)$, then by Theorem 2.6, there exists $\overset{\wedge}{\omega} \in \overset{\wedge}{\Omega}\,{}_r^s$ such that

$$\begin{aligned} x &= \varphi_{r,b}^s(\overset{\wedge}{\omega}) \\ &= \varphi_r^s(a, \overset{\wedge}{\omega}). \end{aligned}$$

Because x is arbitrary, this shows that φ_r^s is onto. A similar argument shows that φ_r is onto.

To show that φ_r^s is continuous, let $(a, \overset{\wedge}{\omega}) \in \widetilde{\Omega}\,{}_r^s$. By assumption, λ_r is continuous, and by Theorem 2.7, the function

$$\overset{\wedge}{\phi}\,{}_r^s : \underset{\sim}{\mathrm{R}} \times \overset{\wedge}{\Omega}\,{}_r^s \to \Lambda_r^s$$

is continuous, therefore, for $(s, \overset{\wedge}{\omega}) \in \widetilde{\Omega}\,{}_r^s$, the function

$$\begin{aligned} (a, \overset{\wedge}{\omega}) \mapsto \overset{\wedge}{\phi}\,{}_r^s(\lambda_r(a), \overset{\wedge}{\omega}) &= \overset{\wedge}{\phi}\,{}_r^s(b, \overset{\wedge}{\omega}) \\ &= \varphi_{r,b}^s(\overset{\wedge}{\omega}) \\ &= \varphi_r^s(a, \overset{\wedge}{\omega}). \end{aligned}$$

is continuous. Thus, φ_r^s is continuous. A similar argument shows that φ_r is continuous. □

3. Construction of the Measures $\boldsymbol{\mu}_{\square}^{\square}$

In this section we use standard results from topology and measure theory to construct the families of regular Borel measures mentioned in (2) and (3) of the introduction.

Theorem 3.1. (Ascoli's Theorem) *Let X be a locally compact Hausdorff space; let (Y, d) be a metric space. Let $\mathcal{C}(X, Y)$ be the space of all continuous functions from X to Y, and consider $\mathcal{C}(X, Y)$ in the compact-open topology. A subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and the subset*

$$\mathcal{F}_x = \{\, f(x) \mid f \in \mathcal{F} \,\}$$

of Y has compact closure for each $x \in X$.

Proof. (See [11], Theorem 6.1.) $\square$

Theorem 3.2. *Let $0 \le r < s$ be arbitrary, and let $(a, b) \in L_r^s$. Then each of the following function spaces is equicontinuous:*

$$\Lambda_{r,a}^{s,b}, \quad \Lambda_{r,a}^{s}, \quad \Lambda_{r}^{s,b}, \quad \Lambda_{r,a}, \quad \Lambda_{r}^{s}, \quad \Lambda_{r}.$$

Proof. We show that $\Lambda_{r,a}^{s,b}$ is equicontinuous. The proof that remaining spaces are equicontinuous is similar. We show that $\Lambda_{r,a}^{s,b}$ is equicontinuous at each point $t_0 \in [r, s]$. Let $\epsilon > 0$ be arbitrary, and set $\delta = \dfrac{\epsilon}{c}$. Then for all $x \in \Lambda_{r,a}^{s,b}$ and all $t \in [r, s] \cap (t_0 - \delta, t_0 + \delta)$, we have

$$|x(t) - x(t_0)| \le c|t - t_0| < c\delta = \epsilon.$$

Thus, $\Lambda_{r,a}^{s,b}$ is equicontinuous at the arbitrary point $t_0 \in [r, s]$, so $\Lambda_{r,a}^{s,b}$ is equicontinuous. $\square$

Theorem 3.3. *Let $0 \le r < s$ be arbitrary. Then for arbitrary $(a, b) \in L_r^s$, $\Lambda_{r,a}^{s,b}$, $\Lambda_{r,a}^{s}$, $\Lambda_{r}^{s,b}$, and Λ_r^s are compact in the uniform topology. If $a \in \mathbb{R}$ is arbitrary, then $\Lambda_{r,a}$ is compact in the compact-open topology.*

Proof. To prove that $\Lambda_{r,a}^{s,b}$ is compact in the the uniform topology, note first that the compact-open topology on $\Lambda_{r,a}^{s,b}$ coincides with the topology of compact convergence, and because $[r, s]$ is compact, the topology of compact convergence on $\Lambda_{r,a}^{s,b}$ coincides with the uniform topology (see [11], Theorem 4.6). Therefore, it suffices to prove that $\Lambda_{r,a}^{s,b}$ is compact in the compact-open topology. To this end, let $t \in [r, s]$, and let $x \in \Lambda_{r,a}^{s,b}$. Then we have

$$\begin{aligned} |x(t)| &= |(x(t) - x(r)) + x(r)| \\ &\leq |x(t) - x(r)| + |a| \\ &\leq c(t - r) + |a|. \end{aligned} \tag{1}$$

Now define $\mathcal{F} = \Lambda_{r,a}^{s,b}$, and define $\mathcal{F}_t = \{\, x(t) \,|\, x \in \mathcal{F} \,\}$. Then (1) implies that $\mathcal{F}_t$ is bounded, and hence it has compact closure. By Theorem 3.2, $\mathcal{F}$ is equicontinuous, and hence Ascoli's theorem implies that $\mathcal{F}$ has compact closure in the compact-open topology, that is, $\mathcal{F} = \Lambda_{r,a}^{s,b}$ is compact in the compact-open topology. A similar argument shows that the spaces $\Lambda_{r,a}^{s}$ and Λ_r^s are compact in the uniform topology.

Now let $a \in \mathbb{R}$ be arbitrary. Define $\mathcal{G} = \Lambda_{r,a}$. Let $t \in [r, +\infty)$ be arbitrary, and define $\mathcal{G}_t = \{\, x(t) \,|\, x \in \mathcal{G} \,\}$. Then (1) implies that $\mathcal{G}_t$ has compact closure. By Theorem 3.2, $\mathcal{G}$ is equicontinuous, and hence by Ascoli's theorem, $\mathcal{G} = \Lambda_{r,a}$ is compact in the compact-open topology. □

Theorem 3.4. *Let $0 \leq r < s$ be arbitrary. Then the space Λ_r^s is locally compact in the uniform topology, and the space Λ_r is locally compact in the compact-open topology.*

Proof. As in the proof of Theorem 3.3, the uniform topology on Λ_r^s coincides with the compact-open topology, hence it suffices to show that Λ_r^s is locally compact in the compact-open topology. To this end, let $x_0 \in \Lambda_r^s$ be arbitrary. Let $\epsilon > 0$ be arbitrary, and define $C = \{r\}$, $U = (x_0(r) - \epsilon, x_0(r) + \epsilon)$. Then the set

$$S(C, U) = \{\, x \in \Lambda_r^s \,|\, x(C) \subseteq U \,\} = \{\, x \in \Lambda_r^s \,|\, |x(r) - x_0(r)| < \epsilon \,\}$$

is a basis element in the compact-open topology on Λ_r^s, and this basis element contains x_0. We claim that $S(C, U)$ has compact closure; and hence, because $x_0 \in \Lambda_r^s$ is arbitrary, Λ_r^s is locally compact in the compact-open topology. To prove the claim, let $x \in S(C, U)$, then we have

$$\begin{aligned} |x(r)| &= |(x(r) - x_0(r)) + x_0(r)| \\ &\leq |x(r) - x_0(r)| + |x_0(r)| \\ &< \epsilon + |x_0(r)|. \end{aligned} \tag{1}$$

Hence, for any $t \geq r$, we have

$$|x(t)| = |(x(t) - x(r)) + x(r)| \tag{2}$$

$$\leq |x(t) - x(r)| + |x(r)|$$
$$< c(t-r) + |x_0(r)| + \epsilon.$$

Define $\mathcal{F} = S(C, U)$. Then (2) implies that $\mathcal{F}_t$ has compact closure. By Theorem 3.2, $\mathcal{F}$ is equicontinuous, and hence, because $t \geq r$ is arbitrary, Ascoli's theorem gives that $\mathcal{F} = S(C, U)$ has compact closure in the compact-open topology. Thus, Λ_r^s is locally compact in the compact-open topology. A similar argument shows that Λ_r is locally compact in the compact-open topology. □

Theorem 3.5. *Let $0 \leq r < s$ be arbitrary. Give the space Λ_r^s the uniform topology, and give the space Λ_r the compact-open topology. Assume that the function $\lambda_r : \underline{R} \to \underline{R}$ in Definition 2.10 has the property that if C is any compact subset of $\underline{R}$, then $\lambda_r^{-1}(C)$ is compact. Then for any compact subset A of Λ_r^s and any compact subset B of Λ_r, the following sets are compact:*

$$(\varphi_r^s)^{-1}(A), \quad \varphi_r^{-1}(B).$$

Proof. Let $A \subseteq \Lambda_r^s$ be compact. By Theorem 3.2, Λ_r^s is equicontinuous, and hence A is also equicontinuous. Ascoli's theorem then implies that the set

$$A_r = \{\, x(r) \,|\, x \in A \,\}$$

has compact compact closure. Therefore there exists a $d > 0$ such that $|x(r)| \leq d$ for all $x \in A$. Now let $(a, \overset{\wedge}{\omega}) \in (\varphi_r^s)^{-1}(A)$. Then $\varphi_r^s(a, \overset{\wedge}{\omega}) \in A$. By Definition 2.10, with $b = \lambda_r(a)$, we have

$$\varphi_r^s(a, \overset{\wedge}{\omega})(r) = \varphi_{r,b}^s(\overset{\wedge}{\omega})(r) = b = \lambda_r(a).$$

It follows that $|\lambda_r(a)| = |\varphi_r^s(a, \overset{\wedge}{\omega})(r)| \leq d$, i.e., $a \in \lambda_r^{-1}([-d, d])$. The set $\lambda_r^{-1}([-d, d])$ is, by assumption, compact, hence there exists an $h > 0$, depending only on d, such that $\lambda_r^{-1}([-d, d]) \subseteq [-h, h]$, which implies that $a \in [-h, h]$. Because $(a, \overset{\wedge}{\omega}) \in (\varphi_r^s)^{-1}(A)$ is arbitrary, we see that

$$(\varphi_r^s)^{-1}(A) \subseteq [-h, h] \times \overset{\wedge}{\Omega}{}_r^s.$$

By Theorem 2.10, the function φ_r^s is continuous, there $(\varphi_r^s)^{-1}(A)$ is a closed subset of the compact set $[-h, h] \times \overset{\wedge}{\Omega}{}_r^s$. It follows that $(\varphi_r^s)^{-1}(A)$ is compact. Similarly, $\varphi_r^{-1}(B)$ is compact whenever $B \subseteq \Lambda_r$ is compact in the compact-open topology. □

Definition 3.1. Let X and Y be locally compact Hausdorff spaces, and let ψ be a continuous function from X onto Y. Let $(X, \mathcal{M}_\mu, \mu)$ be a measure space constructed from a nonnegative linear functional on X via the Daniell approach (see [7]: Section 9). Assume that one of the following conditions holds.

1. (a) $\psi^{-1}(F)$ is compact in X for every compact subset F of Y.

2. (b) $\mu(X) < +\infty$.

Let $\mathcal{C}(Y)$ be the space of all continuous complex-valued functions on Y, and let $\mathcal{C}_{00}(Y)$ be the set of all $f \in \mathcal{C}(Y)$ such that f has compact support. Then for any $f \in \mathcal{C}_{00}(Y)$, $f \circ \psi \in \mathcal{L}_1(X, \mathcal{M}_\mu, \mu)$, and hence the mapping

$$J(f) = \int_X f \circ \psi(x)\, d\mu(x)$$

is a nonnegative linear functional on $\mathcal{C}_{00}(Y)$. Hence we may construct a measure space $(Y, \mathcal{M}_\nu, \nu)$ from J via the Daniell approach. We then have

$$\int_Y f(y)\, d\nu(y) = \int_X f \circ \psi(x)\, d\mu(x), \quad f \in \mathcal{C}_{00}(Y).$$

The measure ν is said to be the image of the measure μ under the continuous function ψ.

Theorem 3.6. *The measure ν constructed in Definition 3.1 has the following properties:*

1. *(a) for all σ-finite ν-measurable subsets B of Y,*

$$\nu(B) = \mu(\psi^{-1}(B)) = \int_X \chi_B \circ \psi(x)\, d\mu(x);$$

2. *(b) for every $f \in \mathcal{L}_1(Y, \mathcal{M}_\nu, \nu)$, $f \circ \psi \in \mathcal{L}_1(X, \mathcal{M}_\mu, \mu)$ and*

$$\int_Y f(y)\, d\nu(y) = \int_X f \circ \psi(x)\, d\mu(x).$$

Proof. See [7]: Theorem 12.46. □

Theorem 3.7. *Let X be a locally compact Hausdorff space and let ν be a regular measure defined on a σ-algebra $\mathcal{A}$ of subsets of X such that $(X, \mathcal{A}, \nu)$*

is a complete measure space. Suppose that $E \in \mathcal{A}$ if and only if $E \cap F \in \mathcal{A}$ for every compact set $F \subseteq X$. Define I on $\mathcal{C}_{00}(X)$ by

$$I(f) = \int_X f(x)\, d\nu(x).$$

Let $(X, \mathcal{M}_\iota, \iota)$ be the measure space constructed from I via the Daniell approach. Then $\mathcal{A} = \mathcal{M}_\iota$ and $\nu(E) = \iota(E)$ for all $E \in \mathcal{M}_\iota$.

Proof. See [7], Theorem 12.42. □

Definition 3.2. Let $0 \leq r < s$ be arbitrary, and let $(a, b) \in L_r^s$ be arbitrary. Define

$$m_r^s, \quad \hat{m}{}_r^s, \quad \overset{\vee}{m}{}_r^s, \quad \hat{m}_r, \tag{1}$$

to be normalized Lebesgue measure, respectively, on the following product spaces:

$$\Omega_r^s, \quad \overset{\wedge}{\Omega}{}_r^s, \quad \overset{\vee}{\Omega}{}_r^s, \quad \overset{\wedge}{\Omega}_r . \tag{2}$$

Let

$$\tilde{m}{}_r^s, \quad \tilde{m}_r$$

be Lebesgue measure, respectively, on the following product spaces:

$$\overset{\sim}{\Omega}{}_r^s, \quad \overset{\sim}{\Omega}_r . \tag{3}$$

Finally, let

$$\mathcal{L}_r^s, \quad \overset{\wedge}{\mathcal{L}}{}_r^s, \quad \overset{\vee}{\mathcal{L}}{}_r^s, \quad \overset{\wedge}{\mathcal{L}}_r, \quad \overset{\sim}{\mathcal{L}}{}_r^s, \quad \overset{\sim}{\mathcal{L}}_r$$

be the respective σ-algebras of Lebesgue measurable subsets of the product spaces given in (2) and (3). Let $(X, \mathcal{A}, \nu)$ denote any one of the following measure spaces:

$$(\Omega_r^s, \mathcal{L}_r^s, m_r^s), \ (\overset{\wedge}{\Omega}{}_r^s, \overset{\wedge}{\mathcal{L}}{}_r^s, \hat{m}{}_r^s), \ (\overset{\vee}{\Omega}{}_r^s, \overset{\vee}{\mathcal{L}}{}_r^s, \overset{\vee}{m}{}_r^s),$$

$$(\overset{\wedge}{\Omega}_r, \overset{\wedge}{\mathcal{L}}_r, \hat{m}_r), \ (\overset{\sim}{\Omega}{}_r^s, \overset{\sim}{\mathcal{L}}{}_r^s, \tilde{m}{}_r^s), \ (\overset{\sim}{\Omega}_r, \overset{\sim}{\mathcal{L}}_r, \tilde{m}_r).$$

Then $(X, \mathcal{A}, \nu)$ satisfies the hypothesis of Theorem 3.7, and hence in Definition 3.1 we may take $(X, \mathcal{M}_\mu, \mu) = (X, \mathcal{A}, \nu)$. The measures in (1) are probability measures, and hence the we may apply Definition 3.1 to the following functions.

$$\varphi_{r,a}^{s,b} : \Omega_r^s \to \Lambda_{r,a}^{s,b}, \ \varphi_{r,a}^{s} : \overset{\wedge}{\Omega}{}_r^s \to \Lambda_{r,a}^{s}, \ \varphi_r^{s,b} : \overset{\vee}{\Omega}{}_r^s \to \Lambda_r^{s,b}, \ \varphi_{r,a} : \overset{\wedge}{\Omega}_r \to \Lambda_{r,a}.$$

Therefore we may use these functions to construct, respectively, the image measure spaces:

$$(\Lambda^{s,b}_{r,a}, \mathcal{M}^s_r, \mu^{s,b}_{r,a}),\ (\Lambda^s_{r,a}, \overset{\wedge}{\mathcal{M}}{}^s_r, \mu^s_{r,a}),\ (\Lambda^{s,b}_r, \overset{\vee}{\mathcal{M}}{}^s_r, \mu^{s,b}_r),\ (\Lambda_{r,a}, \overset{\wedge}{\mathcal{M}}{}_r, \mu_{r,a}).$$

According to Theorem 3.5, the functions

$$\varphi^s_r : \overset{\sim}{\Omega}{}^s_r \to \Lambda^s_r, \quad \varphi_r : \overset{\sim}{\Omega}_r \to \Lambda_r$$

satisfy condition (a) of Definition 3.1, and hence we may use these functions to construct, respectively, the measure spaces:

$$(\Lambda^s_r, \overset{\sim}{\mathcal{M}}{}^s_r, \mu^s_r),\ (\Lambda_r, \overset{\sim}{\mathcal{M}}{}_r, \mu_r).$$

This completes the definition of the measure spaces

$$(\Lambda^{s,b}_{r,a}, \mathcal{M}^s_r, \mu^{s,b}_{r,a}),\ (\Lambda^s_{r,a}, \overset{\wedge}{\mathcal{M}}{}^s_r, \mu^s_{r,a}),\ (\Lambda^{s,b}_r, \overset{\vee}{\mathcal{M}}{}^s_r, \mu^{s,b}_r),$$
$$(\Lambda_{r,a}, \overset{\wedge}{\mathcal{M}}{}_r, \mu_{r,a}),\ (\Lambda^s_r, \overset{\sim}{\mathcal{M}}{}^s_r, \mu^s_r),\ (\Lambda_r, \overset{\sim}{\mathcal{M}}{}_r, \mu_r).$$

4. Uniform Measure on $\boldsymbol{\Lambda}^{\square}_{\square}$

In this section we select the functions

$$\lambda^{s,b}_{r,a} : [0,1] \to I^{s,b}_{r,a}, \quad \lambda^s_{r,a} : [0,1] \to I^s_{r,a}, \quad \lambda_r : \underline{R} \to \underline{R}$$

to be, respectively, the unique affine mappings of $[0,1]$ onto $I^{s,b}_{r,a}$, $I^s_{r,a}$, and the identity mapping on $\underline{R}$. These mappings give rise to "the uniform probability measure on" $\Lambda^{\square}_{\square}$ and "Lebesgue measure" on $\Lambda^{\square}_{\square}$.

Definition 4.1. Let $0 \le r < s$ be arbitrary. Let (a,b) be any pair in L^s_r. For $\xi \in [0,1]$, define

$$\lambda^{s,b}_{r,a}(\xi) = \begin{cases} [(a-b) + c(s-r)]\xi + b - \frac{1}{2}c(s-r), & a \le b; \\ [(b-a) + c(s-r)]\xi + a - \frac{1}{2}c(s-r), & b \le a, \end{cases}$$

$$\lambda^s_{r,a}(\xi) = 2c(s-r)\xi + a - c(s-r).$$

Then the affine function

$$\lambda^{s,b}_{r,a} : [0,1] \to I^{s,b}_{r,a}$$

satisfies the conditions of Definition 2.3; and the affine function

$$\lambda^{s}_{r,a} : [0,1] \to I^{s}_{r,a}$$

satisfies the conditions of Definition 2.8. Hence we may use the functions $\lambda^{s,b}_{r,a}$ and $\lambda^{s}_{r,a}$ to construct the mappings

$$\varphi^{s,b}_{r,a} : \Omega^{s}_{r} \to \Lambda^{s,b}_{r,a},\ \varphi^{s}_{r,a} : \overset{\wedge}{\Omega}{}^{s}_{r} \to \Lambda^{s}_{r,a},\ \varphi^{s,b}_{r} : \overset{\vee}{\Omega}{}^{s}_{r} \to \Lambda^{s,b}_{r},\ \varphi_{r,a} : \overset{\wedge}{\Omega}_{r} \to \Lambda_{r,a}.$$

By Definition 3.2, these mappings give rise to the following image measure spaces.

$$(\Lambda^{s,b}_{r,a}, \mathcal{M}^{s}_{r}, \mu^{s,b}_{r,a}),\ (\Lambda^{s}_{r,a}, \overset{\wedge}{\mathcal{M}}{}^{s}_{r}, \mu^{s}_{r,a}),\ (\Lambda^{s,b}_{r}, \overset{\vee}{\mathcal{M}}{}^{s}_{r}, \mu^{s,b}_{r}),\ (\Lambda_{r,a}, \overset{\wedge}{\mathcal{M}}_{r}, \mu_{r,a}). \qquad (1)$$

We call each measure $\lambda^{\square}_{\square}$ in (1) **the uniform probability measure on** $\Lambda^{\square}_{\square}$. Define

$$\lambda_r : \underline{R} \to \underline{R}$$

by $\lambda_r(\xi) = \xi$, $\xi \in \underline{R}$. Then λ_r satisfies the conditions of Definition 2.10. Hence, according to Definition 3.2, the functions

$$\lambda^{s,b}_{r,a} : [0,1] \to I^{s,b}_{r,a},\ \lambda^{s}_{r,a} : [0,1] \to I^{s}_{r,a},\ \lambda_r : \underline{R} \to \underline{R}$$

give rise to the following image measure spaces:

$$(\Lambda^{s}_{r}, \widetilde{\mathcal{M}}{}^{s}_{r}, \mu^{s}_{r}),\ (\Lambda_{r}, \widetilde{\mathcal{M}}_{r}, \mu_{r}). \qquad (2)$$

The measures $\lambda^{\square}_{\square}$ in (2) are called **Lebesgue measure on** $\Lambda^{\square}_{\square}$.

5. The Relativistic Feynman and Wiener Measures on $\Lambda^{\square}_{\square}$

Consider a spinless nonrelativistic quantum mechanical particle of mass m moving in the one-dimensional space $\underline{R}$ under the action of a potential $V(x, \dot{x})$. The quantum dynamics of the particle is given by

$$e^{-\frac{i}{\hbar}tH}$$

in the Hilbert space $L^2(\mathbb{R})$, where $\hbar$ is Plank's constant divided by 2π and $H = H_0 + V$, where

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}.$$

In his 1942 doctoral dissertation, R.P. Feynman (see [5]) picked up on an idea of P.M. Dirac [3] and transformed it into a novel, Lagrangian formulation of quantum mechanics. Let $L(x,\dot{x})$ be the **Lagrangian** of of the particle and let $S(x)$ be the **action** corresponding to $L(x,\dot{x})$, i.e.,

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x,\dot{x}), \quad S(x) = \int_0^t L(x,\dot{x})\,ds.$$

Feynman postulated that, formally,

$$(e^{-\frac{i}{\hbar}tH}\psi)(b) = \int_{x(t)=b} e^{\frac{i}{\hbar}S(x)}\,\psi(x(0))\,\mathcal{D}(x), \tag{1}$$

where the "integration" is over the space $C_0^{t,b}$ of all continuous paths $x : [0,t] \to \mathbb{R}$ such that $x(t) = b$. $\mathcal{D}(x)$ is supposed to be "Lebesgue measure" on $C_0^{t,b}$. Formally,

$$\mathcal{D}(x) = \mathcal{N} \prod_{0 \le s \le t} dx(s), \tag{2}$$

where $\mathcal{N}$ is a formal normalization factor. The main obstruction to making rigorous mathematical sense of the heuristic statement (1) is that the purely formal definition (2) can not be directly replaced by a mathematically well-defined σ-finite translation invariant Borel measure on $C_0^{t,b}$—simply because, as stated in the introduction, no such measure exists. More generally, the problem is to find a mathematically rigorous definition of "Feynman integrals" of the form

$$\int_{C_0^{t,b}} \exp\left\{i\sigma \int_0^t \dot{x}^2(s)\,ds\right\} f(x)\,\mathcal{D}(x), \tag{3}$$

where f is a complex-valued functional on $C_0^{t,b}$ and σ is a real parameter. Mathematicians have attacked this problem by different methods since the 1950's. For detailed up to date treatment of these efforts see [8]. If $\sigma = \frac{i}{2}$ in (3), then we obtain expressions of the form

$$\int_{C_0^{t,b}} \exp\left\{-\frac{1}{2}\int_0^t \dot{x}^2(s)\,ds\right\} f(x)\,\mathcal{D}(x).$$

A mathematically rigorous meaning can be assigned to expressions of this form by replacing the formal expression

$$\exp\Big\{-\frac{1}{2}\int_0^t \dot{x}^2(s)\,ds\Big\}\,\mathcal{D}(x)$$

by $dw_0^{t,b}(x)$, where $w_0^{t,b}$ is **Wiener measure** on $C_0^{t,b}$. The counterpart of (1) for the Wiener measure is the rigorous **Feynman-Kac formula** (see [9], Theorem 4.2, p. 268)

$$(e^{-\frac{1}{\hbar}tH}\psi)(b) = \int_{C_0^{t,b}} \exp\Big\{-\int_0^t V(x(s))\,ds\Big\}\,dw_0^{t,b}(x).$$

Now consider a spinless relativistic quantum mechanical particle of rest mass $m_0 > 0$ moving in the one dimensional space $\underline{\mathrm{R}}$ under the action of a potential $V(x,\dot{x})$. Let c be the velocity of light. One of the basic postulates of Einstein's special theory of relativity is that the velocity of the particle can not be greater than c. Thus, for all $t \geq 0$, we must have

$$|x(s)-x(r)| \leq c|s-r|, \quad 0 \leq r,s \leq t.$$

This implies that $x \in \Lambda_0^t$. The relativistic Lagrangian and relativistic action can be written as

$$L_{rel}(x,\dot{x}) = -m_0c^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - V(x,\dot{x}),$$

$$S_{rel}(x) = \int_0^t \Big(-m_0c^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - V(x,\dot{x})\Big)ds,$$

where we have used the fact that because $x \in \Lambda_0^t$, $\dot{x}$ exists a.e. on $[0,t]$. Therefore, in the relativistic case, if we replace $L(x,\dot{x})$ by $L_{rel}(x,\dot{x})$, and $C_0^{t,b}$ by a suitable subspace of paths $\mathcal{X} \subseteq \Lambda_0^{t,b}$, then (3) becomes a "relativistic Feynman integral" of the form

$$\int_{\mathcal{X}} \exp\Big\{i\sigma\int_0^t \sqrt{1-\frac{\dot{x}^2}{c^2}}\,ds\Big\} f(x)\,\mathcal{D}(x), \tag{4}$$

where $|b| \le c$, f is complex-valued on $\mathcal{X}$, and σ is a real parameter. A mathematically rigorous meaning can be assigned to expressions of the form (4) by interpreting defining $\mathcal{X}$ to be $\Lambda_0^{t,b}$ and defining $\mathcal{D}(x)$ to be **the uniform probability measure** $\mu_0^{t,b}(x)$ on $\Lambda_0^{t,b}$, obtaining expressions of the form

$$\int_{\Lambda_0^{t,b}} \exp\left\{ i\sigma \int_0^t \sqrt{1 - \frac{\dot{x}^2}{c^2}}\, ds \right\} f(x)\, d\mu_0^{t,b}(x). \tag{5}$$

The integral in (5) exists if σ is any complex parameter and f is any functional on $\Lambda_0^{t,b}$ such that

$$f \in L_1(\Lambda_0^{t,b}, \overset{\vee}{\mathcal{M}}{}_0^t, \mu_0^{t,b}).$$

This leads to the following definition.

Definition 5.1. Let $0 \le r < s$ be arbitrary and let $(a,b) \in L$ be arbitrary. Consider the following measure spaces defined in Definition 4.1:

$$(\Lambda_{r,a}^{s,b}, \mathcal{M}_r^s, \mu_{r,a}^{s,b}),\ (\Lambda_{r,a}^{s}, \overset{\wedge}{\mathcal{M}}{}_r^s, \mu_{r,a}^{s}),\ (\Lambda_{r}^{s,b}, \overset{\vee}{\mathcal{M}}{}_r^s, \mu_{r}^{s,b}),\ (\Lambda_{r,a}, \overset{\wedge}{\mathcal{M}}{}_r, \mu_{r,a}); \tag{6}$$

$$(\Lambda_{r}^{s}, \widetilde{\mathcal{M}}{}_r^s, \mu_{r}^{s}),\ (\Lambda_{r}, \widetilde{\mathcal{M}}{}_r, \mu_{r}). \tag{7}$$

Let $\lambda_{\square}^{\square}$ denote any of the measures on $\Lambda_{\square}^{\square}$ in (6) or (7). Let σ be an arbitrary real parameter. Then for any $f \in L_1(\Lambda_{\square}^{\square})$ we define **the relativistic Feynman integral** of f with respect to $\lambda_{\square}^{\square}$ to be the following integral:

$$\int_{\Lambda_{\square}^{\square}} \exp\left\{ i\sigma c^2 \int_r^s \sqrt{1 - \frac{\dot{x}^2}{c^2}}\, dt \right\} f(x)\, d\lambda_{\square}^{\square}(x).$$

For $\sigma \ge 0$, we define **the relativistic Wiener integral** of f with respect to $\lambda_{\square}^{\square}$ to be the following integral:

$$\int_{\Lambda_{\square}^{\square}} \exp\left\{ -\sigma c^2 \int_r^s \sqrt{1 - \frac{\dot{x}^2}{c^2}}\, dt \right\} f(x)\, d\lambda_{\square}^{\square}(x).$$

It is possible to define $\mathcal{X}$ in (4) in such a way the each $x \in \mathcal{X}$ has a continuous derivative $\dot{x}$ on $[0,t]$ and $\ddot{x}$ exists a.e. on $[0,t]$. This is done in the following manner. Let $a > 0$ be an arbitrary constant (a is the *maximum acceleration* of the particle on $[0,t]$). Define

$$\mathcal{Y} = \{ y \in C_0^t \mid \sup_{0 \le s \le t} |y(s)| \le c, \text{ and for all } r,s \in [0,t],\ |y(s) - y(r)| \le a|s-r| \}.$$

Define a $\psi : \mathcal{Y} \to \Lambda_0^{t,b}$ as follows: for $y \in \mathcal{Y}$,

$$\psi(y)(s) = b - \int_s^t y(u)\,du, \quad s \in [0,t].$$

Define $\mathcal{X} = \psi(\mathcal{Y})$. For $y \in \mathcal{Y}$, set $x = \psi(y)$. Then $\ddot{x}$ exists a.e. on $[0,t]$ and $|\,\ddot{x}\,| \le a$ a.e. on $[0,t]$. Using Theorem 3.1, it is easy to see that $\mathcal{Y}$ is compact, and hence $\mathcal{X}$ is a closed subspace of $\Lambda_0^{t,b}$, which gives that $\mathcal{X}$ is itself compact. It follows from these observations that $\psi : \mathcal{Y} \to \mathcal{X}$ is continuous and onto. Using the methods of sections (2)-(4), we can construct **uniform probability measure** ν on $\mathcal{Y}$. Define the regular Borel probability measure μ on $\mathcal{X}$ to be the continuous image of ν under the mapping ψ.

Definition 5.2. Let $\mathcal{X}$, $\mathcal{Y}$, μ, and ν be defined as above. Let σ be an arbitrary real parameter. Then for any $f \in L_1(\mathcal{X}, \mu)$ we define **the relativistic Feynman integral** of f with respect to μ to be the following integral:

$$\int_{\mathcal{X}} \exp\left\{ i\sigma c^2 \int_0^t \sqrt{1 - \frac{\dot{x}^2}{c^2}}\, ds \right\} f(x)\, d\mu(x).$$

For $\sigma \ge 0$, we define **the relativistic Wiener integral** of f with respect to $\mu$to be the following integral:

$$\int_{\mathcal{X}} \exp\left\{ -\sigma c^2 \int_0^t \sqrt{1 - \frac{\dot{x}^2}{c^2}}\, ds \right\} f(x)\, d\mu(x).$$

If we want the space $\mathcal{Y}$ to depend only on the acceleration $a > 0$, we first define a space of functions $\mathcal{Z}$ as follows:

$$\mathcal{Z} = \{\, y \in C_0^t \,|\, |y(s) - y(r)| \le a|s - r|, \text{ for all } r, s \in [0,t] \,\}.$$

Now define a ϕ on $\mathcal{Z}$ as follows: for $y \in \mathcal{Z}$,

$$\phi(y)(s) = b - \int_s^t y(u)\,du, \quad s \in [0,t].$$

Set $\mathcal{W} = \phi(\mathcal{Z})$. Then for any $x \in \mathcal{W}$, if $x = \phi(y)$, $y \in \mathcal{Z}$, $\dot{x}$ is continuous on $[0,t]$. Moreover, $\ddot{x}$ exists a.e. on $[0,t]$ and $|\,\ddot{x}\,| \le a$ a.e. on $[0,t]$. As in the case of the space $\mathcal{Y}$ above, we can construct the **uniform probability measure** ρ on $\mathcal{Z}$. Define the regular Borel probability measure η on $\mathcal{W}$ to be the continuous image of ρ under the mapping ϕ.

Definition 5.3. Let $\mathcal{W}$, $\mathcal{Z}$, η, and ρ be defined as above. Let σ be an arbitrary real parameter. Then for any $f \in L_1(\mathcal{W}, \eta)$ we define **the Feynman integral** of f with respect to η to be the following integral:

$$\int_{\mathcal{W}} \exp\left\{ i\sigma \int_0^t \dot{x}^2(s)\, ds \right\} f(x)\, d\eta(x).$$

For $\sigma \geq 0$, we define **the Wiener integral** of f with respect to η to be the following integral:

$$\int_{\mathcal{W}} \exp\left\{ -\sigma \int_0^t \dot{x}^2(s)\, ds \right\} f(x)\, d\eta(x).$$

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