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# DOUBLE PYRAMIDAL CENTRAL CONFIGURATION WITH HONEYCOMB BASE

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**Abstract:** In this paper,we discuss a double pyramidal central configuration with honeycomb base.

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**Key Words:** $N$-body problem, double pyramidal, central configuration, honeycomb base

## 1. Introduction

Celestial mechanics is a subject with a tremendously long and varied history. A very old problem in Celestial Mechanics is the study of central configurations of the $N$-body problem.One of the reasons why central configurations are interesting is that they allow to obtain explicit homographic solutions of the $N$-body problem (see [3, 7, 9, 11]).

The problem of finding the central configurations of a system of N particles in the plane has been the subject of many papers (see [1, 2, 4, 5, 6, 8, 10, 12, 13, 14]) during the last two hundred years. One of the reasons why central configurations are interesting is that they allow us to construct exact solutions of the $N$-body problem.

The classical Newtonian $N$-body problems study the motion of N point masses under the action of Newton's gravitation law of attraction. Let $q_i \in R^d, i = 1, \cdots, N$, be the positions of the bodies and $m_j > 0, j = 1, \cdots, N$, the

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respective masses.We assume that mass, time, and distance units are chosen in such a way that the masses $m_i, i = 1, \cdots, N$, the gravitational constant, and the period of the system are all equal to one. The motion of $N$ masses in the plane is described by the following nonlinear system of second order differential equations

$$\ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \quad i = 1, \cdots, N. \tag{1.1}$$

The function $U(q)$ is the potential function that generates an attractive force between each pair of bodies. In particular, we are interested in homogeneous Newtonian potential of degree $-1$ which is the most important in the celestial mechanics:

$$U(q) = U(q_1, \cdots, q_N) = \sum_{1 \leq i < j \leq N} \frac{1}{|q_i - q_j|}. \tag{1.2}$$

Excluding the singularities of the equations, the configuration space for the n masses is is the space of all distinct position vectors for which the center of mass is fixed at the origin, i.e.,:

$$X = \{q = (q_1, \cdots, q_n) \in R^2 : \sum_{i=1}^{N} m_i q_i = 0, q_i \neq q_j, \text{ for } i \neq j\}. \tag{1.3}$$

The configuration $q = (q_1, \cdots, q_n) \in X$ is called a central configuration if the acceleration vectors of the bodies satisfy:

$$\sum_{j=1, j \neq k}^{N} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k \tag{1.4}$$

for a constant $\lambda$. i.e., if the acceleration vector of every particle is directed towards the center of mass and its modulus is proportional to the distance from the particle to the center of mass. Furthermore $\lambda = \frac{U}{I}$ and $I$ is the moment of inertial of the system, i.e.

$$I = \sum_{i=1}^{N} m_i |q_i|^2. \tag{1.5}$$

Recently there are some interesting results in pyramidal central configurations (see [10, 13]), here we are interested in the double pyramidal central configuration for a 15-body problem, where the base of this central configuration is honeycomb-like. we consider the sufficient and necessary condition when the configuration is a central one for some mass vectors.

## 2. Some Results and their Proofs

We assume that the radiuses of the three circles with the same origin are $r_0$, $\sqrt{3}r_0$, $2r_0$ respectively, and the masses located in the same circle are at the vertices of a regular polygon. The regular polygon are regular triangle, regular hexagon and regular triangle from inner to outside. If we change perspective, the paragraph will be a honeycomb-like configuration.

Without loss of generality we can assume that

$$q_i = (4\cos\frac{2(i-1)\pi}{3}, 4\sin\frac{2(i-1)\pi}{3}, 0), \quad i = 1, 2, 3, \tag{2.1}$$

$$q_i = (2\sqrt{3}\cos(\frac{\pi}{6} + \frac{(i-4)\pi}{3}), 2\sqrt{3}\sin(\frac{\pi}{6} + \frac{(i-4)\pi}{3}), 0), \quad i = 4, \cdots, 9, \tag{2.2}$$

$$q_i = (2\cos(\frac{\pi}{3} + 2\frac{(i-10)\pi}{3}), 2\sin(\frac{\pi}{3} + 2\frac{(i-10)\pi}{3}), 0), \quad i = 10, 11, 12, \tag{2.3}$$

$$q_{13} = (0, 0, 0), \quad q_{14} = (0, 0, h), \quad q_{15} = (0, 0, -h), \tag{2.4}$$

where $h(h > 0)$ be the distance from $q_{14}$ or $q_{15}$ to the honeycomb-like base.

**Theorem 1.** *Under the assumptions of* (2.1) – (2.4), *if* $m_1, \cdots, m_{15}$ *form a double pyramidal central configuration with a honeycomb-like base, then* $m_{14} = m_{15}$.

*Proof.* Let

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1) \tag{2.5}$$

From $(1.4)_{k=1} \cdot \mathbf{k}$ and the assumptions of (2.1) – (2.4), we have:

$$\frac{hm_{14}}{(\sqrt{16 + h^2})^3} - \frac{hm_{15}}{(\sqrt{16 + h^2})^3} = 0 \tag{2.6}$$

then

$$m_{14} = m_{15}. \quad \square \tag{2.7}$$

**Theorem 2.** *Under the assumptions of* (2.1) – (2.4), *if* $m_1, \cdots, m_{15}$ *form a double pyramidal central configuration with a honeycomb-like base, then*

$$m_2 = m_3, m_4 = m_9, m_5 = m_8, m_6 = m_7, m_{10} = m_{12}. \tag{2.8}$$

*Proof.* Let

$$a=\frac{2}{(4\sqrt{3})^3},\quad b=\frac{1}{(\sqrt{28})^3},\quad c=\frac{1}{(\sqrt{52})^3}. \tag{2.9}$$

Under the assumptions of (2.1) – (2.4), by (1.4) we have

$$A\alpha=0, \tag{2.10}$$

where

$$\alpha=(m_1,m_2,\cdots,m_{11},m_{12})^T, \tag{2.11}$$

and

$$A=\begin{pmatrix}
0 & a & -a & \frac18 & 2b & c & -c & -2b & -\frac18 & 4a & 0 & -4a\\
4a & 0 & -4a & 8b & \frac12 & -\frac12 & -8b & -4c & 4c & 16a & -16a & 0\\
4a & -4a & 0 & 4c & -4c & -8b & -\frac12 & \frac12 & 8b & 0 & -16a & 16a\\
\frac12 & -8b & 4c & 0 & -24a & -\frac{1}{36} & 0 & \frac{1}{36} & 24a & -\frac14 & -2b & \frac{1}{16}\\
4b & -\frac14 & -2c & 12a & 0 & -12a & -\frac{1}{72} & 0 & \frac{1}{72} & \frac18 & -\frac{1}{32} & b\\
4c & \frac12 & -8b & \frac{1}{36} & 24a & 0 & -24a & -\frac{1}{36} & 0 & \frac{1}{16} & -\frac14 & -2b\\
4c & -8b & \frac12 & 0 & -\frac{1}{36} & -24a & 0 & 24a & \frac{1}{36} & -2b & -\frac14 & \frac{1}{16}\\
4b & -2c & -\frac14 & \frac{1}{72} & 0 & -\frac{1}{72} & -12a & 0 & 12a & b & -\frac{1}{32} & \frac18\\
\frac12 & 4c & -8b & 24a & \frac{1}{36} & 0 & -\frac{1}{36} & -24a & 0 & \frac{1}{16} & -2b & -\frac14\\
16a & -16a & 0 & \frac14 & -\frac14 & -\frac{1}{16} & -2b & 2b & \frac{1}{16} & 0 & -8a & 8a\\
8a & 0 & -8a & b & \frac{1}{32} & \frac18 & -\frac18 & -\frac{1}{32} & -b & 4a & 0 & -4a\\
16a & 0 & -16a & \frac{1}{16} & 2b & -2b & -\frac{1}{16} & -\frac14 & \frac14 & 8a & -8a & 0\\
\frac{1}{16} & -\frac{1}{32} & \frac{1}{32} & 12a & 0 & -12a & -12a & 0 & 12a & \frac18 & -\frac14 & \frac18\\
0 & \frac{1}{32} & -\frac{1}{32} & 4a & 8a & 4a & -4a & -8a & -4a & \frac18 & 0 & -\frac18
\end{pmatrix}. \tag{2.12}$$

With the elementary row transformation of matrix $A$, we can obtain the following matrix equation:

$$B\beta=0, \tag{2.13}$$

where

$$\beta=(m_2-m_3,m_4-m_9,m_5-m_8,m_6-m_7,m_{10}-m_{12})^T, \tag{2.14}$$

and

$$B=\begin{pmatrix}
a & \frac18 & 2b & c & 4a\\
-4a & 4c-8b & -4c-\frac12 & \frac12-8b & -16a\\
-8b-\frac12 & -\frac{1}{36} & -\frac{1}{36}-24a & -24a & -2b-\frac{1}{16}\\
\frac14-2c & \frac{1}{72}-12a & 0 & 12a-\frac{1}{72} & b-\frac18\\
4c+8b & 24a & \frac{1}{36}+24a & \frac{1}{36} & \frac{5}{16}.
\end{pmatrix}. \tag{2.15}$$

With the simple computation by *MATLAB*, we have

$$\det(B) \neq 0 \tag{2.16}$$

thus (2.13) has the unique trivial solution $\beta = (0, 0, 0, 0, 0)$, i.e.

$$m_2 = m_3, \quad m_4 = m_9, \quad m_5 = m_8, \quad m_6 = m_7, \quad m_{10} = m_{12}. \quad \square \tag{2.17}$$

**Theorem 3.** *Under the assumptions of* (2.1) – (2.4), *if* $m_1, \cdots, m_{15}$ *form a double pyramidal central configuration with a honeycomb-like base, then*

$$m_1 = m_2 = m_3, m_4 = m_5 = \cdots = m_9, m_{10} = m_{11} = m_{12}. \tag{2.18}$$

*Proof.* Let

$$a = \frac{2}{(4\sqrt{3})^3}, \quad b = \frac{1}{(\sqrt{28})^3}, \quad c = \frac{1}{(\sqrt{52})^3}. \tag{2.19}$$

By (2.1) – (2.4), the matrix $A$ and the system (1.4), similar to the proof of Theorem 2.2, under the elementary transformation of matrix $A$ we have

$$C\gamma = 0, \tag{2.20}$$

where

$$\gamma = (m_1 - m_3, m_4 - m_7, m_5 - m_6, m_8 - m_9, m_{10} - m_{11})^T, \tag{2.21}$$

and

$$C = \begin{pmatrix} 4a & 8b & \frac{1}{2} & -4c & 16a \\ 4a & 4c + \frac{1}{2} & 8b - 4c & \frac{1}{2} - 8b & 16a \\ 16a & \frac{1}{16} & 2b & -\frac{1}{4} & 8a \\ 16a & 2b + \frac{1}{4} & -\frac{3}{16} & 2b - \frac{1}{16} & 8a \\ \frac{1}{16} & 0 & 24a & -24a & \frac{1}{4}. \end{pmatrix}. \tag{2.22}$$

With the simple computation by *MATLAB*, we have

$$det(C) \neq 0, \tag{2.23}$$

thus (2.20) has the unique trivial solution $\gamma = (0, 0, 0, 0, 0)$, i.e.

$$m_1 = m_3, \quad m_4 = m_7, \quad m_5 = m_6, \quad m_8 = m_9, \quad m_{10} = m_{11}. \quad \square \tag{2.24}$$

From Theorem 2.2 we have $m_2 = m_3$, $m_4 = m_9$, $m_5 = m_8$, $m_6 = m_7$, $m_{10} = m_{12}$, then

$$m_1 = m_2 = m_3, \quad m_4 = m_5 = \cdots = m_9, \quad m_{10} = m_{11} = m_{12}. \tag{2.25}$$

In the following we give a numerical result based on the computation by *MATLAB*.

**Theorem 4.** *Under the assumptions of* (2.1) – (2.4), *and let* $m_1 = m_2 = m_3 = \mu_1$, $m_4 = m_5 = \cdots = m_9 = \mu_2$,$m_{10} = m_{11} = m_{12} = \mu_3$, *then for suitable* $h \in (2.5620, 3.2176)$, $m_1, \cdots, m_{15}$ *form a double pyramidal central configuration with a honeycomb-like base.*

*Proof.* Under the assumptions of (2.1)−(2.4) and the respective masses from above, we can obtain the following matrix equations about $\chi = (\mu_1, \mu_2, \cdots, \mu_5, \lambda)^T$:

$$D\chi = 0, \tag{2.26}$$

where

$$D = \begin{pmatrix} \frac{1}{16\sqrt{3}} & \frac{1}{4} + \frac{1}{7\sqrt{7}} + \frac{7}{52\sqrt{13}} & \frac{1}{4\sqrt{3}} + \frac{1}{36} & \frac{1}{16} & \frac{8}{\sqrt{16+h^2}^3} & -4 \\ \frac{1}{8} - \frac{1}{56\sqrt{7}} + \frac{3}{104\sqrt{13}} & \frac{1}{72} + \frac{5}{96\sqrt{3}} & \frac{1}{32} + \frac{1}{56\sqrt{7}} & \frac{1}{24\sqrt{3}} & \frac{2}{\sqrt{12+h^2}^3} & -1 \\ \frac{1}{4} - \frac{1}{14\sqrt{7}} + \frac{1}{52\sqrt{13}} & 0 & -\frac{3}{32} - \frac{1}{56\sqrt{7}} & 0 & 0 & 0 \\ \frac{1}{36} & -\frac{3}{16} + \frac{5}{28\sqrt{7}} & \frac{1}{4\sqrt{3}} & \frac{1}{4} & \frac{4}{\sqrt{4+h^2}^3} & -2 \\ \frac{3}{\sqrt{16+h^2}^3} & \frac{6}{\sqrt{12+h^2}^3} & \frac{3}{\sqrt{4+h^2}^3} & \frac{1}{h^3} & \frac{1}{4h^3} & -1 \end{pmatrix}. \tag{2.27}$$

By straightforward computations one finds that the rank of $D$ i.e. $r(D) < 6$, then there exists nontrivial solutions for the matrix equations (1.4).

In the following we give numerical evidence that the matrix equations have the positive solutions (see Table 1).

From above, we can find that when we choose some suitable $h \in (2.5620, 3.2176)$, $m_1, \cdots, m_{15}$ form a double pyramidal central configuration with a honeycomb-like base.

These numerical results provide us evidence that for all $h \in (2.5620, 3.2176)$, $m_1, \cdots, m_{15}$ form a double pyramidal central configuration with a honeycomb-like base. □

| h | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\mu_5$ | $\lambda$ |
|---|---|---|---|---|---|---|
| 2.5618 | 1.1408 | 5.9571 | 2.5919 | -0.0010 | 19.8158 | 1.0000 |
| 2.5619 | 1.1409 | 5.9572 | 2.5921 | 0.0000 | 19.8189 | 1.0000 |
| 2.5620 | 1.1409 | 5.9573 | 2.5923 | 0.0011 | 19.8179 | 1.0000 |
| 2.6000 | 1.1721 | 5.9869 | 2.6630 | 0.4042 | 19.4168 | 1.0000 |
| 2.7000 | 1.2724 | 6.0933 | 2.8908 | 1.5190 | 18.0086 | 1.0000 |
| 2.8000 | 1.3994 | 6.2428 | 3.1794 | 2.7070 | 16.0380 | 1.0000 |
| 2.9000 | 1.5534 | 6.4383 | 3.5293 | 3.9634 | 13.4347 | 1.0000 |
| 3.0000 | 1.7347 | 6.6829 | 3.9412 | 5.2853 | 10.1198 | 1.0000 |
| 3.1000 | 1.9435 | 6.9793 | 4.4156 | 6.6714 | 6.0049 | 1.0000 |
| 3.2000 | 2.1802 | 7.3308 | 4.9535 | 8.1222 | 0.9894 | 1.0000 |
| 3.2176 | 2.2248 | 7.3986 | 5.0549 | 8.3844 | 0.0051 | 1.0000 |
| 3.2177 | 2.2508 | 7.4384 | 5.1139 | 8.5358 | -0.5742 | 1.0000 |

Table 1

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