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A GENERATING SET OF THE AUTOMORPHISM GROUP OF FREE NILPOTENT LIE ALGEBRA OF CLASS 4 AND RANK 2

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Abstract: Let $Aut\,(L_{2,4})$ be the automorphism group of free nilpotent Lie algebra of class 4 and rank 2. We find a generating set of this $Aut\,(L_{2,4})$ group.

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1. Introduction

In 1990 Drensky and Gupta [7] studied a generating set of free nilpotent Lie algebra and free metabelian Lie algebra. Papistas [13] find a generating set of free nilpotent Lie algebra of rank $n \geq 2$ and over rational field. Let F_n, be the free Lie algebra of rank n generated by $X = \{x_1, x_2, ..., x_n\}$ $(n \geq 2)$ over a field K of characteristic zero. Cohn [6] proved that the automorphism group of F_n is generated by elementary automorphisms. This result is similar to the well-known result of Nielsen [11]. Bachmuth [3] is studied generators of automorphism group of free group and free metabelian group. Let $F_{n,c}$ be the free nilpotent group of class c and rank n. Goryaga at [8] for $n \geq 3 \cdot 2^{c-2} + c$ and $c \geq 2$, give a set of c generators for $Aut(F_{n,c})$. Andreadakis [2] obtained a simple set of $Aut(F_{n,c})$ under the condition $n \geq c \geq 2$. Bryant and Gupta [4] gave a minimum set of generators for the $Aut(F_{n,c})$ under the condition that

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$n \geq c-1 \geq 2$. Drensky and Gupta [7] showed that the automorphism group of free nilpotent Lie algebra with the condition $n \geq c \geq 2$ is generated by GL_n and

$$\delta : \begin{array}{l} x_1 \to x_1 + [x_1, x_2] \\ x_k \to x_k, \quad k \geq 2 \end{array}$$

In this study we give a generating set of free nilpotent Lie algebra of class 4 and rank 2.

2. A Generating Set of Aut$(L_{2,4})$

Definition 2.1. Let F_n be the free Lie algebra generated by $X = \{x_1, ..., x_n\}$ over a constructive field K of characteristic zero. For any positive integer n, c, let $L_{n,c} = F_n \diagup \gamma_{c+1}(F_n)$ where $\gamma_{c+1}(F_n)$ is the $(c+1)$th term of the lower central series of F_n. Then $L_{n,c}$ is a free nilpotent-of-class-c Lie algebra with rank n.

Since $\gamma_{c+1}(F_n) \lhd \gamma_c(F_n)$ there is

$$(F_n \diagup \gamma_c(F_n)) \cong (F_n \diagup \gamma_{c+1}(F_n)) \diagup (\gamma_c(F_n) \diagup \gamma_{c+1}(F_n))$$

from this $L_{n,c-1} \cong L_{n,c} \diagup I$ where $I = (\gamma_c(F_n) \diagup \gamma_{c+1}(F_n))$. So there is a natural homomorphism $L_{n,c} \longrightarrow L_{n,c-1}$. So an automorphism of $L_{n,c}$ determine one automorphism of $L_{n,c-1}$. Since algebra $L_{n,c-1}$ is isomorphic to algebra $L_{n,c} \diagup \gamma_c(L_{n,c})$ there is a natural epimorphism $\phi : Aut(L_{n,c}) \longrightarrow Aut(L_{n,c-1})$. The kernel of this epimorphism occurs special automorphisms.

By $\ker\phi$ we denote the kernel of ϕ homomorhism and we define the following automorphisms.

$$\zeta_{12} : \begin{array}{l} x_1 \to x_1 + [x_1, x_2] \\ x_2 \to x_2 \end{array} \text{ and } \zeta_{121} : \begin{array}{l} x_1 \to x_1 + [x_1, x_2, x_1] \\ x_2 \to x_2 \end{array}$$

$$\zeta_{i_1 i_2 i_3 i_4} : \begin{array}{l} x_1 \to x_1 + [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] \\ x_2 \to x_2 \end{array}, \text{ where } i_1 \neq i_2 \text{and } i_1, i_2, i_3, i_4 \in \{1, 2\}$$

Lemma 2.2. *The kernel of homomorphism $\phi : Aut(L_{n,c}) \longrightarrow Aut(L_{n,c-1})$ consist of all central automorphisms of $L_{n,c}$ and it is generated by all automorphisms of the form $\varsigma^{-1}\zeta_{i_1 i_2 \ldots i_c}\varsigma$, where $\varsigma \in GL_n$ and*

$$\zeta_{i_1 \ldots i_c} : \begin{array}{l} x_1 \to x_1 + [x_{i_1}, x_{i_2}, ..., x_{i_n}] \\ x_j \to x_j, \quad j \neq 1. \end{array}$$

Proof. See [12].

Theorem 2.3. $Aut\,(L_{2,1}) = \langle GL_2 \rangle$.

Theorem 2.4. $Aut\,(L_{2,2}) = \langle GL_2, \zeta_{12} \rangle$.

Proof. The proof is obtained by the result of [6].

Theorem 2.5. $Aut\,(L_{2,3}) = \langle GL_2, \zeta_{12}, \zeta_{121} \rangle$.

Proof. See [12].

Theorem 2.6. $Aut\,(L_{2,4}) = \langle GL_2, \zeta_{12}, \zeta_{121}, \zeta_{1212} \rangle$.

Proof. Let $\phi : Aut\,(L_{2,4}) \to Aut\,(L_{2,3})$ be the natural epimorphism. Note that $\phi\,(GL_2) = GL_2$ and $\phi\,(\zeta) = \overline{\zeta}$, where $\overline{\zeta}$ is defined for $L_{2,3}$ in the same way as ζ is defined for $L_{24,}$. Thus $Aut\,(L_{2,4})$ is generated by GL_2 and $\ker\phi$. Let $H = \langle GL_2, \zeta_{12}, \zeta_{121}, \zeta_{1212} \rangle$. We shall prove that $\ker\phi \subseteq H$

Let $\alpha \in Aut\,(L_{2,4})$. Then it is of the form

$$\alpha : \begin{array}{l} x_1 \to ax_1 + bx_2 + c\,[x_1, x_2] + d\,[[x_1, x_2]\,, x_{i_3}] + e\,[[x_1, x_2]\,, x_1, x_1] \\ \quad + f\,[[x_1, x_2]\,, x_1, x_2] + g\,[[x_1, x_2]\,, x_2, x_1] + h\,[[x_1, x_2]\,, x_2, x_2] \end{array}$$

$$\begin{array}{l} x_2 \to a'x_1 + b'x_2 + c'\,[x_1, x_2] + d'\,[[x_1, x_2]\,, x_{i_3}] + e'\,[[x_1, x_2]\,, x_1, x_1] \\ \quad + f'\,[[x_1, x_2]\,, x_1, x_2] + g'\,[[x_1, x_2]\,, x_2, x_1] + h'\,[[x_1, x_2]\,, x_2, x_2] \end{array}$$

where $i_3, i_4 \in \{1, 2\}$ and $a, b, c, d, e, f, g, h, a', b', c', d', e', f', g', h' \in K$ with $ab' - ba' \neq 0$.

If $\alpha \in \ker\phi$ then $a' = c' = d' = 0$, $b = c = d = 0, a = b' = 1$. Thus

$$\alpha : x_1 \to ax_1 + \sum_{i_3, i_4 = 1,2} e_{i_3,i_4}[[[x_1, x_2]\,, x_{i_3}]\,, x_{i_4}]x_2$$

$$\to a'x_1 + \sum_{i_3, i_4 = 1,2} e'_{i_3,i_4}[[[x_1, x_2]\,, x_{i_3}]\,, x_{i_4}],$$

where $e_{i_3,i_4}, e'_{i_3,i_4} \in K$. Now we show $\alpha \in H$.From Lemma 2.1, it is obvious that α can be written by the automorphisms of the form $\delta^{-1}\zeta_{i_1 i_2 i_3 i_4}\delta$, where and $\delta \in GL_2$. It is enough to show that the automorphisms

$$\zeta_{i_1 i_2 i_3 i_4} \text{ and } \zeta^{-1}_{i_1 i_2 i_3 i_4} \; : \; \begin{array}{l} x_1 \to x_1 - [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] \\ x_2 \to x_2 \end{array}$$

can be written by $\zeta_{12}, \zeta_{121}, \zeta_{1212}$ and automorphisms from GL_2. For

$$\omega : \begin{array}{l} x_1 \to x_1 \\ x_2 \to x_2 + x_1 \end{array}, \psi : \begin{array}{l} x_1 \to x_1 \\ x_2 \to -x_2 \end{array} \; \tau : \begin{array}{l} x_1 \to x_1 + x_2 \\ x_2 \to x_1 \end{array} \; \upsilon : \begin{array}{l} x_1 \to x_2 \\ x_2 \to x_1 \end{array},$$

$$\mu_{-1}: \begin{array}{l} x_1 \to -x_1 \\ x_2 \to x_2 \end{array}, \quad \upsilon: \begin{array}{l} x_1 \to x_1 + x_2 \\ x_2 \to x_2 \end{array}$$

We have:

$$\begin{array}{ll}
\zeta_{1212}^{-1}=\mu_{-1}\zeta_{1212}\mu_{-1}, & \zeta_{1222} = [\zeta_{1212}, \upsilon^{-1}], \\
\zeta_{1221} = \zeta_{1212}, & \zeta_{1221}^{-1} = \mu_{-1}\zeta_{1221}\mu_{-1} = \mu_{-1}\zeta_{1212}\mu_{-1}, \\
\zeta_{1211} = [\omega^{-1}, \zeta_{1221}^{-1}] = [\omega^{-1}, \zeta_{1212}^{-1}] & \zeta_{2112} = \mu_{-1}\zeta_{1212}\mu_{-1}, \\
\zeta_{2112} = \mu_{-1}\zeta_{1221}\mu_{-1} = \mu_{-1}\zeta_{1212}\mu_{-1}, & \zeta_{2122} = \zeta_{1222}^{-1} = \upsilon^{-1}\zeta_{1212}\upsilon\zeta_{1212}^{-1}, \\
\zeta_{1211}^{-1} = \psi\zeta_{1211}\psi = \psi\left[\omega^{-1}, \zeta_{1212}^{-1}\right]\psi, & \zeta_{2111} = \psi\zeta_{1211}\psi = \psi\left[\omega^{-1}, \zeta_{1212}^{-1}\right]\psi.
\end{array}$$

Hence $\alpha \in H$. Thus $\ker\phi \subseteq H$. From here

$$Aut\,(L_{2,4}) = \langle GL_2, \zeta_{12}, \zeta_{121}, \ker\phi\rangle \subseteq \langle GL_2, \zeta_{12}, \zeta_{121}, \zeta_{1212}\rangle$$

and

$$Aut\,(L_{2,4}) = \langle GL_2, \zeta_{12}, \zeta_{121}, \zeta_{1212}\rangle\,.$$

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