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ON $k\omega$-OPEN SETS AND $k\omega$-CONTINUOUS FUNCTIONS

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Abstract: In the present paper we introduce a new type of sets, called $k\omega$-open sets and we investigate some basic properties and their relationships with some other types of sets.

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1. Introduction and Preliminaaries

Hdeib [10] introduced the notions of ω-closed sets and ω-closed mappings in 1982 and ω-continuous mappings [11]in 1989. Then K.Y. Al-Zoubi [2] introduced generalized ω-closed sets in topological spaces. In continuation, Ahmad Al Omari and Mohd Salmi Md Noorani [1] provided a relatively new notion of generalized closed set, namely, regular generalized ω-closed sets, regular generalized ω-continuous mapping, a-ω continuous mapping and regular generalized w-maps in 2007. K. Chandrasekhara Rao and D. Narasimhan [5, 6] introduced semi star generalized ω-closed sets and semi star generalized ω-continuity in topological spaces.

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Let (X, τ) or simply X denote a topological space. The closure and interior of B relative to A is written as $cl_B(A)$ and $int_B(A)$ respectively. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed if it contains all its condensation points.

The complement of ω-closed set is called ω-open. It is well known that a subset A of a space (X, τ) is ω-open if and only if for each $x \in A$, there exists $U \in \tau$ such that $x \in U$ and $U \cap A^C$ is countable. The family of all ω-open subsets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in a manner similar to $cl(A)$ and $int(A)$, respectively, will be denoted by $cl_\omega(A)$ and $int_\omega(A)$, respectively. The ω-closure and ω-interior of A relative to B can be defined in a similar manner, will be denoted by $cl_\omega(A_B)$ and $int_\omega(A_B)$, respectively. A^C denotes the complement of A in X unless explicitly stated. We shall require the following known definitions.

Definition 1.1. A set A of a topological space (X, τ) is called:

(a) *generalized ω-closed* ($g\omega$-closed) if $cl_\omega(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X,

(b) *generalized ω-open* ($g\omega$-open) if $X - A$ is $g\omega$-closed.

(c) *regular generalized ω-closed* ($rg\omega$-closed) if $cl_\omega(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X,

(d) *regular generalized ω-open* ($rg\omega$-open) if $X - A$ is $rg\omega$-closed.

(e) semi star generalized ω-closed ($s^*g\omega$-closed) if $cl_\omega(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

(f) *semi star generalized ω-open* ($s^*g\omega$-open) if $X - A$ is $s^*g\omega$-closed.

2. $k\omega$-Open Sets

Definition 2.1. A subset A of a topological space (X, τ) is called $k\omega$-open set if it is both open and ω-compact subset of X. The complement of $k\omega$-open sets is called $k\omega$-closed sets.

It is easy to see that each $k\omega$-open set is open (resp. ω-open, α-open, β-open, semi-open, pre-open, b-open). But the converse is not true in general. The following example supports our claim.

Example 2.2. Consider the usual topological space (R, τ_U). Clearly R is an open subset of R and hence it is ω-open, α-open, β-open, semi-open, pre-open, b-open. But it is not compact and hence it is not ω-compact. Therefore, it is not $k\omega$-open.

Corollary 2.3. *A subset H of a topological space (X, τ) is $k\omega$-closed if and only if H is closed and for each family $\{F_\lambda / \lambda \in \Lambda\ \}$of ω-closed subsets of X such that $\bigcap_{\lambda \in \Lambda} F_\lambda \subseteq H$, then there exists a finite subset Λ_H of Λ such that $\bigcap_{\lambda \in \Lambda_H} F_\lambda \subseteq H$.*

The following diagram is obtained from the definition of $k\omega$-closed sets:

$$
\begin{array}{llll}
 & s^*g\omega\text{-closed} \Rightarrow & g\omega\text{-closed} \Rightarrow & rg\omega\text{-closed} \\
 & \Uparrow & \Uparrow & \Uparrow \\
 & \omega\text{-closed} & g\text{-closed} \Rightarrow & rg\text{-closed} \\
 & \Uparrow & \Uparrow & \\
k\omega\text{-closed} \Rightarrow & \text{closed} \Rightarrow & s^*g\text{-closed} \Rightarrow & gs\text{-closed} \\
 & \Downarrow & \Downarrow & \Uparrow \\
 & \text{semi closed} \Rightarrow & sg\text{-closed} &
\end{array}
$$

Lemma 2.4. The finite union (intersection) of ω-compact subsets of a space (X, τ) is an ω-compact subset.

Corollary 2.5. (1) The finite union (intersection) of $k\omega$-open sets is also an $k\omega$-open set.

(2) The finite union (intersection) of $k\omega$-closed sets is also an $k\omega$-closed set.

The following example shows that the infinite union of $k\omega$-open sets need not be an $k\omega$-open set.

Example 2.6. Consider the discrete topology τ_{dis} on the set of all natural numbers N, then $\{\{n\}/n \in N\}$ is a an infinite family of $k\omega$-open subsets of N, but $N = \bigcup_{n \in N} \{n\}$, is not ω-compact, hence it is not $k\omega$-open.

Question: Is the infinite intersection of $k\omega$-open set also an $k\omega$-open set?

Theorem 2.7. Let (X, τ) be a T_2-space. Then every $k\omega$-open subset of X is clopen.

Proof. Let A be a $k\omega$-open subset of X. Then A is both open and $k\omega$-compact. Since every $k\omega$-compact subset is compact, we have A is compact subset of X. Since compact subset of T_2-space is closed, we have A is closed. Hence it is clopen. □

Remark 2.8. It is not necessary that any hereditary compact space to be ω-compact? That is it is not necessary that to be a hereditary compact space to be a hereditary ω-compact. As the following Example shows.

Example 2.9. Consider the co-finite topology τ_{cof} on the set of all natural numbers N, then (N, τ_{cof}) is a hereditary compact space, but N is not ω-compact.

Theorem 2.10. Every hereditary compact T_2-space is an ω-space.

Proof. Let (X, τ) be a hereditary compact T_2-space. Let A be an ω-open subset of X. Let $x \in V$. Then there exist an open subset G and a countable subset C of X such that $x \in G - C \subseteq A$. Hence $x \notin C$. Since X is hereditary compact space, we have C is compact. Since a compact subset of a T_2-space can be separated by two disjoint open sets, there exists two disjoint open sets U and V such that $x \in U \subseteq V^C \subseteq C^C$. Consequently, $x \in G \cap U \subseteq G \cap V^C \subseteq G \cap C^C$. Hence A is open in X. Therefore, X is an ω-space. □

Remark 2.11. If we take a look at the above examples, it is easy to see that (N, τ_{cof}) is a hereditary compact non T_2-space which is not ω-space. However, the space (R, τ_U) is a T_2-space which is not hereditary compact but it is not ω-space. This means that the conditions hereditary compact and T_2-space can not be dropped in the Theorem 2.10.

Theorem 2.12. There is no locally countable infinite space (X, τ) for which the set of all $k\omega$-open sets form a topology on X.

Proof. Let (X, τ) be a locally countable infinite space, we have $\tau^\omega = \tau_{dis}$. Since X is an infinite set, so it is not ω-compact. Therefore, it is not an $k\omega$-open set. Thus the set of all $k\omega$-open sets does not form a topology on X. □

Corollary 2.13. If X is any countable infinite set and τ any topology on X, then X is not a $k\omega$-open set.

Corollary 2.14. There is no hereditary compact T_2-space (X, τ) such that X is a countable infinite set.

Proof. Suppose that there is a hereditary compact T_2-space (X, τ) such that X is a countable infinite set. Then by Theorem 2.10, (X, τ) is an ω-space. That is, $\tau = \tau^\omega$. But since X is a countable infinite set, then (X, τ) is locally-countable space. Therefore, $\tau^\omega = \tau_{dis}$. Thus $\tau = \tau_{dis}$. But since (X, τ) is hereditary compact, then (X, τ_{dis}) is compact space which is impossible. □

The following result contains a sufficient condition that to make the collection of all $k\omega$-open subsets of a topological space (X, τ) form a topology on X.

Theorem 2.15. If (X, τ) is a finite or anti-locally countable hereditary compact T_2-space, then the collection of all $k\omega$-open subsets of X form a topology on X.

Proof. If X is finite, then the theorem followed in virtue of the fact that every finite space is hereditary ω-compact. If X is an anti-locally countable hereditary compact T_2-space, then theorem is followed by using Theorem 2.10. $\square$

Theorem 2.16. Let (X, τ) be any topological space and $A \subseteq Y \subseteq X$. If A is a $k\omega$-open subset of X, then A is a $k\omega$-open subset of Y.

Proof. Let A be a $k\omega$-open subsets of X and $A \subseteq Y \subseteq X$. Then A is open and ω-compact subset of X. Since $A = A \cap Y$, we have A is an open subset of Y. Since A is an ω-compact subset of X, then in view of Theorem [2, part b of Lemma 3.1], it is easy to see that A is an ω-compact subset of Y. Therefore, A is a $k\omega$-open subset of Y. $\square$

The following example shows that the converse of the above theorem is not true in general.

Example 2.17. Consider the topological space (N, τ_{cof}), and suppose that Y is any finite non-empty subset of N, then the induced topology $(\tau_{cof})_Y = \tau_{dis}$. Hence each subset of Y is a $k\omega$-open subset of Y. Especially, Y is a $k\omega$-open subset of itself. But the only $k\omega$-open subset of N is the empty set.

Remark 2.18. It is easy to see that in any finite (or anti-locally countable hereditary compact τ_2-space, the converse of the above theorem is true.

Definition 2.19. Let (X, τ) be a topological space and $A \subseteq X$, the a point x in A (resp. in X) is called the $k\omega$-interior (resp. $k\omega$-closure) point of A, if there exists a $k\omega$-open set (resp. for each $k\omega$-open set) G such that $x \in G$, then $G \subseteq A$ (resp. $G \cap A \neq \phi$). The set of all $k\omega$-interior (resp. $k\omega$-closure) points of A is called the $k\omega$-interior (resp, $k\omega$-closure) of A and will be denoted by $k\omega$-int(A) (resp. $k\omega$-Cl(A)).

Theorem 2.20. Let (X, τ) be a topological space and A, $B \subseteq X$, then the following statements hold.

(1) $k\omega\text{-int}(A) = \bigcup\{G/G \text{ is } k\omega\text{-open and } G \subseteq A\}$.

(2) $k\omega$-Cl$(A) = \bigcap\{F/F$ is $k\omega$-closed and $A \subseteq F$ $\}$.

(3) $k\omega$-int$(A) \subseteq$ int(A) and Cl$(A) \subseteq k\omega$-Cl(A).

(4) $k\omega$-int(A) is open and $k\omega$-Cl(A) is closed.

(5) $k\omega$-int$(A \cap B)$= $k\omega$-int$(A) \cap k\omega$-int(B) and
$k\omega$-Cl$(A \cup B)$ = $k\omega$-Cl$(A) \cup k\omega$-Cl(B).

(6) $k\omega$-int$(A \cup B) \subseteq k\omega$-int$(A) \cup k\omega$-int$(B)$ and
$k\omega$-Cl$(A \cap B) \subseteq k\omega$-Cl$(A) \cap k\omega$-Cl$(B)$.

(7) $k\omega$-Cl$(X - A) = X - [k\omega$-int$(A)]$ and $k\omega$-int$(X - A) = X - [k\omega$-Cl$(A)]$.

Proof. Straightforward. □

Remark 2.21. (1) The empty set is always $k\omega$-open.

(2) The whole space is always $k\omega$-closed.

(3) The $k\omega$-int(A) need not be $k\omega$-open.

(4) The empty set is not necessary that to be $k\omega$-closed.

(5) $k\omega$-Cl(A) is not necessary $k\omega$-closed.

The proof of first part of the above remark is clear. However, the following example clarifies each of the other parts of the same remark.

Example 2.22. Let (N, τ_{dis}) be a topological space which is given in Example 2.6.

(2) Since $k\omega$-int$(N) = N$, but N is not $k\omega$-open, so $k\omega$-int(N) is not $k\omega$-open.

(3) Since N is not $k\omega$-open, then the empty set is not $k\omega$-closed.

(4) Since $k\omega$-Cl$(\phi) = \phi$ and ϕ is not $k\omega$-closed by part (2), so $k\omega$-Cl(ϕ) is not $k\omega$-closed.

3. $k\omega$-Continuous Functions

Definition 3.1. A function $f{:}(X,\tau) \to (Y,\sigma)$ is said to be a $k\omega$-continuous function at a point x in X, if for each open subset G in Y containing $f(x)$, there exists a $k\omega$-open subset U of X containing x such that $f(U) \subseteq G$. Also f is called $k\omega$-continuous function if it is $k\omega$-continuous at each points x of X.

Theorem 3.2. A function $f{:}(X,\tau) \to (Y,\sigma)$ is $k\omega$-continuous, if the pre-image of each open set V in Y is a $k\omega$-open set in X.

Proof. Suppose that the condition of the theorem holds. Let $x \in X$ and U be any open set in Y containing $f(x)$, then $f^{-1}(U)$ is a $k\omega$-open set in X. Since $f(x) \in U$, we have $x \in f^{-1}(U)$ and $f(f^{-1}(U)) \subseteq U$, hence f is a $k\omega$-continuous function. □

Corollary 3.3. A function $f{:}(X,\tau) \to (Y,\sigma)$ is $k\omega$-continuous, if the pre-image of each closed set V in Y is a $k\omega$-closed set in X.

The following example shows that the converse of the above theorem and Corollary are not true in general.

Example 3.4. Let $f{:}(N,\tau_{dis}) \to (N,\tau_{dis})$ be the identity function, then it is clear that f is $k\omega$-continuous function but $f^{-1}(N)$=N is not $k\omega$-open in N and $f^{-1}(\phi) = \phi$ is not $k\omega$-closed in N.

The following result is a simple characterization of $k\omega$ -continuous functions.

Theorem 3.5. A function $f{:}(X,\tau) \to (Y,\sigma)$ is $k\omega$-continuous at a point x in X if and only if for each closed subset F of Y not contains $f(x)$, there exists a $k\omega$-closed subset H of X which is not containing x such that $f^{-1}(F) \subseteq H$.

Proof. Let $f{:}(X,\tau) \to (Y,\sigma)$ be a $k\omega$-continuous function at a point x in X, and let F be any closed subset of Y such that $f(x) \notin F$. Then $Y - F$ is an open subset of Y containing $f(x)$, so by $k\omega$-continuity of f at the point x, there exists a $k\omega$-open set U contains x in X such that $f(U) \subseteq Y - F$. Hence $U \subseteq f^{-1}(Y - F)$. Then $f^{-1}(F) \subseteq X - U$ and hence the set $X - U$ is the required $k\omega$-closed subset of X.

Conversely, suppose that the condition of the theorem holds, and let G be any open subset of X containing $f(x)$. Then $Y - G$ is a closed subset of Y such that $f(x) \notin Y - G$. Consequently by our hypothesis, there exists a $k\omega$-closed subset H of X such that $x \notin H$ and $f^{-1}(Y - G) \subseteq H$. This implies that

$X - H \subseteq f^{-1}(G)$. Therefore, $f(X - H) \subseteq G$. Since $x \in X - H$ and $X - H$ is $k\omega$-closed subset of X, we have f is $k\omega$-continuous at x. □

For further study we recollect the following definitions.

Definition 3.6. A map $f : X \to Y$ is called

(a) ω-irresolute if the inverse image of ω-closed set Y is ω-closed in X.

(b) pre ω-closed if image of ω-closed set in X is ω-closed in Y.

(c) R map if inverse image of regular closed in Y is regular closed in X.

(d) $g\omega$-closed if image of closed set in X is $g\omega$-closed in Y.

(e) $g\omega$-continuity if inverse image of ω-closed is gw-closed.

(f) $g\omega$-irresolute if the inverse image of $g\omega$-closed set Y is $g\omega$-closed in X.

(g) $rg\omega$-closed if image of closed set in X is $rg\omega$-closed in Y.

(h) $rg\omega$-continuity if inverse image of ω-closed is $rg\omega$-closed.

(i) $rg\omega$-irresolute if the inverse image of $rg\omega$-closed set Y is $rg\omega$-closed in X.

Definition 3.7. A function $f{:}(X,\tau) \to (Y,\sigma)$ is $k\omega$-irresolute if the pre-image of each $k\omega$-closed set in Y is $k\omega$-closed in X.

Definition 3.8. A function $f{:}(X,\tau) \to (Y,\sigma)$ is $k\omega$-closed if the image of each closed set in X is $k\omega$-closed in Y.

Concerning composition of functions, we observe the following. If f, g are $k\omega$-irresolute, then gof is also $k\omega$-irresolute. If f is $k\omega$-irresolute and g is $k\omega$-continuous, then gof is also $k\omega$-continuous. The composition of two $k\omega$-continuous functions is not $k\omega$-continuous. Since every $k\omega$-closed set is closed, ω-closed, $s^*g\omega$-closed, $g\omega$-closed and $rg\omega$-closed, we have every $k\omega$-continuous function is continuous, ω-continuous, $s^*g\omega$-continuous, $g\omega$-continuous and $rg\omega$-continuous.

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