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COMMON FIXED POINT OF MAPPINGS SATISFYING RATIONAL INEQUALITY IN COMPLEX VALUED METRIC SPACE

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Abstract: Azam, Fisher and Khan [A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, *Numerical Functional Analysis and optimization,* **32**, No. 3 (2011), 243-253] introduced the notion of complex valued metric spaces and obtained common fixed point result for mappings in the context of complex valued metric spaces. In this paper, existence of common fixed point is established for two mappings on complex valued metric space. Our results unify, generalize and complement the comparable results from the current literature.

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1. Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach contraction principle which gives an answer on existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed

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point theorem in all of analysis. This principal is constructive in nature and is one of the most useful tools in the study of nonlinear equations. There are a lot of generalizations of the Banach contraction mapping principle in the literature. These generalization were made either by wakening the contractive condition or by imposing some additional conditions on ambient space. There have been a number of generalizations of metric spaces such as, rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) .Recently, A. Azam, B. Fisher and M. Khan [1] obtianed the generalization of Banach contraction principal introducing the concept of complex valued metric space. Common fixed point problem for two maps under several variants of noncomutativity has been studied by many authors. The purpose of this paper is to study common fixed points of two mappings satisfying a rational inequality, without exploiting any type of commutativity condition in the framework of complex valued metric space. The results presented in this paper substantially extend and strengthen the results given in [1].

Consistent with Azam, Fisher and Khan [1], the following definitions and results will be needed in the sequel.

Let C be the set of complex numbers and let $z_1, z_2 \in C$. Define a partial order $\leq$ on C as follows:

$z_1 \leq z_2$ if and only if Re $(z_1) \leq \text{Re}(z_2)$, $\text{Im}(z_1) \leq \text{Im}(z_2)$.

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

$$(1) \quad \text{Re}(z_1) = \text{Re}(z_2),\ \ \text{Im}(z_1) < \text{Im}(z_2),$$

$$(2) \quad \text{Re}(z_1) < \text{Re}(z_2),\ \ \text{Im}(z_1) = \text{Im}(z_2),$$

$$(3) \quad \text{Re}(z_1) < \text{Re}(z_2),\ \ \text{Im}(z_1) < \text{Im}(z_2),$$

$$(4) \quad \text{Re}(z_1) = \text{Re}(z_2),\ \ \text{Im}(z_1) = \text{Im}(z_2).$$

In particular, we will write $z_1 \lesssim z_2$ if one of (1), (2) and (3) is satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

Definition 1.1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to C$ satisfies:

(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in C$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$. A subset A in X is called open whenever each point of A is an interior point of A. The family $F = \{B(x,r) : x \in X,\ 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in C$, $B(x,r) \cap (A \setminus X) \neq \phi$. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in C$, with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

If for every $c \in C$, with $0 \prec c$, there is an $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X,d). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued metric space.

Lemma 1.2. (see [1]) *Let (X,d) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 1.3. (see [1]) *Let (X,d) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

2. Main Results

We begin with a common fixed point theorem for two functions on complex valued metric space. It may regarded as an extension of Banach fixed point theorem.

Theorem 2.1. *Let (X,d) be a complete complex valued metric space. Let mappings $S,T : X \to X$ satisfy*

$$d(Sx,Ty) \precsim \frac{a[d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)]}{d(x,Ty) + d(y,Sx)} \tag{2.1}$$

for all $x,y \in X$, where $0 \leq a < 1$. Then S and T have a unique common fixed point.

Proof. For any arbitrary point x_0 in X, construct sequence $\{x_n\}$ in X such that

$$\begin{aligned} x_{n+1} &= Sx_n \text{ and} \\ x_{n+2} &= Tx_{n+1}, \text{ for } n = 0, 1, 2, ... \end{aligned}$$

Now

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Sx_n, Tx_{n+1}) \\ &\leq \frac{a[d(x_n, Sx_n)d(x_n, Tx_{n+1}) + d(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+1})]}{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})} \\ &= \frac{a[d(x_n, x_{n+1})d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+1})]}{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})} \\ &\leq a\frac{d(x_n, x_{n+1})d(x_n, x_{n+2})}{d(x_n, x_{n+2})} \leq ad(x_n, x_{n+1}) \end{aligned}$$

for all $n \geq 0$ and consequently,

$$d(x_{n+1}, x_{n+2}) \leq ad(x_n, x_{n+1})... \leq a^n d(x_0, x_1).$$

Now for $m > n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq a^n d(x_0, x_1) + a^{n+1} d(x_0, x_1) + ... + a^{m-1} d(x_0, x_1) \\ &\leq \frac{a^n}{1-a} d(x_0, x_1). \end{aligned}$$

This implies that

$$|d(x_m, x_n)| \leq \frac{a^n}{1-a} |d(x_0, x_1)|$$

which on taking limit as $n \to \infty$ gives that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists an element u in X such that $\{x_n\}$ converges to u. Now we show that $Su = u$. If not, then there exist z in C such that $d(Su, u) = z > 0$. From (2.1), we get

$$\begin{aligned} z &\leq d(u, x_{n+2}) + d(x_{n+2}, Su) \\ &\leq d(u, x_{n+2}) + d(Su, Tx_{n+1}) \\ &\leq d(u, x_{n+2}) + \frac{a[d(u, su)d(u, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Su)]}{d(u, Tx_{n+1}) + d(x_{n+1}, Su)} \\ &= d(u, x_{n+2}) + \frac{a[d(u, su)d(u, x_{n+2}) + d(x_{n+1}, x_{n+2})d(x_{n+1}, Su)]}{d(u, x_{n+2}) + d(x_{n+1}, Su)}, \end{aligned}$$

which implies that

$$|z| \leq |d(u, x_{n+2})| + \frac{a[|z||d(u, x_{n+2})| + |d(x_{n+1}, x_{n+1})||d(x_{n+1}, Su)|]}{|d(u, x_{n+2})| + |d(x_{n+1}, Su)|}.$$

On taking limit as $n \to \infty$, we obtain that $|z| \leq 0$, a contradiction. So $|z| = 0$. Hence $Su = u$. Similarly, we obtain $Tu = u$. Now we show that S and T have unique common fixed point. For this, assume that u^* is another common fixed point of S and T. Then

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\leq \frac{a[d(u, Su)d(u, Tu^*) + d(u^*, Tu^*)d(u^*, Sx)]}{d(u, Tu^*) + d(u^*, Sx)}, \end{aligned}$$

implies $d(u, u^*) \leq 0$ and hence $u = u^*$.

Corollary 2.2. *Let (X, d) be a complete complex valued metric space and let the mappings $T : X \to X$, satisfying*

$$d(T^n x, T^n y) \leq a \frac{[d(x, T^n x)d(x, T^n y) + d(y, T^n y)d(y, T^n x)]}{d(x, T^n y) + d(y, T^n x)}$$

for all $x, y \in X$, where $0 \leq a < 1$ and $n \in N$. Then T has a unique common fixed point.

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