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THREE-DIMENSIONAL MHD STAGNATION POINT-FLOW OF A NEWTONIAN AND A MICROPOLAR FLUID

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Abstract: The steady three-dimensional stagnation-point flow of an electrically conducting Newtonian or micropolar fluid in the presence of a uniform external magnetic field $\mathbf{H}_0$ is analysed and some physical situations are examined.

In particular, we prove that, if we impress an external magnetic field $\mathbf{H}_0$, and we neglect the induced magnetic field, then the steady three-dimensional MHD stagnation-point flow is possible if, and only if, $\mathbf{H}_0$ has the direction of one of the coordinate axes.

In all cases it is shown that the governing nonlinear partial differential equations admit similarity solutions. We find that the flow has to satisfy an ordinary differential problem whose solution depends on $\mathbf{H}_0$ through the Hartmann number M^2.

Finally, the skin-friction components along the axes are computed.

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Key Words: Newtonian fluids, Micropolar fluids, MHD flow, three-dimensional stagnation-point flow

1. Introduction

An important example of flow past a body, where the three velocity components appear, is the three-dimensional stagnation-point flow. Such a flow occurs when a jet of fluid impinges on a rigid wall.

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The steady three-dimensional stagnation-point flow of a Newtonian fluid has been studied by Homman ([9]), Howarth ([10]), Davey and Schoffield ([1], [14]). In the similarity transformations which reduce the Navier-Stokes equations to an ODE nonlinear system two constant parameters a, b appear. As it is known, if both a and b are positive then there is a nodal point of attachment and if one and only one of a, b is negative, such that $a+b>0$, then there is a saddle point of attachment ([10], [1], [3], [14]). Further suitable boundary conditions have to be assigned. The ODE system obtained depends upon a parameter $c=\dfrac{b}{a}$ which is a measure of three-dimensionality.

Guram and Anwar Kamal ([6]) studied the steady three-dimensional stagnation-point flow of a micropolar fluid. We recall that the micropolar fluids introduced by Eringen ([4]) physically represent fluids consisting of rigid randomly oriented particles suspended in a viscous medium which have an intrinsic rotational micromotion (for example biological fluids in thin vessels, polymeric suspensions, slurries, colloidal fluids). Extensive reviews of the theory and its applications can be found in [5] and [11].

The aim of this paper is to study how the steady three-dimensional stagnation-point flow of an electrically conducting Newtonian and micropolar fluid is influenced by a uniform external electromagnetic field $(\mathbf{E}_0, \mathbf{H}_0)$.

As it is customary in the literature, we assume that at infinity, the velocity $\mathbf{v}$ and the pressure p of the fluid under consideration approach the flow of an inviscid fluid for which the stagnation-point is shifted from the origin ([2], [12] and [13]). Moreover for the micropolar fluid we suppose that at infinity the microrotation $\mathbf{w}$ is given by $\mathbf{w}=\frac{1}{2}\nabla\times\mathbf{v}$, i.e. the micropolar fluid behaves like a classical fluid far from the wall.

For this reason, first of all, we study the steady three-dimensional stagnation-point flow of an inviscid fluid in the presence of a uniform external magnetic field $\mathbf{H}_0$. As it is customary in MHD, we assume that the magnetic Reynolds number is very small, so that the induced magnetic field is negligible in comparison with the imposed field. Under this assumption we prove that $\mathbf{H}_0$ has to be parallel to one of the coordinate axes.

Then we consider the same problems for a Newtonian and a micropolar fluid. Taking into account the results obtained for an inviscid fluid, we find that the flow has to satisfy an ordinary differential problem depending on two parameters: c and M^2 (M^2 is the Hartmann number).

Moreover we analyse the skin-friction components τ_1, τ_3 along x_1 and x_3 axes, which are physically interesting.

The paper is organized in this way:

In Section 2, we study the three-dimensional stagnation-point flow of an inviscid fluid in the presence of a uniform external electromagnetic field. We prove in Theorem 2 that, if we impress an external magnetic field $\mathbf{H}_0$, and we neglect the induced magnetic field, then the steady three-dimensional MHD stagnation-point flow of such a fluid is possible for any value of $c > -1$ if, and only if, $\mathbf{H}_0$ has the direction parallel to one of the axes. Further if $c = 1$, or $c = -\frac{1}{2}$, we prove that the magnetic field can be parallel also to the plane Ox_1x_3, or Ox_2x_3 respectively.

In Section 3, we determine the nonlinear ODE problems for a Newtonian fluid corresponding to the cases treated in Theorem 2. Moreover we prove that if $c = 1$ or $c = -\frac{1}{2}$, then the three-dimensional MHD stagnation point flow is not possible when $\mathbf{H}_0$ is parallel to the plane Ox_1x_3 or Ox_2x_3, unlike what occurs to the inviscid fluid.

In Section 4, we obtain the nonlinear ODE problems for a micropolar fluid corresponding to the cases treated in Theorem 2. Also for this fluid the three-dimensional MHD stagnation point flow does not exist if $c = 1$ and $\mathbf{H}_0$ is parallel to the plane Ox_1x_3 or $c = -\frac{1}{2}$ and $\mathbf{H}_0$ is parallel to the plane Ox_2x_3.

2. Preliminaries

Consider the steady three-dimensional MHD flow of an electrically conducting homogeneous incompressible inviscid fluid near a stagnation point filling the half-space $\mathcal{S}$, given by

$$\mathcal{S} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, x_2 > 0\}. \tag{1}$$

The boundary of $\mathcal{S}$, i.e. $x_2 = 0$, is a rigid, fixed, non-electrically conducting wall.

As it is well known in the three-dimensional stagnation-point flow the velocity field is given by

$$v_1 = ax_1, \quad v_2 = -(a+b)x_2, \quad v_3 = bx_3, \qquad (x_1, x_2, x_3) \in \mathcal{S}, \tag{2}$$

where a, b are constants.

We suppose $a > 0$, $b \neq 0$, and $c = \dfrac{b}{a}$. We exclude the case $c \leq -1$ because we impose the condition $v_2 < 0$, so that the fluid moves towards the wall $x_2 = 0$.

Remark 1. If $c = 1$, the velocity is axial symmetric with respect to x_2 axis:

$$v_1 = ax_1, \quad v_2 = -2ax_2, \quad v_3 = ax_3.$$

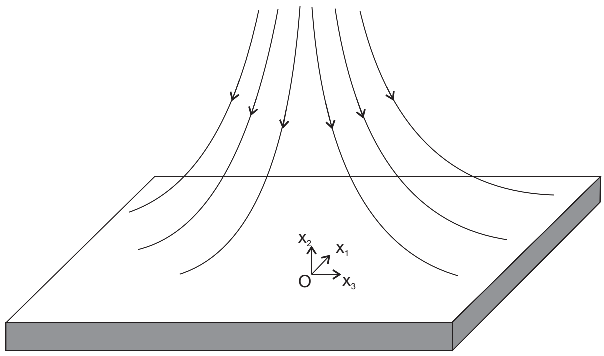

Figure 1: Flow description.

The equations governing such a flow in the absence of external mechanical body forces are:

$$\begin{aligned}
&\rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu_e (\nabla \times \mathbf{H}) \times \mathbf{H}, \\
&\nabla \cdot \mathbf{v} = 0, \\
&\nabla \times \mathbf{H} = \sigma_e (\mathbf{E} + \mu_e \mathbf{v} \times \mathbf{H}), \\
&\nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0 \qquad \text{in } \mathcal{S} \qquad (3)
\end{aligned}$$

where $\mathbf{v}$ is the velocity field, p is the pressure, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, ρ is the mass density (constant > 0), μ_e is the magnetic permeability, σ_e is the electrical conductivity ($\mu_e, \sigma_e =$ constants > 0).

We suppose that a uniform external magnetic field $\mathbf{H}_0$ is impressed and that the electric field is absent. As it is customary in the literature, we assume that the magnetic Reynolds number is very small, so that the induced magnetic field is negligible in comparison with the imposed field. Then

$$(\nabla \times \mathbf{H}) \times \mathbf{H} \simeq \sigma_e \mu_e (\mathbf{v} \times \mathbf{H}_0) \times \mathbf{H}_0. \qquad (4)$$

Now our aim is to prove the following:

Theorem 2. *Let a homogeneous, incompressible, electrically conducting inviscid fluid occupy the half-space* $\mathcal{S}$. *If we impress an external magnetic field* $\mathbf{H}_0$, *and we neglect the induced magnetic field, then the steady three-dimensional MHD stagnation-point flow of such a fluid is possible for all* $c > -1$ *if, and only if,* $\mathbf{H}_0$ *is parallel to one of the coordinate axes.*

Proof. For brevity sake, we will denote by $\mathbf{H}$ the external magnetic field:

$$\mathbf{H} = H_1 \mathbf{e}_1 + H_2 \mathbf{e}_2 + H_3 \mathbf{e}_3, \qquad (5)$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical base of $\mathbb{R}^3$.

On substituting the approximation (4) in $(3)_1$, taking into account that the velocity field $\mathbf{v}$ is given by (2), we get:

$$\begin{aligned}
\frac{\partial p}{\partial x_1} &= -\rho a^2 x_1 + \sigma_e a[cB_1B_3x_3 - (B_2^2 + B_3^2)x_1 - (1+c)B_1B_2x_2],\\
\frac{\partial p}{\partial x_2} &= -\rho a^2 (1+c)^2 x_2 + \sigma_e a[B_1B_2x_1 + (1+c)(B_1^2 + B_3^2)x_2 + cB_2B_3x_3],\\
\frac{\partial p}{\partial x_3} &= -\rho a^2 c^2 x_3 + \sigma_e a[-(1+c)B_2B_3x_2 - c(B_1^2 + B_2^2)x_3 + B_1B_3x_1],
\end{aligned} \tag{6}$$

where $\mathbf{B} = \mu_e \mathbf{H}$.

It is possible to find a function $p = p(x_1, x_2, x_3)$ satisfying (6) if and only if

$$\frac{\partial^2 p}{\partial x_i \partial x_j} = \frac{\partial^2 p}{\partial x_j \partial x_i}, \quad i, j = 1, 2, 3, \ i \neq j. \tag{7}$$

On the other hand we have

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = -\sigma_e a(1+c)B_1B_2, \quad \frac{\partial^2 p}{\partial x_2 \partial x_1} = \sigma_e a B_1 B_2, \tag{8}$$

$$\frac{\partial^2 p}{\partial x_1 \partial x_3} = \sigma_e a c B_1 B_3, \quad \frac{\partial^2 p}{\partial x_3 \partial x_1} = \sigma_e a B_1 B_3, \tag{9}$$

$$\frac{\partial^2 p}{\partial x_2 \partial x_3} = \sigma_e a c B_2 B_3, \quad \frac{\partial^2 p}{\partial x_3 \partial x_2} = -\sigma_e a(1+c) B_2 B_3. \tag{10}$$

Therefore, since c is arbitrary (> -1), conditions (8), (9), (10) are satisfied if and only if $\mathbf{B} = B\mathbf{e}_1$ or $\mathbf{B} = B\mathbf{e}_2$ or $\mathbf{B} = B\mathbf{e}_3$.

Finally if $\mathbf{B} = B\mathbf{e}_1$ then we deduce

$$p = -\rho \frac{v^2(x_1, x_2, x_3)}{2} + \frac{a}{2}\sigma_e B^2[(1+c)x_2^2 - cx_3^2] + p_0; \tag{11}$$

if $\mathbf{B} = B\mathbf{e}_2$ then we deduce

$$p = -\rho \frac{v^2(x_1, x_2, x_3)}{2} - \frac{a}{2}\sigma_e B^2[x_1^2 + cx_3^2] + p_0; \tag{12}$$

if $\mathbf{B} = B\mathbf{e}_3$ then we deduce

$$p = -\rho \frac{v^2(x_1, x_2, x_3)}{2} - \frac{a}{2}\sigma_e B^2[x_1^2 - (1+c)x_2^2] + p_0. \tag{13}$$

□

Remark 3. The results obtained in Theorem 2 hold for any $c > -1$. We remark that if $c = 1$, then it is possible to consider also the magnetic field parallel to the plane Ox_1x_3 as one can see from (9). In this case the pressure becomes

$$p = -\rho \frac{v^2(x_1, x_2, x_3)}{2} + a\sigma_e(B_1^2 + B_3^2)x_2^2 - \frac{a}{2}\sigma_e(B_3x_1 - B_1x_3)^2 + p_0.$$

Moreover, if $c = -\frac{1}{2}$ then it is possible to consider the magnetic field parallel to the plane Ox_2x_3, as one can see from (10). The corresponding pressure is

$$p = -\rho \frac{v^2(x_1, x_2, x_3)}{2} - \frac{a}{2}\sigma_e(B_2^2 + B_3^2)x_1^2 + \frac{a}{4}\sigma_e(B_3x_2 - B_2x_3)^2 + p_0.$$

Remark 4. As it is well known, in the absence of the external magnetic field the stagnation point is the point where the pressure assumes its maximum. We notice that, from (11), (12), (13), the pressure along the wall $x_2 = 0$ takes its maximum in the stagnation-point.

Remark 5. In order to study the three-dimensional stagnation-point flow for other fluids, it is convenient to consider a more general flow. More precisely, we suppose the fluid impinging on the flat plane $x_2 = C$ and

$$v_1 = ax_1, \quad v_2 = -(a+b)(x_2 - C), \quad v_3 = bx_3, \qquad (x_1, x_3) \in \mathbb{R}^2, \quad x_2 \geq C, \quad (14)$$

with C some constant.

In this way, the stagnation point is not the origin but the point $(0, C, 0)$.

As it is easy to verify, in the cases of Theorem 2 and Remark 3 the pressure must be modified by replacing x_2 with $x_2 - C$.

3. Newtonian Fluids

Consider the steady three-dimensional stagnation-point flow of an electrically conducting homogeneous incompressible Newtonian fluid towards a flat surface coinciding with the plane $x_2 = 0$, the flow being confined to the half-space $\mathcal{S}$. In the absence of external mechanical body forces, the MHD equations for such a fluid are

$$\begin{aligned} &\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \triangle \mathbf{v} + \frac{\mu_e}{\rho}(\nabla \times \mathbf{H}) \times \mathbf{H}, \\ &\nabla \cdot \mathbf{v} = 0, \end{aligned}$$

$$\nabla \times \mathbf{H} = \sigma_e(\mathbf{E} + \mu_e \mathbf{v} \times \mathbf{H}),$$
$$\nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0 \qquad \text{in } \mathcal{S} \tag{15}$$

where ν is the kinematic viscosity.

As far as the boundary condition for $\mathbf{v}$ is concerned, we prescribe the adherence condition, i.e.

$$\mathbf{v}|_{x_2=0} = \mathbf{0}. \tag{16}$$

We search $\mathbf{v}$ in the following form:

$$\forall (x_1, x_2, x_3) \in \mathcal{S}$$

$$v_1 = a x_1 f'(x_2), \quad v_2 = -[a f(x_2) + b g(x_2)], \quad v_3 = b x_3 g'(x_2), \tag{17}$$

where f, g are sufficiently regular unknown functions. As for an inviscid fluid, we suppose $a > 0, \ b \neq 0$, so that if $b > 0$ then we have a flow in the neighborhood of a nodal point of attachment, while if $b < 0$ and $c = \dfrac{b}{a} > -1$, then we have a flow in the neighborhood of a saddle point of attachment.

The condition (16) supplies

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 0, \quad g'(0) = 0. \tag{18}$$

Moreover, as is customary when studying the stagnation-point flow for viscous fluids, we assume that at infinity, the flow approaches the flow of an inviscid fluid, whose velocity is given by (14).

Therefore, to (18) we also must append the following conditions

$$\lim_{x_2 \to +\infty} f'(x_2) = 1, \quad \lim_{x_2 \to +\infty} g'(x_2) = 1. \tag{19}$$

The constant C in (14) is related to the behaviour of f and g at infinity. Actually, if

$$\lim_{x_2 \to +\infty} [f(x_2) - x_2] = -A, \quad \lim_{x_2 \to +\infty} [g(x_2) - x_2] = -B \tag{20}$$

with A, B some constants, then

$$\lim_{x_2 \to +\infty} [f(x_2) + c g(x_2) - (1+c) x_2] = -(1+c) C, \tag{21}$$

where

$$C = \frac{A + cB}{1 + c}, \quad c > -1.$$

The constants A, B, C are not assigned a priori, but their values can be found by solving the problem.

In order to study the influence of a uniform external electromagnetic field, we continue to use the approximation (4), where $\mathbf{v}$ is given by (17). As a result of the Theorem 2, we consider the following three cases.

3.1. Case I

$$\mathbf{H}_0 = H_0\mathbf{e}_1.$$

$$(\nabla \times \mathbf{H}) \times \mathbf{H} \simeq \sigma_e\mu_e a\{[H_0^2(f+cg)]\mathbf{e}_2 - cH_0^2 g' x_3\mathbf{e}_3\}. \tag{22}$$

We substitute (17), and (22) in $(15)_1$ to obtain

$$\begin{aligned}
&ax_1\left[\nu f''' + af''(f+cg) - af'^2\right] = \frac{1}{\rho}\frac{\partial p}{\partial x_1},\\
&-\nu a(f''+cg'') - a^2(f'+cg')(f+cg) + \frac{\sigma_e a}{\rho}B_0^2(f+cg) = \frac{1}{\rho}\frac{\partial p}{\partial x_2},\\
&acx_3\left[\nu g''' + ag''(f+cg) - acg'^2 - \frac{\sigma_e}{\rho}B_0^2 g'\right] = \frac{1}{\rho}\frac{\partial p}{\partial x_3}.
\end{aligned} \tag{23}$$

Then, by integrating $(23)_2$, we find

$$\begin{aligned}
p = &-\frac{1}{2}\rho a^2[f(x_2)+cg(x_2)]^2 - \rho a\nu[f'(x_2)+cg'(x_2)]\\
&+\sigma_e a B_0^2\int_0^{x_2}[f(s)+cg(s)]ds + P(x_1,x_3),
\end{aligned}$$

where the function $P(x_1, x_3)$ is determined supposing that, far from the wall, the pressure p has the same behaviour as for an inviscid fluid, whose velocity is given by (14) and the pressure is given by (11) replacing x_2 by $x_2 - C$.

Therefore, by virtue of (19), and (20), we get

$$P(x_1,x_3) = -\rho\frac{a^2}{2}({x_1}^2 + c^2x_3^2) - \frac{a}{2}\sigma_e B_0^2 c x_3^2 + p_0^*,$$

where p_0^* is a suitable constant. Finally, the pressure field assumes the form

$$p = -\rho\frac{a^2}{2}\{x_1^2 + [f(x_2)+cg(x_2)]^2 + c^2x_3^2\} - \rho a\nu[f'(x_2)+cg'(x_2)]$$

$$+ \sigma_e a B_0^2 \left\{ \int_0^{x_2} [f(s) + cg(s)]ds - \frac{c}{2} x_3^2 \right\} + p_0, \tag{24}$$

where the constant p_0 is the pressure at the origin.

In consideration of (24), we obtain the ordinary differential system

$$\begin{aligned} &\frac{\nu}{a} f''' + (f + cg) f'' - f'^2 + 1 = 0, \\ &\frac{\nu}{a} g''' + (f + cg) g'' - cg'^2 + c + M^2(1 - g') = 0, \end{aligned} \tag{25}$$

where

$$M^2 = \frac{\sigma_e B_0^2}{\rho a}$$

is the Hartmann number. To these equations we append the boundary conditions (18), and (19).

As far as the other two cases are concerned, if we proceed as previously we get

3.2. Case II

$$\mathbf{H}_0 = H_0 \mathbf{e}_2.$$

$$\begin{aligned} p = &- \rho \frac{a^2}{2} \{x_1^2 + [f(x_2) + cg(x_2)]^2 + c^2 x_3^2\} - \rho a \nu [f'(x_2) + cg'(x_2)] \\ &- \sigma_e a B_0^2 (x_1^2 + c x_3^2) + p_0, \end{aligned} \tag{26}$$

$$\begin{aligned} &\frac{\nu}{a} f''' + (f + cg) f'' - f'^2 + 1 + M^2(1 - f') = 0, \\ &\frac{\nu}{a} g''' + (f + cg) g'' - cg'^2 + c + M^2(1 - g') = 0. \end{aligned} \tag{27}$$

3.3. Case III

$$\mathbf{H}_0 = H_0 \mathbf{e}_3.$$

$$p = - \rho \frac{a^2}{2} \{x_1^2 + [f(x_2) + cg(x_2)]^2 + c^2 x_3^2\} - \rho a \nu [f'(x_2) + cg'(x_2)]$$

$$+\sigma_e a B_0^2 \left\{ \int_0^{x_2} [f(s)+cg(s)]ds - \frac{x_1^2}{2} \right\} + p_0, \tag{28}$$

$$\begin{aligned} &\frac{\nu}{a} f''' + (f+cg)f'' - f'^2 + 1 + M^2(1-f') = 0, \\ &\frac{\nu}{a} g''' + (f+cg)g'' - cg'^2 + c = 0. \end{aligned} \tag{29}$$

We have the following:

Theorem 6. *Let a homogeneous, incompressible, electrically conducting Newtonian fluid occupy the half-space $\mathcal{S}$. If we impress the external magnetic field $\mathbf{H}_0$ parallel to one of the coordinate axes and if we neglect the induced magnetic field, then the steady three-dimensional MHD stagnation-point flow of such a fluid has the form*

$$v_1 = ax_1 f'(x_2), \quad v_2 = -[af(x_2) + bg(x_2)], \quad v_3 = bx_3 g'(x_2), \quad \mathbf{E} = \mathbf{0},$$

and

1. *if $\mathbf{H}_0 = H_0\mathbf{e}_1$, then the pressure field is given by (24) and (f, g) satisfies problem (25), (18), and (19);*

2. *if $\mathbf{H}_0 = H_0\mathbf{e}_2$, then the pressure field is given by (26) and (f, g) satisfies problem (27), (18), and (19);*

3. *if $\mathbf{H}_0 = H_0\mathbf{e}_3$, then the pressure field is given by (28) and (f, g) satisfies problem (29), (18), and (19).*

Remark 7. We remark that from (24), (26), (28) the pressure along the wall $x_2 = 0$ takes its maximum again in the stagnation-point.

We now analyse the cases considered in Remark 3.

Proposition 8. *Let a homogeneous, incompressible, electrically conducting Newtonian fluid occupy the half-space $\mathcal{S}$. If we neglect the induced magnetic field and we suppose either*

$$i) \quad c = 1, \quad \mathbf{H_0} \text{ parallel to the plane } Ox_1x_3,$$

or

$$ii) \quad c = -\frac{1}{2}, \quad \mathbf{H_0} \text{ parallel to the plane } Ox_2x_3,$$

then there is no solution to the problem of the steady three-dimensional MHD stagnation-point flow.

Proof. i) If $c = 1$ and the external magnetic induction field is $\mathbf{B} = B_1\mathbf{e}_1 + B_3\mathbf{e}_3$ ($B_1, B_3 \neq 0$), then from equations (15), (17), proceeding as in Case I, after some calculations, we deduce:

$$\begin{aligned} &\nu f''' + a(f+g)f'' - af'^2 + B_3^2\frac{\sigma_e}{\rho}(1-f') + a = 0,\\ &\frac{\sigma_e a}{\rho}B_1B_3[g'-1] = 0,\\ &\nu g''' + a(f+g)g'' - ag'^2 + B_1^2\frac{\sigma_e}{\rho}(1-g') + a = 0,\\ &\frac{\sigma_e a}{\rho}B_1B_3[f'-1] = 0. \end{aligned} \tag{30}$$

From $(30)_2, (30)_4$ one has $f' = g' = 1$ for all $x_2 \geq 0$ which contradicts the boundary conditions (18).

ii) Then we examine the case $c = -\dfrac{1}{2}$ and $\mathbf{B} = B_2\mathbf{e}_2 + B_3\mathbf{e}_3$ ($B_2, B_3 \neq 0$). By proceeding as above, we obtain

$$\begin{aligned} &\nu f''' + a(f - \frac{g}{2})f'' - af'^2 + (B_2^2 + B_3^2)\frac{\sigma_e}{\rho}(1-f') + a = 0,\\ &\nu g''' + a(f - \frac{g}{2})g'' + \frac{a}{2}g'^2 + B_2^2\frac{\sigma_e}{\rho}(1-g') - \frac{a}{2} = 0,\\ &\frac{\sigma_e a}{\rho}B_2B_3(f - g + A - B) = 0. \end{aligned} \tag{31}$$

From $(31)_3$ evaluated at $x_2 = 0$ and (18), we obtain $f = g$. So $(31)_1$, $(31)_2$ become

$$\begin{aligned} &\nu f''' + a\frac{f}{2}f'' - af'^2 + (B_2^2 + B_3^2)\frac{\sigma_e}{\rho}(1-f') + a = 0,\\ &\nu f''' + a\frac{f}{2}f'' + \frac{a}{2}f'^2 + B_2^2\frac{\sigma_e}{\rho}(1-f') - \frac{a}{2} = 0. \end{aligned} \tag{32}$$

By subtracting $(32)_1$ from $(32)_2$, we arrive at

$$(f'-1)\left[\frac{3}{2}(f'+1) + B_3^2\frac{\sigma_e}{a\rho}\right] = 0. \tag{33}$$

At $x_2 = 0$, (33) gives the absurdum

$$\frac{3}{2} + \frac{\sigma_e}{a\rho}B_3^2 = 0.$$

□

Remark 9. If $c = 1$, $f = g$, $\mathbf{H}_0 = H_0\mathbf{e}_2$, the axial symmetric case is obtained.

Finally, it is convenient to write the boundary value problems in Theorem 6 in dimensionless form in order to reduce the number of the parameters. To this end we put

$$\eta = \sqrt{\frac{a}{\nu}} x_2, \quad \varphi(\eta) = \sqrt{\frac{a}{\nu}} f\left(\sqrt{\frac{\nu}{a}}\eta\right), \quad \gamma(\eta) = \sqrt{\frac{a}{\nu}} g\left(\sqrt{\frac{\nu}{a}}\eta\right). \tag{34}$$

So we can rewrite equations (25) as:

$$\begin{aligned} &\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 = 0, \\ &\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c + M^2(1 - \gamma') = 0; \end{aligned} \tag{35}$$

equations (27) as

$$\begin{aligned} &\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 + M^2(1 - \varphi') = 0, \\ &\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c + M^2(1 - \gamma') = 0; \end{aligned} \tag{36}$$

equations (29) as

$$\begin{aligned} &\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 + M^2(1 - \varphi') = 0, \\ &\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c = 0. \end{aligned} \tag{37}$$

Of course we obtain three different ordinary differential problems by adjoining the boundary conditions in dimensionless form:

$$\begin{aligned} &\varphi(0) = 0, \quad \varphi'(0) = 0, \\ &\gamma(0) = 0, \quad \gamma'(0) = 0, \\ &\lim_{\eta \to +\infty} \varphi'(\eta) = 1, \quad \lim_{\eta \to +\infty} \gamma'(\eta) = 1. \end{aligned} \tag{38}$$

Remark 10. If $M = 0$, then equations (35), (36), (37) reduce to

$$\begin{aligned} &\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 = 0, \\ &\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c = 0 \end{aligned} \tag{39}$$

which are the dimensionless equations governing the flow of a Newtonian fluid in the absence of $\mathbf{H}_0$. We recall that in [1] it is proved that the problem (39), (38) does not admit solution for $c < -1$. As far as existence of solutions is concerned, we refer to [7], [8].

Remark 11. It is physically interesting to determine the skin-friction components τ_1, τ_3 along x_1 and x_3 axes :

$$\tau_1 = \mu \left(\frac{\partial v_1}{\partial x_2}\right)_{x_2=0} = \rho(\nu)^{1/2} a^{3/2} x_1 \varphi''(0),$$
$$\tau_3 = \mu \left(\frac{\partial v_3}{\partial x_2}\right)_{x_2=0} = c\rho(\nu)^{1/2} a^{3/2} x_3 \gamma''(0). \tag{40}$$

Formally τ_1, τ_3 have the same expression as in the absence of $\mathbf{H}_0$, but of course $\varphi''(0), \gamma''(0)$ depend on $\mathbf{H}_0$ through the Hartmann number M^2.

Remark 12. It is easy to verify that regular solutions to problem (36), (38) are invariant under the following transformation

$$\varphi\left(\eta, \frac{1}{c}, \frac{M^2}{c}\right) = \sqrt{c}\,\gamma\left(\frac{\eta}{\sqrt{c}}, c, M^2\right),$$
$$\gamma\left(\eta, \frac{1}{c}, \frac{M^2}{c}\right) = \sqrt{c}\,\varphi\left(\frac{\eta}{\sqrt{c}}, c, M^2\right), \qquad c > 0.$$

Moreover we notice that by means of the previous transformation the regular solutions to problem (35), (or (37)), (38) are transformed in the solutions to problem (37), (or (35)), (38).

4. Micropolar Fluids

Consider now the steady three-dimensional stagnation-point flow of an electrically conducting homogeneous incompressible micropolar fluid towards a flat surface coinciding with the plane $x_2 = 0$, the flow being confined to the half-space $\mathcal{S}$. In the absence of external mechanical body forces and body couples, the MHD equations for such a fluid are ([11])

$$\begin{aligned}
&\mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + (\nu+\nu_r)\triangle\mathbf{v} + 2\nu_r(\nabla\times\mathbf{w}) + \frac{\mu_e}{\rho}(\nabla\times\mathbf{H})\times\mathbf{H},\\
&\nabla\cdot\mathbf{v} = 0,\\
&I\mathbf{v}\cdot\nabla\mathbf{w} = \lambda\triangle\mathbf{w} + \lambda_0\nabla(\nabla\cdot\mathbf{w}) - 4\nu_r\mathbf{w} + 2\nu_r(\nabla\times\mathbf{v}),\\
&\nabla\times\mathbf{H} = \sigma_e(\mathbf{E} + \mu_e\mathbf{v}\times\mathbf{H}),\\
&\nabla\times\mathbf{E} = \mathbf{0}, \quad \nabla\cdot\mathbf{E} = 0, \quad \nabla\cdot\mathbf{H} = 0 \qquad \text{in } \mathcal{S}
\end{aligned} \tag{41}$$

where $\mathbf{w}$ is the microrotation field, ν is the kinematic newtonian viscosity coefficient, ν_r is the microrotation viscosity coefficient, λ, λ_0 (positive constants) are material parameters related to the coefficient of angular viscosity and I is the microinertia coefficient.

We notice that in [4], [5], eqs. $(41)_{1,3}$ are slightly different, as they are deduced as a special case of much more general model of microfluids. For the details, we refer to [11], p.23.

As far as the boundary conditions are concerned, we prescribe the appropriate boundary condition for the velocity $\mathbf{v}$ and the microrotation $\mathbf{w}$, i.e.

$$\mathbf{v}|_{x_2=0} = \mathbf{0}, \quad \mathbf{w}|_{x_2=0} = \mathbf{0} \text{ (strict adherence condition).} \tag{42}$$

Other boundary conditions are possible. We refer to Eringen ([4], p.17-18) for a complete discussion. However in our studies we will always assume the strict adherence condition.

We search $\mathbf{v}$, $\mathbf{w}$ in the following form

$$\begin{aligned} &v_1 = a x_1 f'(x_2), \quad v_2 = -[a f(x_2) + b g(x_2)], \quad v_3 = b x_3 g'(x_2), \\ &w_1 = -c x_3 F(x_2), \quad w_2 = 0, \quad w_3 = x_1 G(x_2), \qquad (x_1, x_2, x_3) \in \mathcal{S}, \end{aligned} \tag{43}$$

where f, g, F, G are sufficiently regular unknown functions and $c = \dfrac{b}{a}$.

The conditions (42) supply

$$\begin{aligned} &f(0) = 0, \quad f'(0) = 0, \quad g(0) = 0, \quad g'(0) = 0, \\ &F(0) = 0, \quad G(0) = 0. \end{aligned} \tag{44}$$

Moreover, we assume that at infinity, the flow approaches the flow of an inviscid fluid, whose velocity is given by (14). Therefore, to (41) we must append also the following conditions

$$\begin{aligned} &\lim_{x_2 \to +\infty} f'(x_2) = 1, \quad \lim_{x_2 \to +\infty} g'(x_2) = 1, \\ &\lim_{x_2 \to +\infty} F(x_2) = 0, \quad \lim_{x_2 \to +\infty} G(x_2) = 0. \end{aligned} \tag{45}$$

Conditions $(45)_{3,4}$ mean that at infinity, $\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v} = \mathbf{0}$, i.e. the micropolar fluid behaves like an ideal fluid.

The constant C is related to the asymptotic behaviour of f and g at infinity as in the Newtonian case. So equation (21) continues to hold.

In order to study the influence of a uniform external electromagnetic field, we continue to use the approximation (4), where $\mathbf{v}$ is given by $(43)_{1,2,3}$. As a result of the Theorem 2, we consider the following three cases.

4.1. Case I

$$\mathbf{H}_0 = H_0\mathbf{e}_1.$$

$$(\nabla \times \mathbf{H}) \times \mathbf{H} \simeq \sigma_e\mu_e a\{[H_0^2(f+cg)]\mathbf{e}_2 - cH_0^2 g' x_3\mathbf{e}_3\}. \tag{46}$$

We substitute (43) in $(41)_{1,3}$ to obtain

$$\begin{aligned}
&ax_1\left[(\nu+\nu_r)f''' + af''(f+cg) - af'^2 + \frac{2\nu_r}{a}G'\right] = \frac{1}{\rho}\frac{\partial p}{\partial x_1},\\
&-(\nu+\nu_r)a(f''+cg'') - a^2(f'+cg')(f+cg) - 2\nu_r(cF+G)\\
&+\frac{\sigma_e a}{\rho}B_0^2(f+cg) = \frac{1}{\rho}\frac{\partial p}{\partial x_2},\\
&cx_3\left[(\nu+\nu_r)g''' + ag''(f+cg) - cg'^2 + \frac{2\nu_r}{a}F' - \frac{\sigma_e}{\rho}B_0^2 g'\right] = \frac{1}{\rho}\frac{\partial p}{\partial x_3},\\
&c\{\lambda F'' + Ia[F'(f+cg) - cFg'] - 2\nu_r(2F+ag'')\} = 0,\\
&\lambda G'' + Ia[G'(f+cg) - Gf'] - 2\nu_r(2G+af'') = 0.
\end{aligned} \tag{47}$$

Since we are interested in three-dimensional flow, we assume $c \neq 0$ and so equation $(47)_4$ can be replaced by

$$\lambda F'' + Ia[F'(f+cg) - cFg'] - 2\nu_r(2F+ag'') = 0. \tag{48}$$

Then, by integrating $(47)_2$, we find

$$\begin{aligned}
p = &-\frac{1}{2}\rho a^2[f(x_2)+cg(x_2)]^2 - \rho a(\nu+\nu_r)[f'(x_2)+cg'(x_2)]\\
&-2\nu_r\rho\int_0^{x_2}[cF(s)+G(s)]ds + \sigma_e a B_0^2\int_0^{x_2}[f(s)+cg(s)]ds\\
&+P(x_1,x_3),
\end{aligned}$$

where the function $P(x_1, x_3)$ is determined supposing that, far from the wall, the pressure p has the same behaviour as for an inviscid fluid, whose velocity is given by (14) and the pressure is given by (11) replacing x_2 by $x_2 - C$.

Therefore, under the assumption $F, G \in L^1([0, +\infty))$, by virtue of (45), (20), we get

$$P(x_1,x_3) = -\rho\frac{a^2}{2}({x_1}^2 + c^2x_3^2) - \frac{a}{2}\sigma_e B_0^2 c x_3^2 + p_0^*,$$

where p_0^* is a suitable constant. Finally, the pressure field assumes the form

$$\begin{aligned} p = & -\rho\frac{a^2}{2}\{x_1^2+[f(x_2)+cg(x_2)]^2+c^2x_3^2\} \\ & -\rho a(\nu+\nu_r)[f'(x_2)+cg'(x_2)] - 2\nu_r\rho\int_0^{x_2}[cF(s)+G(s)]ds \\ & +\sigma_e a B_0^2\left\{\int_0^{x_2}[f(s)+cg(s)]ds - \frac{c}{2}x_3^2\right\}+p_0, \end{aligned} \tag{49}$$

where the constant p_0 is the pressure at the origin.

In consideration of (49), we obtain the ordinary differential system

$$\begin{aligned} & \frac{\nu+\nu_r}{a}f''' + (f+cg)f'' - f'^2 + 1 + \frac{2\nu_r}{a^2}G' = 0, \\ & \frac{\nu+\nu_r}{a}g''' + (f+cg)g'' - cg'^2 + c + \frac{2\nu_r}{a^2}F' + M^2(1-g') = 0, \end{aligned} \tag{50}$$

where $M^2 = \dfrac{\sigma_e B_0^2}{\rho a}$ is the Hartmann number. To these equations we append equations $(47)_5$, and (48) and the boundary conditions (44), and (45).

As far as the other two cases are concerned, if we proceed as previously we get

4.2. Case II

$$\mathbf{H}_0 = H_0\mathbf{e}_2.$$

$$\begin{aligned} p = & -\rho\frac{a^2}{2}\{x_1^2+[f(x_2)+cg(x_2)]^2+c^2x_3^2\} \\ & -\rho a(\nu+\nu_r)[f'(x_2)+cg'(x_2)] \\ & - 2\nu_r\rho\int_0^{x_2}[cF(s)+G(s)]ds - \sigma_e a B_0^2(x_1^2+cx_3^2)+p_0, \end{aligned} \tag{51}$$

$$\begin{aligned} & \frac{\nu+\nu_r}{a}f''' + (f+cg)f'' - f'^2 + 1 + \frac{2\nu_r}{a^2}G' + M^2(1-f') = 0, \\ & \frac{\nu+\nu_r}{a}g''' + (f+cg)g'' - cg'^2 + c + \frac{2\nu_r}{a^2}F' + M^2(1-g') = 0. \end{aligned} \tag{52}$$

4.3. Case III

$$\mathbf{H}_0 = H_0\mathbf{e}_3.$$

$$\begin{aligned} p = &- \rho\frac{a^2}{2}\{x_1^2 + [f(x_2) + cg(x_2)]^2 + c^2x_3^2\} \\ &- \rho a(\nu + \nu_r)[f'(x_2) + cg'(x_2)] - 2\nu_r\rho\int_0^{x_2}[cF(s) + G(s)]ds \\ &+ \sigma_e aB_0^2\Big\{\int_0^{x_2}[f(s) + cg(s)]ds - \frac{x_1^2}{2}\Big\} + p_0, \end{aligned} \tag{53}$$

$$\begin{aligned} &\frac{\nu + \nu_r}{a}f''' + (f + cg)f'' - f'^2 + 1 + \frac{2\nu_r}{a^2}G' + M^2(1 - f') = 0, \\ &\frac{\nu + \nu_r}{a}g''' + (f + cg)g'' - cg'^2 + c + \frac{2\nu_r}{a^2}F' = 0. \end{aligned} \tag{54}$$

Thus, we have the following:

Theorem 13. *Let a homogeneous, incompressible, electrically conducting micropolar fluid occupy the half-space $\mathcal{S}$. If we impress the external magnetic field $\mathbf{H}_0$ parallel to one of the coordinate axes and if we neglect the induced magnetic field, then the steady three-dimensional MHD stagnation-point flow of such a fluid has the form*

$$\begin{aligned} &v_1 = ax_1f'(x_2), \quad v_2 = -[af(x_2) + bg(x_2)], \quad v_3 = bx_3g'(x_2), \\ &w_1 = -cx_3F(x_2), \quad w_2 = 0, \quad w_3 = x_1G(x_2), \quad \mathbf{E} = \mathbf{0}, \end{aligned}$$

and

1. *if $\mathbf{H}_0 = H_0\mathbf{e}_1$, then the pressure field is given by (49) and (f, g, F, G) satisfies problem (50),* (47)$_5$, (48), *(44), and (45), provided $F, G \in L^1([0, +\infty))$;*
2. *if $\mathbf{H}_0 = H_0\mathbf{e}_2$, then the pressure field is given by (51) and (f, g, F, G) satisfies problem (52),* (47)$_5$, (48), *(44), and (45), provided $F, G \in L^1([0, +\infty))$;*
3. *if $\mathbf{H}_0 = H_0\mathbf{e}_3$, then the pressure field is given by (53) and (f, g, F, G) satisfies problem (54),* (47)$_5$, (48), *(44), and (45), provided $F, G \in L^1([0, +\infty))$.*

Remark 14. We remark that, from (49), (51), (53), the pressure along the wall $x_2 = 0$ takes its maximum again in the stagnation-point.

Remark 15. If $c = 1$, $f = g$, $F = G$, $\mathbf{H}_0 = H_0\mathbf{e}_2$, the axial symmetric case is obtained.

Now it is convenient to write the boundary value problems in Theorem 13 in dimensionless form in order to reduce the number of the material parameters. To this end we put

$$\eta = \sqrt{\frac{a}{\nu+\nu_r}}x_2, \qquad \varphi(\eta) = \sqrt{\frac{a}{\nu+\nu_r}}f\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right),$$
$$\gamma(\eta) = \sqrt{\frac{a}{\nu+\nu_r}}g\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right), \; \Phi(\eta) = \frac{2\nu_r}{a^2}\sqrt{\frac{a}{\nu+\nu_r}}F\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right),$$
$$\Gamma(\eta) = \frac{2\nu_r}{a^2}\sqrt{\frac{a}{\nu+\nu_r}}G\left(\sqrt{\frac{\nu+\nu_r}{a}}\eta\right). \tag{55}$$

So system (50), $(47)_5$, (48) can be written as

$$\begin{aligned}
&\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 + \Gamma' = 0,\\
&\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c + \Phi' + M^2(1-\gamma') = 0,\\
&\Phi'' + c_3\Phi'(\varphi + c\gamma) - \Phi(c_3c\gamma' + c_2) - c_1\gamma'' = 0,\\
&\Gamma'' + c_3\Gamma'(\varphi + c\gamma) - \Gamma(c_3\varphi' + c_2) - c_1\varphi'' = 0,
\end{aligned} \tag{56}$$

where

$$c_1 = \frac{4\nu_r^2}{\lambda a}, \quad c_2 = \frac{4\nu_r(\nu+\nu_r)}{\lambda a}, \quad c_3 = \frac{I}{\lambda}(\nu+\nu_r). \tag{57}$$

The boundary conditions in dimensionless form become:

$$\begin{aligned}
&\varphi(0) = 0, \quad \varphi'(0) = 0,\\
&\gamma(0) = 0, \quad \gamma'(0) = 0,\\
&\Phi(0) = 0, \quad \Gamma(0) = 0,\\
&\lim_{\eta\to+\infty}\varphi'(\eta) = 1, \quad \lim_{\eta\to+\infty}\gamma'(\eta) = 1,\\
&\lim_{\eta\to+\infty}\Phi(\eta) = 0, \quad \lim_{\eta\to+\infty}\Gamma(\eta) = 0.
\end{aligned} \tag{58}$$

Moreover, equations (52) can be written as

$$\begin{aligned}
&\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 + \Gamma' + M^2(1-\varphi') = 0,\\
&\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c + \Phi' + M^2(1-\gamma') = 0;
\end{aligned} \tag{59}$$

and equations (54) as

$$\varphi''' + (\varphi + c\gamma)\varphi'' - \varphi'^2 + 1 + \Gamma' + M^2(1-\varphi') = 0,$$
$$\gamma''' + (\varphi + c\gamma)\gamma'' - c\gamma'^2 + c + \Phi' = 0. \tag{60}$$

Of course we obtain other two different ordinary differential problems by adjoining equations $(56)_{3,4}$ and the boundary conditions (58).

We now analyse the cases considered in Remark 2. For sake of simplicity we use the dimensionless equations.

Proposition 16. *Let a homogeneous, incompressible, electrically conducting micropolar fluid occupy the half-space $\mathcal{S}$. If we neglect the induced magnetic field and we suppose either*

$$i)\ \ c = 1, \quad \mathbf{H_0} \ \textit{parallel to the plane } Ox_1x_3,$$

or

$$ii)\ \ c = -\frac{1}{2}, \quad \mathbf{H_0} \ \textit{parallel to the plane } Ox_2x_3,$$

then there is no solution to the problem of steady three-dimensional MHD stagnation-point flow.

Proof. i) If $c = 1$ and the external magnetic induction field is $\mathbf{B} = B_1\mathbf{e}_1 + B_3\mathbf{e}_3$ $(B_1, B_3 \neq 0)$, then after some calculations we deduce:

$$\varphi''' + (\varphi + \gamma)\varphi'' - \varphi'^2 + M_3^2(1-\varphi') + 1 + \Gamma' = 0,$$
$$M_1M_3(\gamma' - 1) = 0,$$
$$\gamma''' + (\varphi + \gamma)\gamma'' - \gamma'^2 + M_1^2(1-\gamma') + 1 + \Phi' = 0,$$
$$M_1M_3(\varphi' - 1) = 0, \tag{61}$$

where $M_i^2 = \dfrac{\sigma_e B_i^2}{\rho a}$, $\quad i = 1, 3.$

From $(61)_2$, $(61)_4$, we have $\varphi' = \gamma' = 1$, $\quad \forall \eta \geq 0$, which contradicts the boundary conditions $(58)_{2,4}$.

ii) If $c = -\dfrac{1}{2}$ and $\mathbf{B} = B_2\mathbf{e}_2 + B_3\mathbf{e}_3$ $(B_2, B_3 \neq 0)$, then we arrive at:

$$\varphi''' + \left(\varphi - \frac{\gamma}{2}\right)\varphi'' - \varphi'^2 + (M_2^2 + M_3^2)(1-\varphi') + 1 + \Gamma' = 0,$$
$$\gamma''' + \left(\varphi - \frac{\gamma}{2}\right)\gamma'' + \frac{\gamma'^2}{2} + M_2^2(1-\gamma') - \frac{1}{2} + \Phi' = 0,$$

$$M_2M_3(\varphi - \gamma + \alpha - \beta) = 0, \tag{62}$$

where

$$\alpha = \sqrt{\frac{a}{\nu + \nu_r}}A, \quad \beta = \sqrt{\frac{a}{\nu + \nu_r}}B.$$

From $(62)_3$ evaluated at $\eta = 0$ and using $(58)_{1,3}$, we deduce $\varphi = \gamma$. Therefore we have to solve the following overdetermined ODE system

$$\begin{aligned}
&\varphi''' + \frac{\varphi}{2}\varphi'' - \varphi'^2 + (M_2^2 + M_3^2)(1 - \varphi') + 1 + \Gamma' = 0,\\
&\varphi''' + \frac{\varphi}{2}\varphi'' + \frac{\varphi'^2}{2} + M_2^2(1 - \varphi') - \frac{1}{2} + \Phi' = 0,\\
&\Phi'' + c_3\Phi'\frac{\varphi}{2} + \Phi\left(\frac{c_3}{2}\varphi' - c_2\right) - c_1\varphi'' = 0,\\
&\Gamma'' + c_3\Gamma'\frac{\varphi}{2} - \Gamma(c_3\varphi' + c_2) - c_1\varphi'' = 0,
\end{aligned} \tag{63}$$

together with the boundary conditions (58).

Our aim is to prove that such a problem does not admit solution. To this end, by subtracting $(63)_2$ to $(63)_1$, we obtain that (φ, Φ, Γ) solves the equation

$$\frac{3}{2}(\varphi'^2 - 1) + M_3^2(\varphi' - 1) + \Phi' - \Gamma' = 0. \tag{64}$$

Computing $(64), (63)_{3,4}$ at $\eta = 0$, gives

$$\Phi'(0) - \Gamma'(0) = \frac{3}{2} + M_3^2, \qquad \Phi''(0) = \Gamma''(0) = c_1\varphi''(0). \tag{65}$$

If we differentiate (64), then we have

$$3\varphi'\varphi'' + M_3^2\varphi'' + \Phi'' - \Gamma'' = 0. \tag{66}$$

By means of $(65)_2, (66)$ we deduce

$$\Phi''(0) = \Gamma''(0) = \varphi''(0) = 0. \tag{67}$$

After differentiating $(63)_{3,4}$ we get

$$\begin{aligned}
&\Gamma'''(0) = -c_2(1 + M_2^2 + M_3^2) + (c_1 - c_2)\varphi'''(0),\\
&\Phi'''(0) - \Gamma'''(0) = c_2\Big(\frac{3}{2} + M_3^2\Big),
\end{aligned} \tag{68}$$

where we have used $(65)_1$, and $(63)_1$.

If we differentiate (66), then we obtain

$$3\varphi''^2 + 3\varphi'\varphi''' + M_3^2\varphi''' + \Phi''' - \Gamma''' = 0 \tag{69}$$

from which, taking account of $(67)_3$, follows

$$\varphi'''(0) = -\frac{\Phi'''(0) - \Gamma'''(0)}{M_3^2}. \tag{70}$$

Further, from $(63)_{1,2}$ we get

$$\Phi'(0) = \frac{1}{2} - M_2^2 - \varphi'''(0), \qquad \Gamma'(0) = -[1 + M_2^2 + M_3^2 + \varphi'''(0)]. \tag{71}$$

Differentiating of (69) furnishes:

$$9\varphi''\varphi''' + 3\varphi'\varphi^{IV} + M_3^2\varphi^{IV} + \Phi^{IV} - \Gamma^{IV} = 0. \tag{72}$$

Another differentiation of $(63)_{3,4}$ gives

$$\Phi^{IV}(0) = \Gamma^{IV}(0) = c_1\varphi^{IV}(0),$$

so that (72) yields

$$\Phi^{IV}(0) = \Gamma^{IV}(0) = \varphi^{IV}(0) = 0. \tag{73}$$

If we differentiate (72) and evaluate the resulting equation at $\eta = 0$, we obtain

$$9\varphi'''(0)^2 + M_3^2\varphi^V(0) + \Phi^V(0) - \Gamma^V(0) = 0. \tag{74}$$

On the other hand we can find $\varphi^V(0)$ from $(63)_1$ after two differentiations:

$$\varphi^V(0) = (M_2^2 + M_3^2)\varphi'''(0) - \Gamma'''(0). \tag{75}$$

Moreover, by means of another differentiation of $(63)_{3,4}$ we arrive at

$$\Phi^V(0) - \Gamma^V(0) = c_2[\Phi'''(0) - \Gamma'''(0)] - c_3\varphi'''(0)\Big[2\Phi'(0) + \frac{5}{2}\Gamma'(0)\Big]. \tag{76}$$

Finally on substituting (75), (76) into (74) and taking into account $(68)_1$, (70), (71), we get

$$\begin{aligned}&\frac{9}{2}(2 + c_3)\varphi'''(0)^2 + \Big[M_3^2(M_2^2 + M_3^2 - c_1 + \frac{5}{2}c_3) + \frac{c_3}{2}(9M_2^2 + 3)\Big]\varphi'''(0)\\ &+ c_2M_3^2(1 + M_2^2 + M_3^2) = 0.\end{aligned} \tag{77}$$

The thesis follows easily because, as one can see from (68), (70), $\varphi'''(0)$ does not depend on c_1 and by differentiating (77) with respect to c_1, we obtain

$$M_3^2\varphi'''(0) = 0, \tag{78}$$

which gives the absurdum

$$\frac{3}{2} + \frac{\sigma_e}{a\rho}B_3^2 = 0,$$

as for a Newtonian fluid. □

Remark 17. If $M = 0$, then equations $(56)_{1,2}$ (or (59), or (60)), $(56)_{3,4}$ reduce to equations found by Guram and Anwar Kamal in [6].

Remark 18. The skin-friction components τ_1, τ_3 along x_1 and x_3 axes are given by (39). Actually $\varphi''(0), \gamma''(0)$ depend on $\mathbf{H}_0$ through M^2.

Remark 19. It is easy to verify that regular solutions of problem (59), $(56)_{3,4}$, (58) are invariant under the following transformation

$$\begin{aligned}
\varphi\left(\eta, \frac{1}{c}, \frac{c_1}{c}, \frac{c_2}{c}, c_3, \frac{M^2}{c}\right) &= \sqrt{c}\,\gamma\left(\frac{\eta}{\sqrt{c}}, c, c_1, c_2, c_3, M^2\right),\\
\gamma\left(\eta, \frac{1}{c}, \frac{c_1}{c}, \frac{c_2}{c}, c_3, \frac{M^2}{c}\right) &= \sqrt{c}\,\varphi\left(\frac{\eta}{\sqrt{c}}, c, c_1, c_2, c_3, M^2\right),\\
\Phi\left(\eta, \frac{1}{c}, \frac{c_1}{c}, \frac{c_2}{c}, c_3, \frac{M^2}{c}\right) &= \frac{1}{\sqrt{c}}\,\Gamma\left(\frac{\eta}{\sqrt{c}}, c, c_1, c_2, c_3, M^2\right),\\
\Gamma\left(\eta, \frac{1}{c}, \frac{c_1}{c}, \frac{c_2}{c}, c_3, \frac{M^2}{c}\right) &= \frac{1}{\sqrt{c}}\,\Phi\left(\frac{\eta}{\sqrt{c}}, c, c_1, c_2, c_3, M^2\right), \quad c > 0.
\end{aligned}$$

Moreover we notice that by means of the previous transformation regular solutions of problem $(56)_{1,2}$ (or (60)), $(56)_{3,4}$, (58) are transformed in solutions of problem (60) (or $(56)_{1,2}$), $(56)_{3,4}$, (58).

The previous transformation continues to hold even if $M = 0$.

Remark 20. In all cases here considered the results continue to hold even if there are external conservative body forces by modifying the pressure field appropriately.

Remark 21. The nonlinear boundary value problems here obtained are not solvable by means of analytical functions, but only through numerical methods. In a subsequent paper we will give the numerical integration of these boundary value problems.

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