International Journal of Pure and Applied Mathematics
Volume 73 No. 2 2011, 189-199

SOME SPECIAL SOLUTIONS OF $u'' = h(r, u)$, $r > 0$: COMPACTLY SUPPORTED SOLUTIONS AND DEAD CORE-SOLUTIONS

Tadie
Mathematics Institut
5, Universitetsparken
Copenhagen, 2100, DENMARK
tad@math.ku.dk

Abstract: The type of equations displayed above is often used for the modelling of various physical, chemical and biological phenomena and they have been subject to lots of investigations lately. Beside their importance as solutions of an ODE, the solutions of those equation can provide some lights on the properties of the solutions of some multidimensional elliptic equations. We are looking for decreasing solutions which have compact supports and increasing ones in some $[0, R)$ which have dead-core space. We find out that these various solutions can be obtained just by giving various data to the initial or boundary conditions. For the hypotheses, we require h to be continuous, nonegative and strictly positive whenever the last argument is zero and to be increasing in the second argument.

1. Introduction

Ordinary differential equation of the type

$$u'' = h(t, u), \quad t > t_0 \geq 0 \tag{1.1}$$

is considered. This problem is a very simple model of equations governing many Physical and Technological phenomena in which compactly supported solutions (as in magnetics islands) and dead core solutions (abscence of reactants inside some Chemical media in action). In fact for a link with multidimensional cases,

Received: July 28, 2011

consider for example the radially symmetric Laplacian problem

$$\begin{cases} D_\alpha U := U'' + \frac{\alpha}{r}\, U' = h(r,U); \quad r := |x|, \quad x \in \mathbf{R}^n; \\ \text{where } \alpha := n-1 \geq 1, \quad n \in \mathbf{N} \quad \text{and } \{\}' := \frac{d}{dr}\{\}. \end{cases} \tag{1.2}$$

By means of the transforms

$$\begin{cases} U(r) & = r^{1-\alpha} V(r^b), \quad \text{for } \beta := \frac{b+1-\alpha}{b} \\ D_\alpha U(r) & = b^2 r^{2b-\alpha-1} \left\{ \frac{\beta}{r^b}\, V'(r^b) + V''(r^b) \right\} \\ & := b^2\, t^{(2b-\alpha-1)/b}\, (D_\beta V)(t); \quad t := r^b; \\ & \quad \text{but for} \quad b = \alpha - 1 \text{ and } \nu(\alpha) = \frac{\alpha-3}{\alpha-1} \\ D_\alpha U(r) & = (\alpha-1)^2\, t^{\nu(\alpha)} V''(t). \end{cases} \tag{1.3}$$

Thus such an equation can be very useful even outside one-dimensional situations. We are mainly interested in the existence of *compactly supported* (or dead core at infinity) and of the existence of *dead cores* solutions of the associated problems under some specific hypotheses.

We recall here that a solution $u \in C^2(I_{t_0})$ for the equation

$$Lu := u'' - h(r,u) = 0 \quad r \in I(t_0) := (t_0,\ \infty)$$

is said to be a compactly supported solution if there is a compact subset $K \subset I(t_0)$ such that $u > 0$ in the interior of K and

$u(t) = u'(t) = 0 \quad \forall t \notin K$.

A solution of the same equation would be said to be a dead core one if there is a compact subset $K \subset I(t_0)$ such that

$u \equiv 0$ in K and $u > 0$ in $I(t_0) \setminus K$.

In the sequel we set the following hypotheses on h:

(h1): $h \in C(\ I(t_0) \times \mathbf{R}_+; \quad \mathbf{R}_+^*\ ) := C(\ [t_0,\ \infty) \times [0,\ \infty);\ (0,\ \infty)\ )$ and there is $A > 0$; $h(t,0) > A \quad \forall t > t_0$.

(h2): h is increasing in $u > 0$ for any fixed $t > 0$.

A. Compact Supported Solutions

As a model problem we consider for some $\mu,\ R,\ k\ > 0$ and $\boldsymbol{a > 0}$ the problem

$$\begin{cases} u'' = K(u) := k + u_+^p \quad t \in I := I(a) = (a,\ \infty); \\ u(a) = m > 0; \quad u'(a) = -\mu \end{cases} \tag{1.4}$$

where $u_+(r) := \max\{0,\ u(r)\}$.

Our interest is to look for *monotone decreasing solutions* which have a positive zero $\rho > a$, say, and satisfying $u(\rho) = u'(\rho) = 0$.

In the sequel, a solution will be supposed restricted to its monotone and non negative part.

Some Notations

For $J \subseteq \mathbf{R}$ and the functions $u,\ u_1, u_2, ...u_k \in C(\mathbf{R})$, define

$$\begin{cases} S(u_1, u_2, ...u_k;\ J) := \{r \in J \mid u_i(r) \geq 0 \quad \forall i = 1, 2, ...k\ \}; \\ S(u;\ J) := supp(u) \bigcap \overline{J} = \{r \in J \mid u(r) \geq 0\}. \end{cases} \tag{1.5}$$

We start with some uniqueness results for decreasing solutions of (1.4) having positive zeros.

Theorem 1. *1) If* $p \leq 1$ *or* $k > (p-1)m^p$ *then* (1.4) *has at most one decreasing solution.*

2) If $p > 1$ *or if* K *is Lipschitz continuous then for any* $\rho > a$

$$u'' = k + u^p \quad in \quad (a,\ \rho); \quad u(\rho) = u'(\rho) = 0 \tag{1.6}$$

has at most one decreasing solution.

Proof. 1) Assume that there are two solutions u and v such that $u > v$ in $L := (a,\ T)$, say. Then in L

$$\begin{cases} vu'' - uv'' & = (vu' - uv')' = uv\{\frac{K(u)}{u} - \frac{K(v)}{v}\} \\ & = uv \int_v^u (\{\frac{K(y)}{y}\}')\ < 0 \end{cases}$$

if any of the displayed conditions holds. This then implies that u/v is decreasing in L starting with the value 1 at a. This conflicts with the assumption that $u > v$ in L (see [5]).

2) Suppose that u and v are two solutions of (1.6) with $u > v$ in some $(t,\ \rho) := J$. From the equation, with $H(t) := (u - v)(t)$ and

$\Phi(t) := [K(u) - K(v)](t)$, in J

$H''(r) = \Phi(r); \quad H(\rho) = H'(\rho) = 0$ whence $H(r) = \int_r^\rho (s - r)\Phi(s)ds.$

Because $p > 1$, there is $B > 0$ such that $\Phi(t) \leq B|(u - v)(t)|$ in J. Thus dropping the factor $M\rho, \quad 0 \leq H(r) \leq \int_r^\rho (\rho - s)H(s)ds$ and we have

$$H(r) \leq I_1(r) := \int_r^\rho H(s)ds;$$

$$\int_r^\rho H(s)ds \le I_2(r) := \int_r^\rho I_1(s)ds = \int_r^\rho (s-r)H(s)ds;$$

$$\int_r^\rho (s-r)H(s)ds \le I_3(r) = \int_r^\rho I_2(s)ds = \int_r^\rho (s-r)^2/(2!)\ H(s)ds;$$

$\vdots$

$$\int_r^\rho I_{k-1}(s)ds = \int_r^\rho \{(s-r)^k\}(k!)\ H(s)ds$$

and we finally get that

$$H(r) \le \int_r^\rho \frac{(s-r)^n}{n!}\ H(s)ds \le \frac{m\ \rho^{n+1}}{n!} \qquad \forall n \in \mathbf{N} \tag{1.7}$$

which completes the proof (see [4]). □

2. Main Result

Theorem 2.*For any* $m > 0$, *the set* $M(m)$ *of the functions* $Mu > 0$ *such that* (1.4) *has a decreasing solution* u_M , *say, with a positive zero* $\rho := \rho(Mu) > a$ *is non empty and*

$$Mu_m := \min\ M(m) \ge \sqrt{2m\{k + \frac{m^p}{p+1}\}}. \tag{2.1}$$

1) $\forall Mu \in M(m)$, *if* $p \le 1$ *or* $k > (p-1)m^p$ *then* u_M *is unique.*

2) The solution $U_m \in C^2(\ [a,\ \rho_m))$ *corresponding to* Mu_m *where* ρ_m *denotes its zero solves uniquely*

$$\begin{cases} U'' = k + U^p \quad in \quad (a,\ \rho_m); \quad U(a) = m; \\ U'(a) = Mu_m; \quad U(\rho_m) = U'(\rho_m) = 0; \\ \rho_m = J_m^M u + a := a + \int_0^m du \Big\{ M_1 + 2ku + \frac{2}{p+1}\ u^{p+1} \Big\}^{-1/2} \end{cases} \tag{2.2}$$

where $M_1 = [Mu_m]^2 - 2m\{k + m^p/(p+1)\}$.
3) If $p > 1$ *or if*

$$J_\infty := \sup_{m>0} \int_0^m du \Big/ \sqrt{M_1 + 2ku + \frac{2}{p+1}\ u^{p+1}} < +\infty \tag{2.3}$$

then there is a unique maximal solution $U_* \in C^2(\,(a,\ \rho_*)\,)$ *that satisfies*

$$\begin{cases} U'' = k + U^p \quad in \quad (a,\ \rho_*); \\ U(\rho_*) = U'(\rho_*) = 0 \quad and \quad \lim_{r \searrow a} U(r) = +\infty. \\ \rho_* = J_\infty + a. \end{cases} \tag{2.4}$$

4) If $p \le 1$ *or if* $J_\infty = +\infty$ *then as* $m \nearrow \infty$, ρ_m *tends to* $+\infty$.

3. Some Lemmata

Lemma 3. *Let* $u \in C^2(I)$ *be a solution of* (1.4). *Then a necessary condition for* u *to be monotone decreasing is*

$$[Mu]^2 \ge 2m\{k + \frac{m^p}{p+1}\} := M(m, p). \tag{3.1}$$

Moreover, for any $r_1 > a$, $u' < 0$ *in* $[a,\ r_1]$ *if*

$$Mu > (r_1 - a)\{k + m^p\}. \tag{3.2}$$

Proof. From the equation we have $u'u'' = ku' + u'u^p$ whence $((u')^2)' = 2\{\ ku +\ u^{p+1}/(p+1)\}'$.

So, for $r \in I$, integrating over $(a,\ r)$ gives

$$\begin{cases} u'(r)^2 & = [Mu]^2 - 2m\{k + \frac{m^p}{p+1}\} + 2[ku(r) + \frac{u(r)^{p+1}}{p+1}] \\ & := M_1(Mu, m, p) + 2[ku(r) + \frac{u(r)^{p+1}}{p+1}] \\ & := M_1 + k_1 u(r) + k_2 u(r)^{p+1} \end{cases} \tag{3.3}$$

and if (3.3) holds then u remains strictly decreasing in $r > a$ as $u'(r)^2 > 0$ throughout. If ρ is its positive zero then $u'(\rho)^2 = [Mu]^2 - 2m\{k + m^p/(p+1)\}$. Also from (3.3), since $k_1 u(r) + k_2 u(r)^{p+1}$ is zero at $u = 0$, if $M_1 < 0$, u cannot reach 0. From the equation, if $u \in C^2(a,\ r_1)$ is a solution,then

$u'(r) = -Mu + \int_a^r K(u(s))ds \le -Mu + K(m)\ (r - a)$ and (2.2) follows.

From now on we assume that (3.1) is satisfied i.e. $M_1 \ge 0$.

Let $A > 0$. The function

$$U(r) := m - Mu(r - a) + \frac{A}{2}\ (r - a)^2 \tag{3.4}$$

solves

$u'' = A$ for $r > a$; $u(a) = m,\ u'(a) = -Mu$ and for $\rho > a$

$$\begin{cases} W(r) := (\rho - r)\{ \frac{m}{\rho - a} - \frac{A}{2}(r-a) \}; \\ \text{with} \quad Mu = \frac{m}{\rho-a} + \frac{A}{2}(\rho - a) \end{cases} \tag{3.5}$$

solves $u'' = A$ for $r > a$; $u(a) = m$, $u'(a) = -Mu$ and $u(\rho) = 0$ while

$$\begin{cases} V(r) := m - (r-a)\sqrt{2mA} + \frac{A}{2}(r-a)^2 \quad r \in (a,\ \rho) \quad \text{where} \\ \rho := a + \sqrt{2mA} \quad \text{and} \quad Mu_1 := \sqrt{2mA} \end{cases} \tag{3.6}$$

solves $u'' = A$ $r > a$; $u(a) = m$, $u'(a) = -Mu_1$ and $u(\rho) = u'(\rho) = 0$.

Lemma 4.*Let u be the solution in the last Lemma with $M_1 > 0$. Then ρ, the zero of u is finite. Moreover if $p > 1$ then there is $R_* > 0$ such that for any $m > 0$ the corresponding solution u_m (under the condition $M_1 \geq 0$) satisfies*

$$S(u_m;\ I) \subset [a,\ R_*]. \tag{3.7}$$

Proof. As u is decreasing, from (3.3)

$$J(m) := \int_0^m \frac{du}{\sqrt{M_1 + k_1 u + k_2 u^{p+1}}} = \rho_m - a \tag{3.8}$$

where ρ_m is the zero of u. Because m is finite and $M_1 \geq 0$, $J(m)$ is finite. In fact

$$\int_0^m \frac{du}{\sqrt{(M_1 + k_1 u + k_2 u^{p+1})}} < \int_0^m \frac{du}{\sqrt{(M_1 + k_1 u)}} < \infty.$$

Moreover if $p > 1$, for large m, the integral in (3.8) behaves like $\int_1^m dt/\sqrt{t^{p+1}}$ which converges as $m \nearrow \infty$. Thus

$$\begin{cases} p > 1 \Longrightarrow \\ J(m) := \int_0^m du/\{\sqrt{M_1 + k_1 u + k_2 u^{p+1}}\} = \rho_m - a \\ J(m) < \int_0^1 \frac{dt}{\sqrt{M_1 + k_1 t}} + \int_1^\infty \frac{dt}{\sqrt{t^{p+1}}} := R_* - a. \end{cases} \tag{3.9}$$

Lemma 5.*Let u_1 and u_2 be respectively two decreasing solutions of $w_i'' = K(w_i)$ in $(a,\ R)$; $w_i(a) = m_i$; $w_i'(a) = Mu_i$; $i = 1, 2$.
Then they cannot have more than one common point. Consequently*

1) if $m_1 = m_2$ and $0 < Mu_2 < Mu_1$ then $u_2 > u_1$ and $u_2' > u_1'$ in $S(u_1, u_2;\ R)$.

2) If the two functions solve respectively
$w_i'' = K(w_i)$ *in* $(a,\ \rho_i)$; $w_i(a) = m_i$; $w_i(\rho_i) = w_i'(\rho_i) = 0$; $i = 1, 2$
with $m_1 < m_2$ *then* $u_2 > u_1$ *in* $S(u_1, u_2;\ \rho_2) \equiv [a,\ \rho_1]$.
We also get $u_2 > u_1$ *in* $S(u_1, u_2;\ [a,\ \rho_1]))$ *if* $m_2 > m_1$ *and* u_1 *satisfies* $u_1(\rho_1) = u_1'(\rho_1) = 0$.
So, given different data, two distinct solutions of (1.6) *do not intersect.*

Proof. 1) If for $H = u_2 - u_1$ there are $T, S \in (a,\ R)$ such that $H(T) = H(s) = 0$ and $H > 0$ in $(T,\ S)$ then there is $\xi \in (T,\ S)$; $H'(\xi) = 0$ whence $H''(\xi) > 0$ making $H(\xi)$ a minimum point; that is absurd.
In the case $m_1 = m_2$, they do not have any other intersection. So, as $H'(a) = Mu_1 - Mu_2 > 0$, $H' < 0$ in $S(u_1, u_2,\ R)$.

2) Suppose that there is $T > a$ such that $H(T) := (u_2 - u_1)(T) = 0$ $u_1 > u_2$ in some $(T,\ T_1)$ with $H(T_1) \leq 0$. Then as $H'(T) < 0$ and $H'(T_1) > 0$ $\exists S \in (T,\ T_1)$; $H'(S) = 0$ whence $H''(S) = [K(u_2) - K(u_1)](S) < 0$ which cannot hold as $H(S) < 0$ and cannot be a local maximum.

Theorem 6.*Consider the function* V *in* (3.6) *where* $A = k + m^p$. *Then for any* $Mu \geq \sqrt{2mA}$, (1.4) *has a strictly decreasing solution* $u := u_M \in C^2((a, \rho_m))$ *such that* $0 \leq u \leq V$.

Proof. Let $D := (a,\ \rho)$. The operator T defined on
$E := E_V = \{\phi \in C^1(\overline{D});\quad \phi(a) = m;\quad 0 \leq \phi \leq V;\quad \phi' \leq -Mu_1 \ in\ D\}$
by

$$U(r) = Tu(r) \Longleftrightarrow U'' = K(u)\ in\ S(u;\ D);\quad U(a) = m;\ U'(a) = -Mu.$$

Through easy verification, $TE \subset E$. By classical theory of fixed points, T has a fixed point u say, which is such a solution.

4. Proof of Theorem 2

The existence of solutions u_M and the lower bound for $M(m)$ follow from Theorem 6 and Lemma 3; the uniquensss results from Theorem 1.

2) Lemma 5 shows that the solutions u_M increase as $Mu \in M(m)$ decreases. So, U_m, the solution corresponding to Mu_m is a maximal solution in the sense that $\forall Mu \in M(m)$, $u_M < U_m$ in $S(u_M, U_m;\ [0,\ \rho_{Mu}))$.

If $U_m'(\rho_m) \neq 0$ then by the implicit function theorem, there is a small $\eta > 0$ such that $(\rho_m - \eta,\ \rho_m + \eta) \subset M(m)$ conflicting with the nature of ρ_m. Thus $U_m'(\rho_m) = 0$. (see [6])

3) We know from (3.1) that $Mu_m \nearrow \infty$ as $m \nearrow \infty$. Consider the family

$U := \{U_m := \{U_m \ ; \ D_m := (a, \ \rho_m)\}; \quad m \in \mathbf{N}\}$ where $\forall m \in \mathbf{N}$

(a) $U_m \in C^2((a, \ \rho_m))$ and satisfies

$U'' = k + U^p$ in D_m ; $U(a) = m; \ U(\rho_m) = U'(\rho_m) = 0$;

(b) $D_m \subseteq D_{m+1} \subset [a, \ R_*)$ and $U_m < U_{m+1}$ in $S(U_m, U_{m+1}; \ D_m)$.
Obviously the properties (a) and (b) endow U with the following order relation: $U_m \preceq U_{m+1} \Longleftrightarrow$ (a) and (b) are satisfied.

Thus $(\ U; \ \preceq \ )$ is a chain and the required $U_* := \{U_* \ ; \ (a, \ \rho_*) \ \}$ is the maximal element of U.

B. Dead Cores Solutions

We here consider the following problem:

given $m, \ k, \ p, \ R > 0$ can one find $a \in (0, \ R)$ and $M_u > 0$ such that

$$u'' = K(u) := k + u^p \quad \text{in} \quad D := (0, \ R); \quad u(R) = m; \ u'(R) = M_u \tag{4.1}$$

has an increasing solution $u \in C^2(\ (a, \ R) \ ) \bigcap C^1([a, \ R])$ satisfying

$u(a) = u'(a) = 0$ and $u > 0$ in $A := (a, \ R]$?

It then comes to the following overdetermined problem:

$$\begin{cases} (a1): & u'' = k + u^p \quad \text{in} \quad A := (a, \ R); \quad u(R) = m; \ u'(R) = M_u; \\ (a2): & u(a) = u'(a) = 0 \quad \text{and} \quad u' > 0 \text{ in } A. \end{cases} \tag{4.2}$$

The problem in (4.2) has two initial data in (a1) and two final data in (a2).

5. Main Result

Consider for prescribed $\rho > a \geq 0, \ R, m, M_u > 0$ the problem

$$u'' = K(u) := k + u^p \quad \text{in} \quad A := (a, \ R); \tag{5.1}$$

$$u(R) = m; \ u'(R) = M_u > 0; \tag{5.2}$$

$$u(\rho) = u'(\rho) = 0. \tag{5.3}$$

Theorem 7. *For any m and $R > 0$ the set $N(m)$ of $M_u > 0$ such that (5.1) - (5.2) has an increasing solution u_M say, with a positive zero $\rho \in A$ is non empty provided that for*

$$M_1 = M_1(M_u) := M_u^2 - 2m\{k + m^p/(p+1)\},\ k_1 = 2k \textit{ and } k_2 = 2/(p+1)$$

$$\begin{cases} M_1 \geq 0 \quad \textit{and} \quad J_m^{M_u}(0) := \int_0^m dt/\{\sqrt{M_1 + k_1 t + k_2 t^{p+1}}\} < R; \\ Mu_m := \min\ N(m) \geq \min\{K(0)R,\ m/R\} \\ \rho = R - J_m^{M_u}(0). \end{cases} \tag{5.4}$$

1) (a) *If in addition of* (5.1) *and* (5.2) *the value of* u_M *or that of* u'_M *is prescribed at a point of* $[a,\ R)$ *then the resulted problem has at most one solution.*

(b) *If in addition of* (5.1) *and* (5.3) *the value of* u_M *or that of* u'_M *is prescribed at a point of* $(a,\ R]$ *then the resulted problem has at most one solution.*

(c) *If* K *is Lipstchitz continuous then* (5.1)-(5.2) *or* (5.1)-(5.3) *has at most one solution.*

2) The solution U_m *of* (5.1) - (5.2) *corresponding to* Mu_m *belongs to* $C^2(\ (\rho_m,\ R)\ ) \bigcap C^1([\rho_m,\ R])$, *is unique and solves*

$$\begin{cases} U'' = K(U) \quad \textit{in } (\rho_m,\ R);\ U(R) = m; \\ U'(R) = Mu_m;\ U(\rho_m) = U'(\rho_m) = 0; \\ \rho_m := R - J_m^{Mu_m}(0). \end{cases} \tag{5.5}$$

3) If $p > 1$ *or if*

$$J_\infty(0) := \sup_{m>0} \int_0^m \frac{dt}{\sqrt{M_1(Mu_m) + k_1 t + k_2 t^{p+1}}} < R$$

there is a maximal solution $U_* \in C^2(\ (\rho_*,\ R))$ *which satisfies*

$$\begin{cases} U'' = K(U) := k + U^p \quad \textit{in} \quad (\rho_*,\ R); \\ U(\rho_*) = U'(\rho_*) = 0 \quad \textit{and} \quad \lim_{r \nearrow R} U(r) = +\infty. \\ \text{where} \quad \rho_* = R - J_\infty(0). \end{cases} \tag{5.6}$$

Inspired by Theorem 2, the proof follows from the next lemmae.

Lemma 8. *For the problem* (4.2), *any provided three of the data leads to the uniqueness of the solution.*

Moreover, if K *is Lipschitz continuous (e.g.* $p \geq 1$*) in* $[a,\ R]$ *then the set of data in (a1) or that in (a2) leads to the uniqueness of the solution.*

Proof. The proof is a slight version of that seen in the part A.

It is enough to show that the hypotheses imply that the two solutions have two common points or that their first derivatives have two common points, which cannot hold when they are distinct. In fact if $H'(T) := (u-v)'(t) = H'(S)$ then $H'' = u^p - v^p$ has a zero between T and S whence $u-v$ has a zero there making the second common point.

Lemma 9. *Let u be a solution of (a1). For $k_1 = 2k$, $k_2 := 2/(p+1)$ and $M_1 := M_1(M_u) = M_u^2 - 2m\{k + m^p/(p+1)\}$,*

(i) if

$$J_m^{M_u}(0) := \int_0^m \frac{dt}{\sqrt{M_1(M_u) + k_1 t + k_2 t^{p+1}}} > R\,, \tag{5.7}$$

then u has no positive zero and $u(0) = \alpha > 0$ where $J_m^{M_u}(\alpha) = R$;

(ii) if

$$M_u > \frac{2m}{R} + \frac{R}{2}K(m) := \frac{2m}{R} + \frac{R}{2}(k + m^p) \tag{5.8}$$

then u has a positive zero ρ, say, with

$$J_m^{M_u}(0) = \int_0^m \frac{dt}{\sqrt{M_1(M_u) + k_1 t + k_2 t^{p+1}}} = R - \rho. \tag{5.9}$$

Also if

$$M_u < \min\{kR,\ \frac{m}{R}\} := \min\{K(0)R,\ \frac{m}{R}\} \tag{5.10}$$

then u has no positive zero.

Proof. As in (3.3), $u'(r)^2 = M_1 + k_1 u + k_2 u^{p+1}$ and (i) is deduced from it.

(ii) From the equation, for $r \in (0,\ R)$ $u'(r) = M_u - \int_r^R K(u)ds$ and $u(r) = m - M_u(R-r) + \int_r^R\{\int_s^R K(u(t))dt\}ds = m - M_u(R-r) + \int_r^R (r - s)K(u(s))ds$. Thus we deduce that $u(R/2) \le m + R/2\{-M_u + (k+m^p)R/2\} < 0$
if (5.8) holds and $u'(0) \le M_u - kR$. Also from the equation $u(\rho) = 0$, we have $M_u(R-\rho) = \{m + \int_\rho^R (\rho - s)K(u(s))ds\}$ thus
$M_u > \{m + \int_\rho^R (\rho - s)K(u(s))ds\}/R$ and if (5.10) holds, we cannot have such a ρ.

Lemma 10. *Let $0 < m_1 < m_2$ and $u_1,\ u_2$ the respective solutions of*

$$u_i'' = K(u_i) \text{ in } D := (0,\ R); \quad u_i(R) = m_i\ ; i = 1, 2.$$

If there is $\rho_2 \in D$ such that $u_2(\rho_2) = u_2'(\rho_2) = 0$ then $u_1 < u_2$ in $S(u_1, u_2;\ D)$.

Consequently, given different sets of data, two distinct solutions of (a1)-(a2) do not intersect.

In the same way, if $m_1 = m_2$ *and* $M_{u_1} > M_{u_2}$ *then* $u_2 > u_1$ *in* $S(u_1, u_2;\ D)$.

Proof. Assume that there is $T \in (\rho_2,\ R)$; $H(T) := (u_2 - u_1)(T) = 0$ with $H < 0$ in $(\rho_2,\ T)$. Then $H'(T) > 0 > H'(\rho_2)$ and H' has a zero between ρ_2 and T which cannot hold. In fact

$H'' = u_2^p - u_1^p < 0$ in $(\rho_2,\ T)$ and $H'(\rho_2) < 0$ implies that $H' < 0$ in that interval.

References

[1] J.I. Diaz, M.A. Herrero, Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems, *Proc. Royal Soc. Edinburgh*, **89A** (1981), 249-258.

[2] D. O'Regan, Existence of positive solutions to some singular and nonsingular second order boundary value problems, *J. Diff. Equ.*, **84** (1990), 228-251.

[3] Uwe Schafer, Unique solvability of an ordinary free boundary problem, *Rocky Mountain J. Math.*, **34**, No. 1 (2004), 341-346.

[4] Tadié, Boundary value problems for $(\phi(u'))' + f(t, u, u') = 0$, $t \in [a, b]$ with monotone ϕ and $f(t, ., .)$, *Int. J. Appl. Math.*, **4**, No. 4 (2000), 453-462,

[5] Tadié, On uniqueness conditions for decreasing solutions of semilinear elliptic equations, *Z. Anal. Anwendungen*, **18**, No. 3 (199), 517-523.

[6] Tadié, An ODE approach for $\triangle u + \lambda u^p - u^{-\gamma} = 0$ in $\mathbf{R}^n$; $\gamma \in (0, 1)$, $p > 0$, *Can. Appl. Math. Q.*, **10**, No. 3 (2002), 375-386.

200