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OSCILLATIONS OF NEUTRAL NONCONSTANT DELAY IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH VARIABLE COEFFICIENTS

M.B. Dimitrova[1 §], V.I. Donev[2]
[1,2]Department of Mathematics
Technical University of Sliven
8800, Sliven, BULGARIA

Abstract: This paper is dealing with the oscillatory properties of first order neutral impulsive differential equations and corresponding to them inequalities with variable coefficients. The established sufficient conditions ensure the oscillation of every solution of this type of equations.

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1. Introduction

There are a lot of processes and phenomena, observed in the theory of optimal control, theoretical physics, population dynamics, biotechnology, industrial robotics, etc., that depend on their history and are subject to short-time disturbances. Very convenient instrument for their adequate description as mathematical models are the impulsive differential equations with deviating arguments (IDEDA). But, the theory of IDEDA and especially their oscillation theory , due to theoretical and practical difficulties, are developing rather slowly, in contrast to the theory of impulsive differential equations (see, [1] and [10]) and differential equations with deviating arguments (see,). We note here

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§Correspondence author

that [6] is the first work where IDEDA and their oscillation theory were considered and studied. For some results, concerning IDEDA with advanced or retarded terms, we choose to refer to [2],[8] or [11], respectively.

Much less we know about the IDEDA with neutral term, i.e. impulsive differential equations in which the highest-order derivative of the unknown function appears in the equation with the argument t (the present state of the system), as well as with one or more retarded arguments (the past state of the system). As it is known (see [5]), the appearance of the neutral term in a differential equation can cause or destroy the oscillation of its solutions. Moreover, the study of neutral differential equations in general , presents complications which are unfamiliar for non-neutral differential equations. As for a discussion on more applications and drastic differences in behavior of the solution of neutral differential equations see, for example, [3],[4],[7] and [9]. Note, that in contrast to [3],[4] and [7], the present paper consider more general homogeneous impulsive differential equations with neutral term of delay.

2. Preliminaries

The object of investigation in the present work is the first order impulsive differential equation with variable coefficients and nonconstant neutral delay arguments of the form

$$\frac{d}{dt}[y(t)-c(t)y(h(t))]+p(t)y(\sigma(t))=0,\ t\neq\tau_k,\ k\in N \qquad (E_1)$$

$$\Delta[y(\tau_k)-c_ky(h(\tau_k))]+p_ky(\sigma(\tau_k))=0,\ k\in N$$

as well as the corresponding to it inequalities

$$\frac{d}{dt}[y(t)-c(t)y(h(t))]+p(t)y(\sigma(t))\leq 0,\ t\neq\tau_k,\ k\in N \qquad (N_{1,\leq})$$

$$\Delta[y(\tau k)-c_ky(h(\tau_k))]+p_ky(\sigma(\tau_k))\leq 0,\ k\in N$$

and

$$\frac{d}{dt}[y(t)-c(t)y(h(t))]+p(t)y(\sigma(t))\geq 0,\ t\neq\tau_k,\ k\in N \qquad (N_{1,\geq})$$

$$\Delta[y(\tau k)-c_ky(h(\tau_k))]+p_ky(\sigma(\tau_k))\geq 0,\ k\in N.$$

Here, c_k and p_k are constants and the points $\tau_k\in(0,+\infty)$, $k\in N$ are the moments of impulsive effect (let us call them jump points), where the unknown function reveals its discontinuities of first kind as jumps. In order to manifest these jumps of the unknown function $y(t)$, we use the notation $\Delta[y(\tau k) - c_k y(h(\tau_k))] = \Delta[y(\tau_k)] - c_k\Delta[y(h(\tau_k))]$, $\Delta[y(\tau_k)] = y(\tau_k + 0) - y(\tau_k - 0)$.

Denote by $P_\tau C(R, R)$ the set of all functions $u\colon R \to R$, which satisfy the following conditions:

(i) u is piecewise continuous on $(\tau_k, \tau_{k+1}]$, $k \in N$,

(ii) u is continuous from the left at the points τ_k, i.e.

$$u(\tau_k - 0) = \lim_{t\to\tau_k-0} u(t) = u(\tau_k),$$

(iii) there exists a sequence of reals $\{u(\tau_k + 0)\}_{k=1}^{\infty}$, such that

$$u(\tau_k + 0) = \lim_{t\to\tau_k+0} u(t),$$

(iv) u may have discontinuities of first kind at the jump points τ_k, $k \in N$, that we qualify as down-jumps when $\Delta u(\tau_k) < 0$, or as up-jumps when $\Delta u(\tau_k) > 0$, $k \in N$.

We introduce the following hypotheses, where $R_+ = (0, +\infty)$:

H1. $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$, $\lim\limits_{k\to+\infty} \tau_k = +\infty$, $\max\big\{\tau_{k+1} - \tau_k\big\} < +\infty$, $k \in N$.

H2. $p(t) \in P_\tau C(R_+, R_+)$, $p_k \geq 0$, $k \in N$.

H3. $h(t), \sigma(t) \in C^1(R_+, R_+)$, $h'(t) > 0$, $\lim\limits_{t\to+\infty} h(t) = +\infty$, $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$ and there exists a constant $D > 0$, such that $t > \sigma(t) \geq h(t + D) > h(t)$, for every $t > 0$.

H4. $c(t) \in P_\tau C(R_+, R_+)$, $c(t) \geq 1$, $c_k = c(\tau_k - 0) = c(\tau_k)$.

H5. $\int\limits_0^{+\infty} p(s)ds + \sum\limits_{k=1}^{+\infty} p_k = +\infty$.

Let $\rho(t) = \min\limits_{t\in R^+}\{\sigma(t), h(t)\}$. We say that a real valued function $y(t)$ is a *solution* of the equation (E_1), if there exists a number $T_0 \in R$ such that $y \in P_\tau C([\rho(T_0), +\infty), R)$, the function $z(t) = y(t) - c(t)y(h(t))$ is continuously differentiable for $t \geq T_0$, $t \neq \tau_k$, $k \in N$ and $y(t)$ satisfies (E_1) for all $t \geq T_0$.

Without other mention, we will assume throughout that every solution $y(t)$ of equation (E_1), that is under consideration here, is continuable to the right and is nontrivial. That is, $y(t)$ is defined on some ray of the form $[T_y, +\infty)$ and $\sup\big\{|y(t)|\colon\ t \geq T\big\} > 0$ for each $T \geq T_y$. Such a solution is called a *regular solution* of equation (E_1).

We will say that a real valued function u defined on an interval of the form $[a, +\infty)$ has some property *eventually* , if there is a number $b \geq a$ such that u has this property on the interval $[b, +\infty)$.

A regular solution $y(t)$ of equation (E_1) is said to be *nonoscillatory,* if there exists a number $t_0 \geq 0$ such that $y(t)$ is of constant sign for every $t \geq t_0$. Otherwise, it is called *oscillatory*. Also, note that a *nonoscillatory* solution is called *eventually positive* (*eventually negative*), if the constant sign that determines its *nonoscillation* is positive (negative). Equation (E_1) is called oscillatory, if all its solutions are oscillatory.

Finally, in this article, when we write a functional expression, we will mean that it holds for all sufficiently large values of the argument.

Moreover, in order to assist our investigations on the oscillation of (E_1), we consider the most generalized case of the impulsive differential equations of advanced type

$$z'(t) - Q(t)z(s(t)) = 0,\ t \neq \tau_k \qquad (E_2)$$

$$\Delta z(\tau_k) - q_k z(s(\tau_k)) = 0,\ k \in N$$

and the corresponding to it inequalities which are of the form

$$z'(t) - Q(t)z(s(t)) \leq 0,\ t \neq \tau_k \qquad (N_{2,\leq})$$

$$\Delta z(\tau_k) - q_k z(s(\tau_k)) \leq 0,\ k \in N$$

and

$$z'(t) - Q(t)z(s(t)) \geq 0,\ t \neq \tau_k \qquad (N_{2,\geq})$$

$$\Delta z(\tau_k) - q_k z(s(\tau_k)) \geq 0,\ k \in N$$

under the following hypotheses:

$H2^*$**.** $Q(t) \in P_\tau C(R_+, R_+)$ $q_k \geq 0,\ k \in N$.

$H3^*$**.** $s(t) \in C^1(R_+, R_+),\ s(t) > t,\ \lim\limits_{t\to+\infty}(s(t) - t) < +\infty$ and there exists a constant $D > 0$, such that $s(t) - t \geq D$.

3. Some Useful Lemmas

Consider $y(t)$ as a solution of equation (E_1) and set the auxiliary function

$$z(t) = y(t) - c(t)y(h(t)), \qquad \Delta z(\tau_k) = \Delta y(\tau_k) - c_{\tau_k}\Delta y(h(\tau_k)),,\ \ c_{\tau_k} = c(\tau_k)\ k \in N. \qquad (*)$$

At the beginning, we introduce two lemmas, which investigate the asymptotic behavior of the auxiliary function $z(t)$, when $y(t)$ is a none-oscillatory solution of (E_1). First of them is formulated and proved for eventually positive solution $y(t)$ of the equation (E_1).

Lemma 1. *Let $y(t)$ be an eventually positive solution of (E_1) and the hypotheses $(H1)-(H5)$ are satisfied. Then, :*

(a) $z(t)$, defined by $(*)$, is an eventually decreasing function of t with down-jumps;

(b) $z(t)$, defined by $(*)$, is an eventually negative function, i.e. $z(t) < 0$ for enough large t and $\lim_{t\to+\infty} z(t) = -\infty$.

Proof. (a) Let $y(t)$ be an eventually positive solution of the equation (E_1), i.e. there exists a number $\tilde{t} > 0$ such that $y(t)$ is defined for $t \geq \tilde{t}$ and $y(t) > 0, y(\sigma(t)) > 0, y(h(t)) > 0$ for $t \geq \rho^{-1}(\tilde{t}) = t_0$. From (E_1) and $(*)$, it follows that

$$z'(t) = -p(t)y(\sigma(t)) < 0,\ t \neq \tau_k,\ k \in N,\ t \geq t_0, \tag{1}$$
$$\Delta z(\tau_k) = -p_k y(\sigma(\tau_k)) < 0,\ k \in N,\ \tau_k \geq t_0.$$

Therefore, $z(t)$ is an eventually decreasing function ($z'(t) < 0$) with "down-jumps" ($\Delta z(\tau_k) < 0$) at the points of impulsive effect τ_k, for $t, \tau_k \geq t_0$. The proof of (a) is complete.

(b) It follows from (1) that $\lim_{t\to+\infty} z(t)$ does exist, where $z(t)$ is an eventually strictly decreasing function with down-jumps. So, $\lim_{t\to+\infty} z(t) = L$, where L could be positive constant, zero, negative constant, or $-\infty$.

Assume $\lim_{t\to+\infty} z(t) = L > 0$. If integrate (E_1) from t_0 to t, we obtain

$$\int_{t_0}^{t} z'(r)dr + \int_{t_0}^{t} p(r)y(\sigma(r))dr = 0, \qquad \text{or}$$

$$z(t) - z(t_0) - \sum_{t_0 \leq \tau_k < t} \Delta z(\tau_k) + \int_{t_0}^{t} p(r)y(\sigma(r))dr = 0.$$

But $\Delta z(\tau_k) = -p_k y(\sigma(\tau_k))$, hence

$$z(t) = z(t_0) - \sum_{t_0 \leq \tau_k < t} p_k y(\sigma(\tau_k)) - \int_{t_0}^{t} p(r)y(\sigma(r))dr \tag{2}$$

Because $z(t) = y(t) - c(t)y(h(t))$, we have in this case $L \leq z(t) < y(t)$, what determines $y(t)$ as a bounded function from below. Then (2) reduces to

$$z(t) \leq z(t_0) - L[\sum_{t_0 \leq \tau_k < t} p_k + \int_{t_0}^{t} p(t)dr], \tag{3}$$

which, in view of hypothesis $(H5)$, implies $\lim_{t\to+\infty} z(t) = -\infty$ and contradicts our assumption.

$$\text{Assume} \qquad \lim_{t\to+\infty} z(t) = \lim_{t\to+\infty}[y(t) - c(t)y(h(t))] = L = 0. \tag{4}$$

It is obvious, that then $z(t) > 0$ eventually, i.e. there exists a number $t_1 \geq t_0$, such that we have $y(t) > c(t)y(h(t)) > y(h(t))$ for every $t \geq t_1$. Observe that the last inequality holds as well as for those moments of impulsive effect τ_k , for which $\tau_k > t_1,\ k \in N$. Therefore, our assumption implies that there will exist a strictly increasing sequence $\{y(t_n)\}_{n=1}^{\infty}$ (where $t_n = h^{-1}(t_{n-1})$ and some t_n could be moments of impulsive effect τ_k), which is bounded by a positive number, i.e. $\lim_{n\to+\infty} y(t_n) = K,\ K > 0,$ or which is unbounded, i.e. $\lim_{n\to+\infty} y(t_n) = +\infty$ and for which (4) has to be fulfilled. But,

$$\lim_{t_n\to+\infty} z(t_n) = \lim_{t_n\to+\infty} y(t_n) - c(t_n) \lim_{t_n\to+\infty} y(h(t_n)) = (1 - c(t_n)) \lim_{t_n\to+\infty} y(t_n) < 0$$

and the contradiction with (4) is evident, because $c(t_n) > 1$.

Assume $\lim_{t\to+\infty} z(t) = L < 0$. Then, because $z(t)$ is a decreasing function with down-jumps, for some $t_1 \geq t_0$ there will exist $\delta_\nu > 0$ such that $z(t) < -\delta_\nu$, for every $t \geq t_1$, $t \neq \tau_k,\ k \in N$, i.e.

$$y(t) - c(t)y(h(t)) < -\delta_\nu,\ t \neq \tau_k,\ t \geq t_1.$$

Except that, because the sequence of eventually negative numbers $\big\{z(\tau_k)\big\}_{k=1}^{+\infty}$ is decreasing, for our $\delta_\nu > 0$, there will be such a term τ_ν in the sequence of the impulsive moments $\{\tau_k\}$, whereafter $z(\tau_k) < -\delta_\nu$, for every $\tau_k \geq \tau_\nu$, when $k \geq \nu$, $k \in N,\ \nu \in N$. Hence,

$$y(\tau_k) - c(t)y(h(\tau_k)) < -\delta_\nu, \qquad \tau_k \geq \tau_\nu,\ k \geq \nu,\ k \in N,\ \nu \in N.$$

Denote $t_\nu = \max\{t_1, \tau_\nu\}$ and combine the last two inequalities as

$$y(t) < -\delta_\nu + c(t)y(h(t)), \qquad t \geq t_\nu. \tag{5}$$

It is obvious that the right side of (5) has to be positive , because $y(t)$ is positive. So, we obtain the inequality $0 < -\delta_\nu + c(t)y(h(t))$, which clearly shows, that $y(t)$ is a bounded function from below. Hence, if integrate (E_1) from t_0 to t we can easily get (3), which will imply $\lim_{t\to+\infty} z(t) = -\infty$ and it will contradict our assumption again.

Thus, the above consideration approves $\lim_{t\to+\infty} z(t) = -\infty$. The proofs of (b) and of the Lemma are completed.

The second lemma is only formulated for eventually negative solution $y(t)$ of the equation (E_1). The proof is carried out respectively to the proof of Lemma 1.

Lemma 2. *Let $y(t)$ be an eventually negative solution of (E_1) and the hypotheses $(H1) - (H5)$ are satisfied. Then :*

(a) $z(t)$, defined by $(*)$, is an eventually increasing function of t with up-jumps;

(b) $z(t)$, defined by $(*)$, is an eventually positive function, i.e. $z(t) > 0$ for enough large t and $\lim_{t\to+\infty} z(t) = +\infty$.

4. Oscillation of All Solutions of Equation (E_2)

Our aim in this section is to establish appropriate sufficient conditions for oscillation of the equation (E_2) under the hypotheses $H2^*$ and $H3^*$.

Next three lemmas specify sufficient conditions under which the equation (E_2) is oscillatory. They are very useful for investigation of the oscillation of the solutions of equation (E_1) and will be utilized in the next section.

Lemma 3. *Assume the hypotheses $(H1)$, $(H2^*)$ and $(H3^*)$ are satisfied. Suppose also that:*

$$1. \quad \liminf_{t\to\infty}\Big[\prod_{t\le\tau_k<s(t)}(1+q_k)\int\limits_t^{s(t)} Q(r)dr\Big] > \frac{1}{e}.$$

Then equation (E_2) is oscillatory.

Proof. Assume , for the sake of contradiction, that equation (E_2) has a non-oscillatory solution. Since the negative of a solution of (E_2) is again a solution of (E_2), it suffices to prove the lemma considering this non-oscillatory solution as an eventually positive function. So, suppose that there exists a solution $z(t)$ of the equation (E_2) and a number $t_0 > 0$, such that $z(t)$ is defined for $t \ge t_0$ and $z(t) > 0$ for $t \ge t_0$.

From (E_2) and conditions $H2^*$ and $H3^*$, it follows that

$$z'(t) = Q(t)z(s(t)) > 0, \ t \ne \tau_k, \ k \in N,$$

$$\Delta[z(\tau_k)] = q_k z(s(\tau_k)) > 0, \ k \in N$$

i.e. $z(t)$ is a increasing function for $t \geq t_0$ with "up-jumps" at the points of impulsive effect ($\Delta[z(\tau_k)] > 0$). Therefore, we can rearrange (E_2) dividing by $z(t)$, to obtain

$$\frac{z'(t)}{z(t)} > Q(t)\frac{z(s(t))}{z(t)} > Q(t), \qquad t \neq \tau_k,\ k \in N, \tag{6}$$

$$\Delta[z(\tau_k)] > q_k z(s(\tau_k)) > q_k z(\tau_k), \qquad k \in N$$

From condition 1, it follows that there exist a constant $K > 0$ and $t_1 \geq t_0$, such that

$$m \int_t^{s(t)} Q(r)dr \geq K > \frac{1}{e}, \qquad t \geq t_1, \tag{7}$$

where we denote

$$m = \liminf_{t\to\infty} \prod_{t \leq \tau_k < s(t)} (1 + q_k).$$

Now, we integrate (6) from t to $s(t)$, i.e.

$$\int_t^{s(t)} \frac{[z(r)]'}{z(r)} dr > \int_t^{s(t)} Q(r)dr$$

and obtain

$$\ln \frac{z(s(t))}{z(t)} + \sum_{t \leq \tau_k < s(t)} \ln \frac{z(\tau_k)}{z(\tau_k + 0)} > \int_t^{s(t)} Q(r)dr. \tag{8}$$

Except that, $z(\tau_k + 0) - z(\tau_k) = q_k z(s(\tau_k)) > q_k z(\tau_k)$ and $z(\tau_k + 0) > (1 + q_k)z(\tau_k)$,

i.e. $\dfrac{1}{1+q_k} > \dfrac{z(\tau_k)}{z(\tau_k + 0)}$. So, we have $\ln \dfrac{1}{1+q_k} > \ln \dfrac{z(\tau_k)}{z(\tau_k + 0)}$.

Thus, from (7) and (8) we get

$$\ln[\frac{z(s(t))}{z(t)} \prod_{t \leq \tau_k < s(t)} \frac{1}{1+q_k}] > \int_t^{s(t)} Q(r)dr \ ,$$

i.e.

$$\ln[\frac{z(s(t))}{z(t)} \prod_{t \leq \tau_k < s(t)} \frac{1}{1+q_k}] > \frac{K}{m} \ .$$

Using the inequality $e^x > ex$, we obtain

$$\frac{z(s(t))}{z(t)} \prod_{t\le\tau_k<s(t)} \frac{1}{1+q_k} > \frac{eK}{m},$$

which implies

$$\frac{z(s(t))}{z(t)} > eK.$$

Repeating the above procedure, it follows by induction on (6) that there exists a sequence $\{t_n\}$ where $t_n \to \infty$ as $n \to \infty$, such that

$$\frac{z(s(t))}{z(t)} > (e.K)^n,\ t \ge t_n. \tag{9}$$

Choose n such that

$$\left(\frac{2m}{K}\right)^2 < (eK)^n, \tag{10}$$

which is possible because by (7) we have $eK > 1$ and consider $\hat{t}$, where $\hat{t} \ge t_n$.

Because of (7), there exists a number $\xi \in [\hat{t}, s(\hat{t})]$ such that

$$m\int_{\hat{t}}^{\xi} Q(r)dr \ge \tfrac{K}{2} \qquad \text{and} \qquad m\int_{\xi}^{s(\hat{t})} Q(r)dr \ge \tfrac{K}{2}.$$

If integrate (E_2) over the interval $[\hat{t}, \xi]$ we find

$$z(\xi) - z(\hat{t}) - \sum_{\hat{t}\le\tau_i\le\xi} \Delta[z(\tau_i)] - \int_{\hat{t}}^{\xi} Q(r)z(s(r))dr = 0,$$

or

$$z(\xi) - z(\hat{t}) - \sum_{\hat{t}\le\tau_i\le\xi} q_i z(s(\tau_i)) - \int_{\hat{t}}^{\xi} Q(r)z(s(r))dr = 0.$$

By omitting the second and the third terms and using the increasing nature of $z(t)$ we find

$$z(\xi) > z(s(\hat{t}))\int_{\hat{t}}^{\xi} Q(r)dr$$

i.e.

$$\frac{z(s(\hat{t}))}{z(\xi)} < \frac{2m}{K} \tag{11}$$

Similarly, integrating (E_2) over the interval $[\xi, s(\hat{t})]$, we find

$$\frac{z(s(\xi))}{z(s(\hat{t}))} < \frac{2m}{K} \tag{12}$$

From (11) and (12) we conclude

$$\frac{z(s(\xi))}{z(\xi)} < \left(\frac{2m}{K}\right)^2$$

This and (9) imply that

$$(eK)^n < \frac{z(s(\xi))}{z(\xi)} < \left(\frac{2m}{K}\right)^2, \tag{13}$$

which contradicts to (10) and completes the proof of the lemma.

As an immediate consequence we have the following result:

Corollary 1. *Let the conditions of Lemma 1 are satisfied. Then:*
(i) the inequality $(N_{2,\le})$ *has no eventually negative solutions*
(ii) the inequality $(N_{2,\ge})$ *has no eventually positive solutions*

The proof of Corollary 1 is similar to that of Lemma 1 and therefore it is omitted.

The next lemma provides similar result of that in Lemma 1, utilizing more strong condition.

Lemma 4. *Assume the hypotheses* $(H1)$, $(H2^*)$ *and* $(H3^*)$ *are satisfied. Suppose also that:*

1. *There exists a constant* $L > 0$, *such that* $Q(t) \ge L$, *for* $t \in R_+$
2. $mLD \ge \frac{1}{e}$, *where* $m = \liminf_{t\to\infty} \prod_{t\le\tau_k<s(t)} (1 + q_k)$.

Then the equation (E_2) *is oscillatory.*

Obviously the condition 2 of Lemma 2 implies the condition 1 of Lemma 1. The proof is very similar to that of Lemma 1 and that is why it is omitted.

As an immediate consequence we have the next result:

Corollary 2. *Let the conditions of Lemma 2 are satisfied. Then:*
(i) the inequality $(N_{2,\le})$ *has no eventually negative solutions*
(ii) the inequality $(N_{2,\ge})$ *has no eventually positive solutions*

Using a different approach , we can obtain another sufficient condition under which the equation (E_2) is oscillatory.

Lemma 5. *Assume the hypotheses $(H1)$, $(H2^*)$ and $(H3^*)$ are satisfied. Suppose also that:*

$$1. \quad \liminf_{k\to\infty}[\int_{\tau_k}^{s(\tau_k)} Q(r)dr + \sum_{\tau_k\le\tau_i<s(\tau_k)} q_i] \ge 1.$$

Then equation (E_2) is oscillatory.

Proof. Let assume , for the sake of contradiction, that equation (E_2) has a non-oscillatory solution. Since the negative of a solution of (E_2) is again a solution of (E_2), it suffices to prove the lemma considering this non-oscillatory solution as an eventually positive function. So, let suppose that there exists a solution $z(t)$ of the equation (E_2) and a number $t_0 > 0$, such that $z(t)$ is defined for $t \ge t_0$ and $z(t) > 0$ for $t \ge t_0$.

From (E_2) and conditions $H2^*$ and $H3^*$, it follows that

$$z'(t) = Q(t))z(s(t)) > 0, \ t \ne \tau_k, \ k \in N,$$

$$\Delta[z(\tau_k)] = q_k z(s(\tau_k)) > 0, \ k \in N$$

The last inequalities imply that $z(t)$ is an increasing function for $t \ge t_0$ with "up-jumps" at the points of impulsive effect $(\Delta[z(\tau_k)] \ge 0)$.

Integrating (E_2) from τ_k to $H(\tau_k)$, we obtain

$$z(s(\tau_k)) - z(\tau_k) - \sum_{\tau_k\le\tau_i\le s(\tau_k)} \Delta[z(\tau_i)] - \int_{\tau_k}^{s(\tau_k)} Q(r)z(s(r))dr = 0,$$

or

$$-z(s(\tau_k)) + z(\tau_k) + \sum_{\tau_k\le\tau_i\le s(\tau_k)} q_i z(s(\tau_i)) + \int_{\tau_k}^{s(\tau_k)} Q(r)z(s(r))dr = 0,$$

or

$$-z(s(\tau_k)) + \sum_{\tau_k\le\tau_i\le s(\tau_k)} q_i z(s(\tau_i)) + \int_{\tau_k}^{s(\tau_k)} Q(r)z(s(r))dr < 0,$$

or, because $z(t)$ is increasing function,

$$-z(s(\tau_k)) + z(s(\tau_k)) \sum_{\tau_k\le\tau_i\le s(\tau_k)} q_i + z(s(\tau_k)) \int_{\tau_k}^{s(\tau_k)} Q(r)dr < 0,$$

i.e.

$$z(s(\tau_k))[-1 + \sum_{\tau_k \le \tau_i \le s(\tau_k)} q_i + \int_{\tau_k}^{s(\tau_k)} Q(r)dr] < 0.$$

From the last inequality and because $z(t) > 0$, we obtain

$$\sum_{\tau_k \le \tau_i \le s(\tau_k)} q_i + \int_{\tau_k}^{s(\tau_k)} Q(r)dr < 1,$$

which contradicts to the condition 1 of the lemma and completes the proof. As an immediate consequence we have the next result:

Corollary 3. *Let the conditions of Lemma 3 are satisfied. Then:*
(i) the inequality $(N_{2,\le})$ *has no eventually negative solutions*
(ii) the inequality $(N_{2,\ge})$ *has no eventually positive solutions*

The proof of Corollary 3 is similar to that of Lemma 3 and therefore it is omitted.

5. Oscillation of All Solutions of Equation (E_1)

Now we are ready to establish sufficient conditions under which the equation (E_1) is oscillatory, utilizing the corollaries of the lemmas in the previous section.

Theorem 1. *Assume that the hypotheses* $(H1)$ – $(H5)$ *are satisfied. Suppose also that:*

1. $$\liminf_{t\to\infty}[\prod_{t \le \tau_k < h^{-1}(\sigma(t))} (1 + \frac{p_k}{c'_k}) \int_t^{h^{-1}(\sigma(t))} \frac{p(r)}{c(h^{-1}(\sigma(r)))} dr] \ge \frac{1}{e},$$

where $c'_k = h^{-1}(\sigma(\tau_k))$.

Then the equation (E_1) *is oscillatory.*

Proof. Let assume, for the sake of contradiction, that equation (E_1) has a non-oscillatory solution. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering this non-oscillatory solution as an eventually positive function. So, let suppose that there exists a solution $y(t)$ of the equation (E_1) and a number $t_0 > 0$, such that $y(t), y(h(t))$, and $y(\sigma(t))$ are defined and positive for every $t \ge t_0$. Set

$$z(t) = y(t) - c(t)y(h(t)), \quad t \ne \tau_k, \; k \in N \tag{14}$$

$$\Delta[z(\tau_k)] = \Delta[y(\tau_k) - c_k y(h(\tau_k))] = \Delta[y(\tau_k)] - c_k\Delta[y(h(\tau_k))],\ k \in N$$

From (14) and conditions H2 and H3, it follows that

$$z'(t) = -p(t)y(\sigma(t)) < 0,\ t \neq \tau_k,\ k \in N,$$

$$\Delta[z(\tau_k)] = -p_k y(\sigma(\tau_k)) < 0,\ k \in N$$

The last inequalities implies that $z(t)$ is a non-increasing function for $t \geq t_0$ with "down-jumps" at the points of impulsive effect ($\Delta[z(\tau_k)] < 0$), which means that the sequence $\left\{z(\tau_k)\right\}_{k=1}^{+\infty}$ is decreasing. Hence, there exists $\lim\limits_{t\to+\infty} z(t)$.

Let suppose for the sake of contradiction, that $z(t) > 0$. Note, that then we have $y(t) > z(t)$, as well as $y(t) > c(t)y(h(t)) > y(h(t))$ and $\Delta[y(\tau_k)] \leq c_k\Delta[y(h(\tau_k))]$. We claim in this situation that $y(t)$ is bounded from below. Indeed, if assume $\lim\limits_{t\to+\infty} z(t) = L > 0$, then $y(t) > z(t) \geq L$ and it will be obvious that $y(t)$ is bounded from below. Further, if assume $\lim\limits_{t\to+\infty} z(t) = 0$, it is naturally to presume $\lim\limits_{t\to+\infty} y(t) = 0$. Then there will exist a sequence $\{t_n\}_{n=1}^{\infty}$, such that $\lim\limits_{n\to+\infty} t_n = +\infty$ with $\lim\limits_{n\to+\infty} y(t_n) = 0$ where we denote $y(t_n) = \min\limits_{t_0\leq s\leq t_n}\{y(s)\}$. But it means that $y(t_n) < y(h(t_n))$, which contradicts the notation above, because $y(t) > c(t)y(h(t)) > y(h(t))$ when $z(t) > 0$.

Hence, $y(t)$ is bounded from below and denote $\liminf\limits_{s\to+\infty} y(s) = m > 0$.

Now, if we integrate (E_1) from t_1 to t, for some $t_1 \geq t_0$, we obtain

$$z(t) - z(t_1) - \sum_{t_1\leq\tau_k\leq t} \Delta[z(\tau_k)] + \int_{t_1}^{t} p(r)y(\sigma(r))dr \leq 0 \qquad i.e.$$

$$z(t) - z(t_1) + \sum_{t_1\leq\tau_k\leq t} p_k y(\sigma(\tau_k)) + \int_{t_1}^{t} p(r)y(\sigma(r))dr \leq 0, \quad i.e.$$

$$z(t) \leq z(t_1) - m[\sum_{t_1\leq\tau_k\leq t} p_k + \int_{t_1}^{t} p(r)dr].$$

The last inequality implies, after passing to limit as $t \to +\infty$, that $\lim\limits_{t\to+\infty} z(t) = -\infty$. This is in contradiction with the assumption that $z(t) > 0$ with $\lim\limits_{t\to+\infty} z(t) = L \geq 0$ for $t \geq t_1 \geq t_0$.

Therefore, $z(t) < 0$ for $t \geq t_1 \geq t_0$ with "down-jumps" at the points of impulsive effect $(\Delta[z(\tau_k)] < 0)$. Obviously, then $y(t) - c(t)y(h(t) = z(t) > -c(t)y(h(t))$.

Note, that because of the hypothesis $(H3)$, there exists an inverse function $h^{-1}(*)$, such that $h^{-1}(\sigma(t)) > t,\ t \geq t_1,$ for which

$$z(h^{-1}(\sigma(t))) > -c(h^{-1}(\sigma(t)))y(h(h^{-1}(\sigma(t)))) = -c(h^{-1}(\sigma(t)))y(\sigma(t)).$$

Multiplying both sides at the last inequality by $\dfrac{p(t)}{-c(h^{-1}(\sigma(t)))} < 0$ we obtain

$$\frac{p(t)z(h^{-1}(\sigma(t)))}{-c(h^{-1}(\sigma(t)))} < p(t)y(\sigma(t)) = -z'(t).$$

Hence,

$$z'(t) - \frac{p(t)}{c(h^{-1}(\sigma(t)))}z(h^{-1}(\sigma(t))) < 0, \quad t \neq \tau_k,\ k \in N \tag{15}$$

Except that, denoting $c'_k = h^{-1}(\sigma(\tau_k))$, we have and $z(h^{-1}(\sigma(\tau_k))) > -c'_k y(\sigma(t))$. Multiplying the both sides of the last inequality by $\dfrac{p_k}{-c'_k} < 0$, we obtain

$$\frac{p_k}{-c'_k}z(h^{-1}(\sigma(\tau_k))) < p_k y(\sigma(t)) < -\Delta[z(\tau_k)],$$

i.e.

$$\Delta[z(\tau_k)] - \frac{p_k}{c'_k}z(h^{-1}(\sigma(\tau_k))) < 0 \tag{16}$$

Denote $s(t) = h^{-1}(\sigma(t)),\ Q(t) = \dfrac{p(t)}{c(h^{-1}(\sigma(t)))},\ t \geq t_1,\ q_k = \dfrac{p_k}{c'_k}$. From (15) and (16), one can realize that the negative function $z(t)$ satisfies the impulsive differential inequality of advanced type

$$[z(t)]' - Q(t)z(s(t)) < 0,\ t \neq \tau_k,\ k \in N \tag{17}$$

$$\Delta[z(\tau_k)] - q_k z(s(\tau_k))) < 0,\ k \in N$$

where obviously the hypothesis $(H2^*)$ and $(H3^*)$ are available, with respect to the hypothesis $(H2)$ and $(H3)$.

But, this contradicts to Corollary 1, which claims that (17) has no eventually negative solutions under conditions of the theorem. Therefore, the equation (E_1) has only oscillatory solutions. The proof of the theorem is complete.

Theorem 2. *Assume that the hypotheses* $(H1)$ – $(H5)$ *are satisfied. Suppose also that:*

1. *There exists a constant* $L > 0$, *such that* $\dfrac{p(t)}{c(h^{-1}(\sigma(t)))} \geq L$, *for* $t \in R_+$.
2. *There exists a constant* $D > 0$, *such that* $h^{-1}(\sigma(t)) \geq t + D$.
3. $\liminf\limits_{t\to\infty} \prod\limits_{t\leq\tau_k<h^{-1}(\sigma(t))} (1 + \dfrac{p_k}{c'_k}) \geq \dfrac{1}{eLD}$, *where* $c'_k = h^{-1}(\sigma(\tau_k))$.

Then the equation (E_1) *is oscillatory.*

Proof. Proceeding as in proof of Theorem 1, we get that the negative function $z(t)$, defined by (14), satisfies the impulsive differential inequality of advanced type (17), where we denote

$$s(t) = h^{-1}(\sigma(t)), \quad Q(t) = \frac{p(t)}{c(h^{-1}(s(t)))}, \ t \geq t_0,$$

$$q_k = \frac{p_k}{c'_k}$$

and where the hypothesis $(H2^*)$ and $(H3^*)$ are available, with respect to the hypothesis $(H2)$ and $(H3)$ and Condition 2.

But, this contradicts to Corollary 2, which claims that (17) has no eventually negative solutions under conditions of the theorem. Therefore, the equation (E_1) has only oscillatory solutions. The proof of the theorem is complete.

As an immediate consequence from the last two theorems above, we have the following result:

Corollary 4. *Let the conditions of Theorem 1 or the more strong conditions of Theorem 2 are satisfied. Then:*

(i) the inequality $(N_{1,\leq})$ *has no eventually positive solutions*

(ii) the inequality $(N_{1,\geq})$ *has no eventually negative solutions*

The proof of Corollary 4 is similar to that of Theorem 1 and therefore it is omitted.

Theorem 3. *Assume that the hypotheses* $(H1)$ – $(H5)$ *are satisfied. Suppose also that:*

1. $\liminf\limits_{k\to\infty} [\int\limits_{\tau_k}^{h^{-1}(\sigma(\tau_k))} \dfrac{p(r)}{c(h^{-1}(\sigma(r)))} dr + \sum\limits_{\tau_k\leq\tau_i<h^{-1}(\sigma(\tau_k))} \dfrac{p_i}{c'_i}] \geq 1,$

where $c'_i = h^{-1}(\sigma(\tau_i))$.

Then the equation (E_1) *is oscillatory.*

Proof. Let assume , for the sake of contradiction, that equation (E_1) has a non-oscillatory solution. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering this non-oscillatory solution as an eventually positive function. So, let suppose that there exists a solution $y(t)$ of the equation (E_1) and a number $t_0 > 0$, such that $y(t)$ is defined for $t \geq t_0$ and $y(t) > 0$ for $t \geq t_0$. Proceeding as in proof of Theorem 1, we get that the negative function $z(t)$, defined by (14), satisfies the impulsive differential inequality of advanced type (17), where we denote $s(t) = h^{-1}(\sigma(t)), \quad Q(t) = \dfrac{p(t)}{c(h^{-1}(\sigma(t)))}, \ t \geq t_0, \quad q_k = \dfrac{p_k}{c'_k}.$

It is obvious and that the hypothesis $(H2^*)$ and $(H3^*)$ are available, with respect to the hypothesis $(H2)$ and $(H3)$.

But, this contradicts to Corollary 3, which claims that (17) has no eventually negative solutions under conditions of the theorem. Therefore, the equation (E_1) has only oscillatory solutions. The proof of the theorem is complete.

As an immediate consequence we get the next result:

Corollary 5. *Let the conditions of Theorem 3 are satisfied. Then:*
(i) the inequality $(N_{1,\leq})$ *has no eventually positive solutions*
(ii) the inequality $(N_{1,\geq})$ *has no eventually negative solutions*

The proof of Corollary 5 is similar to that of Theorem 3 and therefore it is omitted.

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