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INDEPENDENT AND VERTEX COVERING NUMBER ON KRONECKER PRODUCT OF P_n

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Abstract: Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number, respectively. The Kronecker Product $G_1 \otimes G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, let G is a simple graph with order m, we prove that, $\alpha(P_n \otimes G) = \max\left\{n\alpha(G), m\lceil \frac{n}{2}\rceil\right\}$ and $\beta(P_n \otimes G) = \min\left\{n\beta(G), m\lfloor \frac{n}{2}\rfloor\right\}$.

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let G_1 and G_2 be graphs. The Kronecker product of graph G_1 and G_2, denote by $G_1 \otimes G_2$, be the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

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Next, we give the definitions about some graph parameters. A subset U of the vertex set $V(G)$ of G is said to be an independent set of G if the induced subgraph $G[U]$ is a trivial graph. An independent set of G with maximum number of vertices is called a maximum independent set of G . The number of vertices of a maximum independent set of G is called the independent number of G , denoted by $\alpha(G)$.

A vertex of graph G is said to cover the edges incident with it, and a vertex cover of a graph G is a set of vertices covering all the edge of G . The minimum cardinality of a vertex cover of a graph G is called the vertex covering number of G , denoted by $\beta(G)$.

By definitions of independent number and vertex covering number, clearly that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$ and $\beta(P_n) = \lfloor \frac{n}{2} \rfloor$.

Proposition 1. Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then
(*i*) $n(V(H)) = n(V(G_1)n(V(G_2)$
(*ii*) $n(E(H)) = 2n(E(G_1)n(E(G_2)$
(*iii*) for every $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.
Note that for any graph G, we have $G_1 \otimes G_2 \cong G_2 \otimes G_1$

Theorem 2. *Let G_1 and G_2 be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

Theorem 3. *Let G_1 and G_2 be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.*

Next we get that general form of graph of Kronecker Product of P_n and a simple graph.

Proposition 4. Let G be connected graph order m, the graph of $P_n \otimes G$ *is* $\bigcup_{i=1}^{n-1} H_i$ where $V(H_i)$=$W_i \cup W_{i+1}$ for $i = 1, 2, ..., n-1$; $W_i = \{(i, 1), (i, 2), ..., (i, m)\}$; $E(H_i) = \{(i, u)(i+1, v)/uv \in E(G)\}$ Moreover, if G has no odd cycle then for each H_1 has exactly two connected components isomorphic to G.

Example

2. Independent Number of the Graph of $C_n \otimes G$

We now state proposition and prove lemma before stating our main results. We begin this section by giving the proposition 5 which show character of

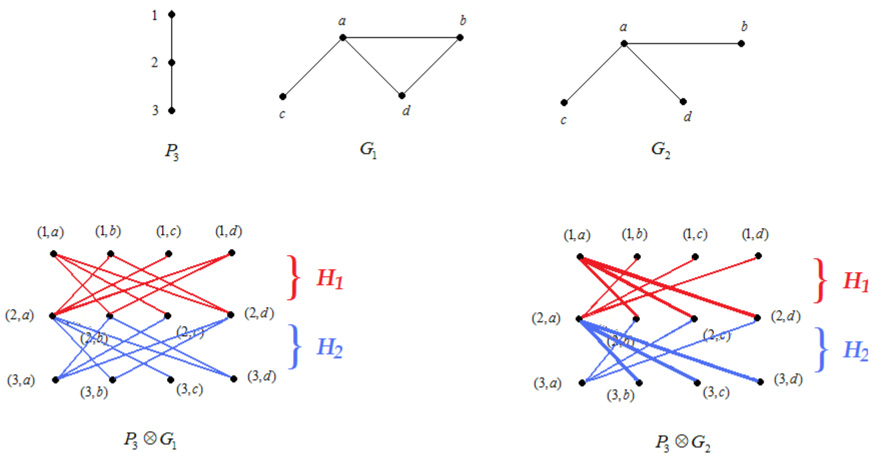

Figure 1: The graph of $P_3 \otimes G_1$ and $P_3 \otimes G_2$

independent set and the lemma 6 which show character of independent set for each H_i.

Proposition 5. Let $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph G if

(i) v_i is not adjacent with v_j for all $i \neq j$ and $i, j = 1, 2, ..., k$

and (ii) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

Lemma 6. Let $P_n \otimes G = \bigcup_{i=1}^{n-1} H_i$. For each H_i, then $\alpha(H_i) = 2\alpha(G)$.

Proof. Suppose G has no odd cycle, by proposition 4, we get H_i=$2G$. So $\alpha(H_i) = 2\alpha(G)$.

if G has odd cycle, for each H_i , vertex $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = P_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, I be the maximum independent set of G . We get

$\overline{H_i} = 2(G - \overline{e})$ then

$$\alpha(\overline{H_i}) = 2\alpha(G-\overline{e}) = \begin{cases} 2[\alpha(G)+1], & \text{if } \overline{e} = xy \text{ then } x \in I, \\ & y \notin I \text{ and is not} \\ & \text{adjecent with vertex } z \in I \\ 2\alpha(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\alpha(G-\overline{e}) = \alpha(G)+1$ be not impossible because the end vertices of edge $\overline{e}$ are in independent set of $G - \overline{e}$, so $\alpha(H_i) = \alpha(\overline{H_i}) - 1$.

Hence $\alpha(H_i) = 2\alpha(G)$. □

Example

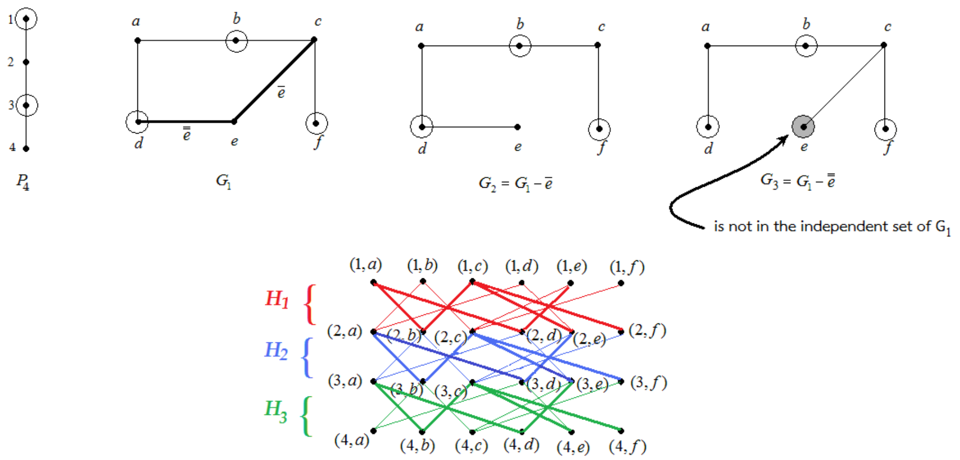

Figure 2: The graph of $P_4 \otimes G_1$

Next, we establish theorem 7 for a maximum independent number of $P_n \otimes G$

Theorem 7. *Let G be connected graph order m, then $\alpha(P_n \otimes G) = max\{n\alpha(G), m\lceil\frac{n}{2}\rceil\}$.*

Proof. Let $V(P_n) = \{u_i, i = 1,2,...,n\}$, $V(G) = \{v_i, i = 1,2,...,m\}$, $S_i = \{(v_i,u_j) \in V(P_n \otimes G)/j = 1,2,...,m\}, i = 1,2,...,n$ and since $\alpha(P_n) = \lceil\frac{n}{2}\rceil$.

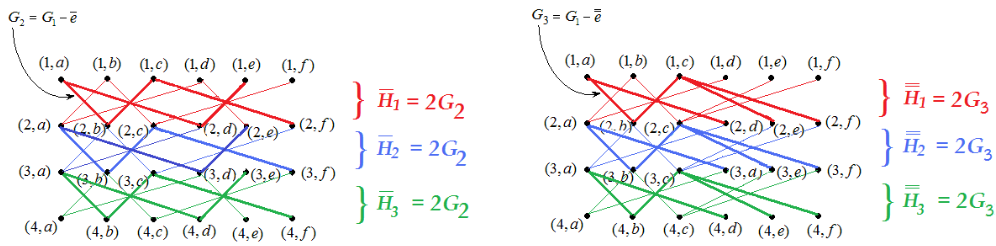

Figure 3: The graph of $P_4 \otimes G_2$ and $C_4 \otimes G_3$

Assume that the maximum independent set of P_n, G be $I_1{=}\{u_1, u_3, ..., u_{2\lceil \frac{n}{2} \rceil - 1}\}$, I_2, respectively.

For each H_i , by lemma 6 we have $\alpha(H_i) = 2\alpha(G)$. Since $P_n{\otimes}G$ is $\bigcup\limits_{i=1}^{n-1} H_i$ which is every H_i and H_{i+1} have $\alpha(G)$ common vertices in their independent set, then $\alpha(P_n \otimes G) \geq n\alpha(G)$.

In the author hand, we get a independent set of $P_n \otimes G$ be $S_1 \cup S_3 \cup ... \cup S_{2\lceil \frac{n}{2} \rceil - 1}$, then $\alpha(C_n \otimes G) \geq m\lceil \frac{n}{2} \rceil$.

Hence $\alpha(P_n \otimes G) \geq max\{n\alpha(G), m\lceil \frac{n}{2} \rceil\}$.

Suppose that $\alpha(P_n \otimes G) >$max$\{n\alpha(G), m\lceil \frac{n}{2} \rceil\}$, then there exists $uv_j(or u_i v) \in V(P_n \otimes G) - W(or S); j = k+1, k+2, ..., m; \quad i = 2, 4, ..., 2\lfloor \frac{n}{2} \rfloor$, which is not adjacent with another vertices in W (or S) , $W = \{uv_k/v_k \in I_2\}$ and $S = \{u_h v/u_h \in I_1\}$. It is not true, because for every H_i has $V(H_i) - W{=}[\bigcup\limits_{j=1}^{m} N(u_i, v_j)] \cup [\bigcup\limits_{j=1}^{m} N(u_{i+1}, v_j)]$.

Hence $\alpha(P_n \otimes G) = max\{n\alpha(G), m\lceil \frac{n}{2} \rceil\}$. □

3. Vertex Covering Number of the Graph of $P_n{\otimes}G$

We begin this section by giving the lemma 8 that shows a relation of independent number and vertex covering number and the lemma 9 that show character of independent set for each H_i.

Lemma 8. (see [2]) Let G be a simple graph with order n. Then $\alpha(G) + \beta(G) = n$

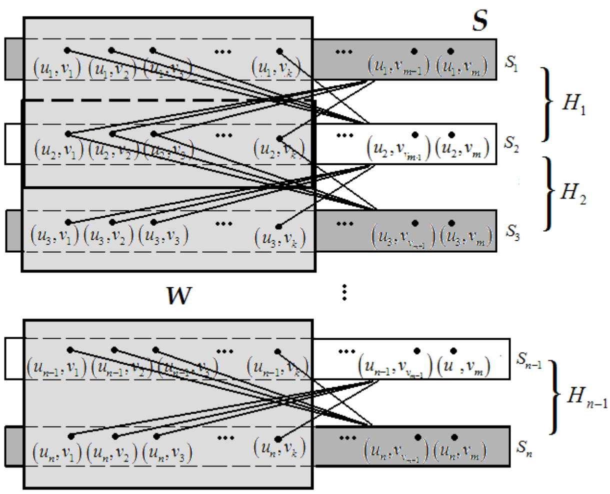

Figure 4: The region of W, S when n is odd

Lemma 9. Let $P_n \otimes G = \bigcup_{i=1}^{n-1} H_i$. For each H_i, then $\beta(H_i) = 2\beta(G)$

Proof. Suppose G has no odd cycle, by proposition 4, we get H_i=$2G$. So $\beta(H_i) = 2\beta(G)$.

If G has odd cycle, for each $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ in $V(H_i)$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v))$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = P_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, C be the minimum vertex covering set of G . We get $\overline{H_i} = 2(G - \overline{e})$ then

$$\beta(\overline{H_i}) = 2\beta(G - \overline{e}) = \begin{cases} 2[\beta(G) - 1], & \text{if } \overline{e} = xy \text{ then } x, y \text{ are adjecent} \\ & \text{with vertices in } C, \\ 2\beta(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\beta(G-\overline{e}) = \beta(G)-1$ be not impossible because edge $\overline{e}$ is not adjacent with vertices in the vertex covering set of $G - \overline{e}$, so $\beta(H_i) = \beta(\overline{H_i}) + 1$.

Hence $\beta(H_i) = 2\beta(G)$. □

Next, we establish theorem 10 for a minimum vertex covering number of $P_n \otimes G$.

Theorem 10. *Let G be connected graph order m, then* $\beta(P_n \otimes G) = min\{n\beta(G), m\lfloor \frac{n}{2} \rfloor\}$

Proof. Let $V(P_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(P_n \otimes G)/j = 1, 2, ..., m\}$, $\quad i = 1, 2, ..., n$ and since $\beta(P_n) = \lfloor \frac{n}{2} \rfloor$. Assume that the maximum vertex covering set of P_n, G be $C_1 = \{u_2, u_4, ..., u_{2\lfloor \frac{n}{2} \rfloor}\}, C_2$, respectively.

For each H_i , by lemma 9 we have $\beta(H_i) = 2\beta(G)$. Since $P_n \otimes G$ is $\bigcup_{i=1}^{n-1} H_i$ which is every H_i and H_{i+1} have $\beta(G)$ common vertices in their vertex covering set, then $\beta(P_n \otimes G) \leq n\beta(G)$.

In the author hand, we get another vertex covering set of $P_n \otimes G$ be $\{S_2, S_4, ..., S_{2\lfloor \frac{n}{2} \rfloor}\}$ then $\alpha(P_n \otimes G) \leq m\lfloor \frac{n}{2} \rfloor$.

Hence $\alpha(P_n \otimes G) \leq max\{n\alpha(G), m\lfloor \frac{n}{2} \rfloor\}$.

Suppose that $\beta(P_n \otimes G) < min\{n\beta(G), m\lfloor \frac{n}{2} \rfloor\}$, then there exists $uv_j(or u_i v) \in V(P_n \otimes G) - W(or S), j = k+1, k+2, ..., m; i = 1, 3, 2\lceil \frac{n}{2} \rceil - 1$, which is not adjacent with another vertices in W (or S) , $W = \{uv_k/v_k \in C_2\}$ and $S = \{u_h v/u_h \in C_1\}$. It is not true, because for every $uv_j(u_i v)$ adjacent with a vertices in W (or S)

Hence $\beta(P_n \otimes G) = min\{n\beta(G), m\lfloor \frac{n}{2} \rfloor\}$. □

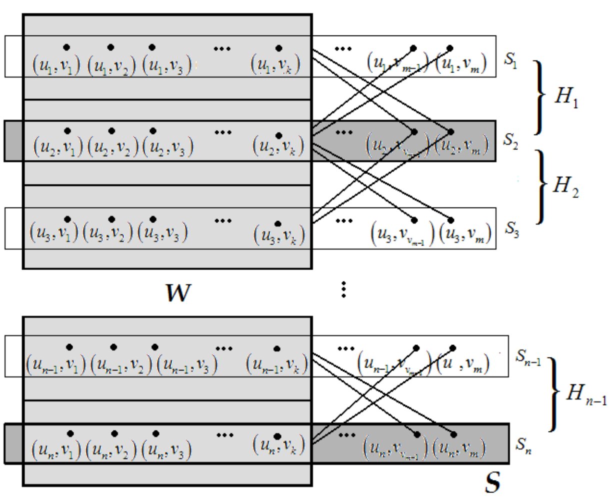

Figure 5: The region of W, S when n is odd

By theorem 7 and lemma 9, we can also show that

$$\begin{aligned}
\alpha(P_n \otimes G) + \beta(P_n \otimes G) &= mn \\
\max\{n\alpha(G), m\lceil \tfrac{n}{2} \rceil\} + \beta(P_n \otimes G) &= mn \\
\beta(P_n \otimes G) &= mn - \max\{n\alpha(G), m\lceil \tfrac{n}{2} \rceil\} \\
&= mn + \min\{-n\alpha(G), -m\lceil \tfrac{n}{2} \rceil\} \\
&= \min\{n(m - \alpha(G)), m(n - \lceil \tfrac{n}{2} \rceil\} \\
&= \min\{n\beta(G)), m\lfloor \tfrac{n}{2} \rfloor\}
\end{aligned}$$

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