International Journal of Pure and Applied Mathematics
Volume 73 No. 2 2011, 235-253

REGIONAL OBSERVABILITY WITH CONSTRAINTS OF THE GRADIENT

A. Boutoulout[1 §], H. Bourray[2], M. Baddi[3]
[1,2,3]TSI Team, MACS Laboratory
Faculty of Sciences
Moulay Ismaïl University
Meknes, MOROCCO

Abstract: The aim of this paper is to explore the concept of observability with constraints of the gradient for distributed parabolic system evolving in spatial domain Ω. It consists in the reconstruction of the initial state gradient and must be between two prescribed functions in a subregion ω of Ω. We give definitions and some properties of this kind of regional observability and we describe two approaches to solve this problem where the first is based on subdifferential techniques and the second uses the lagrangian multiplier method. This last approach leads to an algorithm which is performed by example and simulation.

AMS Subject Classification: 93B07, 93C20
Key Words: distributed systems, parabolic systems, regional observability, gradient reconstruction

1. Introduction

In the case of distributed parameter systems defined on a geometric domain Ω, numerous papers were devoted to the state observation in a portion of the spatial domain Ω on which the system is considered. This concept has been widely developed and survey of these developments can be found in [1], [2], [3], [4], [5] and [6]. Later the concept of the regional gradient observability has been introduced and developed by [7] and concerns the reconstruction of the state gradient only in a critical subregion interior to the system evolution domain without the knowledge of the state.

Received: October 26, 2011

§Correspondence author

Here we are interested to approach the initial state gradient and the reconstructed state between two prescribed functions given only on a subregion ω of the geometric domain where the system is considered (see [8]). This problem is more realistic and more adapted for system analysis than the classical one since, in general, a mathematical model of a real system does not represent exactly the dynamic of a physical process, there can be many reasons for this: the parameters may not be known precisely, the model can be of reduced order, also, in order to simplify calculation, the model can be a linear approximation of a nonlinear process, and since the reconstruction of the initial state gradient is based only on an approximate model. A similar problem occurs in numerical analysis, because of rounding errors, a computed solution can be far from the actual solution. Therefore the solutions of such systems are known approximately as well as their gradients.

The paper is organized as follows: First we provide results on regional observability for distributed parameter system of parabolic type and we give definitions related to regional observability with constraints of the gradient of parabolic systems. The next section is focused on the reconstruction of the initial state gradient by using an approach based on subdifferential tools. the same objective is achieved in Section 4 by applying the multiplier Lagrangian approach which gives a practice algorithm. The last section is devoted to compute the obtained algorithm with numerical example and simulations.

2. Problem Statement

Let Ω be an open bounded subset of $I\!R^n (n = 1, 2, 3)$ with regular boundary $\partial\Omega$. For $T > 0$, we note $Q = \Omega\times]0, T[$, $\Sigma = \partial\Omega\times]0, T[$ and we consider a parabolic system defined by

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} & = & \mathcal{A}y(x,t) & \text{in} & Q \\ y(x,0) & = & y_0(x) & \text{in} & \Omega \\ y(\xi,t) & = & 0 & \text{on} & \Sigma \end{cases} \tag{1}$$

with measurements given by the output function

$$z(t) = Cy(t) \tag{2}$$

where

$$\mathcal{A} = a_0 - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$$

with $a_0, a_{i,j} \in D(Q)$.

Suppose that $\mathcal{A}$ is elliptic, i.e., there exists $\alpha > 0$ such that

$$a_0 \geq \alpha \text{ and } \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha \sum_{j=1}^{n} |\xi_j^2| \text{ a.e. in } Q \quad \forall \xi = (\xi_1, ..., \xi_n) \in I\!R^n$$

This operator is a second order differential linear operator which generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ in the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(y)dxdy$$

and the norm

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega |u(x)|^2 dx$$

the adjoint $\mathcal{A}^*$ is defined by

$$\mathcal{A}^* = a_0 - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ji}\frac{\partial}{\partial x_j})$$

and $C : H_0^1(\Omega) \longrightarrow I\!R^q$, q depends on the number of the considered sensors, is linear and depends on the considered sensor structure. Consider the observation space $\mathcal{O} = L^2(0, T, I\!R^q)$ and assume that $y_0 \in X = H_0^1(\Omega)$.

The system (1) is autonomous, then the output can be expressed by

$$z(t) = CS(t)y_0 = (Ky_0)(t), \quad t \in]0, T[$$

Where the operator K is given by

$$\begin{array}{rcl} K : X & \longrightarrow & \mathcal{O} \\ z & \longrightarrow & CS(.)z \end{array}$$

which is linear bounded with the adjoint K^* given by

$$\begin{array}{rcl} K^* : \mathcal{O} & \longrightarrow & X \\ z^* & \longrightarrow & \displaystyle\int_0^T S^*(t)C^*z^*(t)dt \end{array}$$

For $\omega \subset \Omega$ an open subregion of Ω with positive Lesbegue measure, let χ_ω be the restriction function defined by

$$\begin{array}{rcl} \chi_\omega : (L^2(\Omega))^n & \longrightarrow & (L^2(\omega))^n \\ y & \longrightarrow & \chi_\omega y = y|_\omega \end{array}$$

and χ_ω^* denotes the adjoint operator, given by the rule

$$\begin{array}{rcl} \chi_\omega^* : (L^2(\omega))^n & \longrightarrow & (L^2(\Omega))^n \\ y & \longrightarrow & \chi_\omega^* y = \left\{ \begin{array}{lll} y & in & \omega \\ 0 & in & \Omega \setminus \omega \end{array} \right. \end{array}$$

Consider the operator ∇ given by the formula

$$\begin{array}{rcl} \nabla : H_0^1(\Omega) & \longrightarrow & (L^2(\Omega))^n \\ y & \longrightarrow & \nabla y = (\dfrac{\partial y}{\partial x_1}, ..., \dfrac{\partial y}{\partial x_n}) \end{array}$$

Its adjoint ∇^* is given by

$$\begin{array}{rcl} \nabla^* : (L^2(\Omega))^n & \longrightarrow & H_0^1(\Omega) \\ y & \longrightarrow & \nabla^* y = v \end{array}$$

where v is the solution of the Dirichlet problem

$$\left\{ \begin{array}{lllll} \Delta v & = & -div(y) & \text{in} & \Omega \\ v & = & 0 & \text{on} & \partial\Omega \end{array} \right.$$

Let us recall same definitions about the regional observability of the gradient.

Definition 1.

1. The system (1) together with the output (2) is said to be exactly gradient observable in ω if $Im(\chi_\omega \nabla K^*) = (L^2(\omega))^n$.

2. The system (1) together with the output (2) is said to be weakly gradient observable in ω if $\ker(K\nabla^*\chi_\omega^*) = \{0\}$.

For more details, we refer the reader to [7].

Let $(\alpha_i(.))_{i=1}^n$ and $(\beta_i(.))_{i=1}^n$ be two functions defined in $(L^2(\omega))^n$ such that $\alpha_i(.) \leq \beta_i(.)$ a.e. in ω for all $1 \leq i \leq n$.

In the sequel we set

$$[\alpha(.), \beta(.)] = \big\{(y_1, y_2, ..., y_n) \in (L^2(\omega))^n \ \mid \ \alpha_i(.) \leq y_i(.) \leq \beta_i(.) \text{ a.e. in } \omega \quad \forall i \in \{1, 2, ..., n\}\big\} .$$

Definition 2. The system (1) together with the output (2) is said to be exactly $[\alpha(.), \beta(.)]$-gradient observable in ω if

$$Im(\chi_\omega \nabla K^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Definition 3. The system (1) together with the output (2) is said to be weakly $[\alpha(.), \beta(.)]$-gradient observable in ω if

$$\overline{Im(\chi_\omega \nabla K^*)} \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Definition 4. A sensor (D, f) is said to be $[\alpha(.), \beta(.)]$-gradient strategic in ω if the observed system is weakly $[\alpha(.), \beta(.)]$-gradient observable in ω.

Remark 5.

1. If the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω then it is weakly $[\alpha(.), \beta(.)]$-gradient observable in ω.

2. If the system (1) together with the output (2) is exactly (resp. weakly) $[\alpha(.), \beta(.)]$-gradient observable in ω_1 then it is exactly (resp. weakly) $[\alpha(.), \beta(.)]$-gradient observable in any $\omega_2 \subset \omega_1$.

3. If the system (1) together with the output (2) is exactly gradient observable in ω then it is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω. Indeed, if the system (1) together with the output (2) is exactly gradient observable in ω then $Im(\chi_\omega \nabla K^*) = (L^2(\omega))^n$ which gives

$$Im(\chi_\omega \nabla K^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

that is to say that the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω.

Thus it was shown that the exact observability of the gradient leads always to the $[\alpha(.), \beta(.)]$-gradient observability. In the following example we will show that the reverse of this implication is not true in general in the weak case.

Example 6. Consider the two-dimensional system described in $\Omega =]0,1[\times]0,1[$ by the diffusion equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1, x_2, t) & = & \dfrac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) & \text{in} & Q, \\ y(\zeta, \eta, t) & = & 0 & \text{on} & \Sigma, \\ y(x_1, x_2, 0) & = & y_0(x_1, x_2) & \text{in} & \Omega. \end{cases} \tag{3}$$

The measurements are given by the output

$$z(t) = \int_D y(x_1, x_2, t) f(x_1, x_2) dx_1 dx_2, \tag{4}$$

Where $D = \{1/2\}\times]0,1[$ is the sensor support and $f(x_1, x_2) = \sin(\pi x_2)$ is the function measure. The operator

$$\mathcal{A} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

generates a semigroup $(S(t))_{t\geq 0}$ in $L^2(\Omega)$ given by

$$S(t)y = \sum_{i,j=1}^{\infty} \exp(\lambda_{ij}t)\langle y, \varphi_{ij}\rangle_{L^2(\Omega)}\varphi_{ij},$$

where $\varphi_{ij}(x_1, x_2) = 2\sin(i\pi x_1)\sin(j\pi x_2)$ and $\lambda_{ij} = -(i^2 + j^2)\pi^2$

Proposition 7. *The system (3) together with the output (4) is not weakly gradient observable in Ω but it is weakly $[\alpha(.), \beta(.)]$-gradient observable in Ω.*

Proof. Let us prove that the system (3) together with the output (4) is not weakly gradient observable in Ω.

Let $g_1(x_1, x_2) = (\cos(\pi x_1)\sin(2\pi x_2), \sin(\pi x_1)\cos(2\pi x_2)) \in (L^2(\Omega))^2$.

Show that $g_1 \in \ker(K\nabla^*)$

$$\begin{aligned}
K\nabla^*(g_1) &= \sum_{i,j=1}^{\infty} \exp(\lambda_{ij}t)\langle \nabla^* g_1, \varphi_{ij}\rangle_{L^2(\Omega)}\langle \varphi_{ij}, f\rangle_{L^2(D)} \\
&= \sum_{i,j=1}^{\infty} \exp(\lambda_{ij}t)(2i\pi \int_0^1 \cos(\pi x_1)\cos(i\pi x_1)dx_1 \\
&\quad \int_0^1 \sin(2\pi x_2)\sin(j\pi x_2)dx_2 \\
&\quad +2j\pi \int_0^1 \sin(\pi x_1)\sin(i\pi x_1)dx_1 \int_0^1 \cos(2\pi x_1)\cos(j\pi x_1)dx_1) \\
&\quad \times 2\sin(\frac{i\pi}{2}) \int_0^1 \sin(j\pi x_2)\sin(\pi x_2)dx_2 \\
&= 0
\end{aligned}$$

so the system (3) together with the output (4) is not weakly gradient observable in Ω but we can show that it is weakly $[\alpha(.), \beta(.)]$-gradient observable in Ω, indeed, for $g_2(x_1, x_2) = (0, \sin(\pi x_1)\cos(\pi x_2))$

$$\begin{aligned}
K\nabla^*(g_2) &= \sum_{i,j=1}^{\infty} \exp(\lambda_{ij}t)\langle \nabla^* g_2, \varphi_{ij}\rangle_{L^2(\Omega)}\langle \varphi_{ij}, f\rangle_{L^2(D)} \\
&= 2j\pi \sum_{i,j=1}^{\infty} \exp(\lambda_{ij}t) \int_0^1 \sin(\pi x_1)\sin(i\pi x_1)dx_1 \\
&\quad \int_0^1 \cos(\pi x_2)\cos(j\pi x_2)dx_2 \\
&\quad \times 2\sin(\frac{i\pi}{2}) \int_0^1 \sin(j\pi x_2)\sin(\pi x_2)dx_2 \\
&= \frac{\pi \exp(-2\pi^2 t)}{2} \neq 0
\end{aligned}$$

which shows that g_2 is weakly gradient observable in Ω.

For $\alpha(x_1, x_2) = (-1, x_1 + x_2 - 3)$ and $\beta(x_1, x_2) = (x_1 + 2, 4)$, we have $g_2 \in [\alpha(.), \beta(.)]$, then the system (3) together with the output (4) is weakly $[\alpha(.), \beta(.)]$-gradient observable in Ω. □

Proposition 8. *The system (1) together with the output (2) is exactly* $[\alpha(.), \beta(.)]$*-gradient observable in* ω *if and only if*

$$(\ker \chi_\omega + Im(\nabla K^*)) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Proof.

- Suppose that $(\ker \chi_\omega + Im(\nabla K^*)) \cap [\alpha(.), \beta(.)] \neq \emptyset$ then, there exists $z \in [\alpha(.), \beta(.)]$ such that $z \in \ker \chi_\omega + Im(\nabla K^*)$, then $z = z_1 + z_2$ where $\chi_\omega z_1 = 0$ and $z_2 = \nabla K^*\theta$ with $\theta \in \mathcal{O}$, then $\chi_\omega z = \chi_\omega z_1 + \chi_\omega z_2 = \chi_\omega z_2 = \chi_\omega \nabla K^*\theta$ and $\chi_\omega z \in Im(\chi_\omega \nabla K^*)$ thus

$$Im\,(\chi_\omega \nabla K^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

  which shows that the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω.

- Suppose that the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω, which is equivalent to

$$Im(\chi_\omega \nabla K^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

  then there exists $z \in [\alpha(.), \beta(.)]$ and $\theta \in \mathcal{O}$ such that $\chi_\omega z = \chi_\omega \nabla K^*\theta$ which gives $\chi_\omega(z - \nabla K^*\theta) = 0$.

Let $y_1 = z - \nabla K^*\theta$ and $y_2 = \nabla K^*\theta$, then $z = y_1 + y_2$ with $y_1 \in \ker \chi_\omega$ and $y_2 \in Im(\nabla K^*)$ which shows that $z \in \ker \chi_\omega + Im(\nabla K^*)$ and therefore

$$(\ker \chi_\omega + Im(\nabla K^*)) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

□

Proposition 9. *The system (1) together with the output (2) is weakly $[\alpha(.), \beta(.)]$-gradient observable in ω if and only if*

$$\left(\ker \chi_\omega + \overline{Im(\nabla K^*)}\right) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Proof.

- Suppose that $\left(\ker \chi_\omega + \overline{Im(\nabla K^*)}\right) \cap [\alpha(.), \beta(.)] \neq \emptyset$ then, there exists $z \in [\alpha(.), \beta(.)]$ such that $z \in \ker \chi_\omega + \overline{Im(\nabla K^*)}$ so $z = z_1 + z_2$, where $\chi_\omega z_1 = 0$ and $z_2 = \lim\limits_{n\to+\infty} \nabla K^*\theta_n$ with $\theta_n \in \mathcal{O}$, $\forall n \in I\!N$, then $\chi_\omega z = \chi_\omega z_2 = \chi_\omega(\lim\limits_{n\to+\infty} \nabla K^*\theta_n) = \lim\limits_{n\to+\infty} \chi_\omega \nabla K^*\theta_n$ and $\chi_\omega z \in \overline{Im(\chi_\omega \nabla K^*)}$ therefore
$$\overline{Im(\chi_\omega \nabla K^*)} \cap [\alpha(.), \beta(.)] \neq \emptyset$$
which implies that the system (1) together with the output (2) is weakly $[\alpha(.), \beta(.)]$-gradient observable in ω.

- Suppose that the system (1) together with the output (2) is weakly $[\alpha(.), \beta(.)]$-gradient observable in ω, which is equivalent to
$$\overline{Im(\chi_\omega \nabla K^*)} \cap [\alpha(.), \beta(.)] \neq \emptyset$$
then there exists $z \in [\alpha(.), \beta(.)]$ and θ_n a sequence of elements of $\mathcal{O}$ such that $\chi_\omega z = \lim\limits_{n\to+\infty} \chi_\omega \nabla K^*\theta_n$ which gives $\chi_\omega(z - \lim\limits_{n\to+\infty} \nabla K^*\theta_n) = 0$.
Let $y_1 = z - \lim\limits_{n\to+\infty} \nabla K^*\theta_n$ and $y_2 = \lim\limits_{n\to+\infty} \nabla K^*\theta_n$, then $z = y_1 + y_2$ with $y_1 \in \ker \chi_\omega$ and $y_2 \in \overline{Im(\nabla K^*)}$ which shows that $z \in \ker \chi_\omega + \overline{Im(\nabla K^*)}$ and therefore
$$\left(\ker \chi_\omega + \overline{Im(\nabla K^*)}\right) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

□

3. Subdifferential Approach

In this section, we present an approach which allows the reconstruction of the initial state in ω with constraint of the gradient based on subdifferential techniques. Let consider the distributed parameter system represented by the autonomous state equation

$$\left\{\begin{array}{lll} \dfrac{\partial y(x,t)}{\partial t} & = Ay(x,t) & \text{in} \quad Q \\ y(x,0) & = y_0(x) & \text{in} \quad \Omega \\ y(\xi,t) & = 0 & \text{on} \quad \Sigma \end{array}\right. \tag{5}$$

augmented with the output function

$$z(t) = Cy(t) \tag{6}$$

with the same assumptions as in Section 2. Then our goal is to minimize the reconstruction error between the output equation and the observation model. That is to say to solve the following minimization problem

$$\left\{\begin{array}{l} \min ||Ky - z||^2_{\mathcal{O}} \\ y \in Y \end{array}\right. \tag{7}$$

where $Y = \{y \in H^1_0(\Omega) \mid \chi_\omega \nabla y \in [\alpha(.), \beta(.)]\}$.

Let recall the following notations and some results which are used in the sequel

- $\Gamma_0(H^1_0(\Omega))$ the set of functions $f : H^1_0(\Omega) \longmapsto \tilde{I\!R} =]-\infty, +\infty]$ proper, lower semi-continuous (l.s.c.) and convex in $H^1_0(\Omega)$.
- For $f \in \Gamma_0(H^1_0(\Omega))$ $dom(f) = \{y \in H^1_0(\Omega) \mid f(y) < +\infty\}$ and f^* the polar function of f, given by

$$f^*(y^*) = \sup_{y \in dom(f)} \{\langle y^*, y\rangle - f(y)\}, \forall y^* \in H^1_0(\Omega)$$

For $y^0 \in dom(f)$ the set

$$\partial f(y^0) = \{y^* \in H^1_0(\Omega) \mid f(y) \geq f(y^0) + \langle y^*, y - y^0\rangle, \forall y \in H^1_0(\Omega)\}$$

denotes the subdifferential of f at y^0, then we have the property

$$y_1 \in \partial f(y^*) \text{ if and only if } f(y^*) + f^*(y_1) = \langle y^*, y_1\rangle$$

- For D a nonempty subset of $H_0^1(\Omega)$

$$\Psi_D(y) = \begin{cases} 0 & \text{if } y \in D \\ +\infty & \text{otherwise} \end{cases}$$

  denotes the indicator functional of D.

With these notations the problem in (7) is equivalent to the problem

$$\begin{cases} \inf(\|Ky - z\|_{\mathcal{O}}^2 + \Psi_Y(y)) \\ y \in H_0^1(\Omega) \end{cases} \tag{8}$$

The solution of this problem may be characterized by the following result.

Proposition 10. *If the system (5) together with the output (6) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω, then y^* is a solution of (8) if and only if*

$$y^* \in Y \quad \textit{and} \quad \Psi_Y^*(-2K^*(Ky^* - z)) = -2\|Ky^*\|_{\mathcal{O}}^2 + 2\langle K^* z, y^*\rangle$$

Proof. If the system (5) together with the output (6) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω, then $Y \neq \emptyset$ and the problem (7) has a solution.

Let $f(y) = \|Ky - z\|_{\mathcal{O}}^2$. The function y^* is a solution of (8) if and only if $0 \in \partial(f + \Psi_Y)(y^*)$, but $f \in \Gamma_0(H_0^1(\Omega))$, $\Psi_Y \in \Gamma_0(H_0^1(\Omega))$ (Y is a nonempty closed and convex set) and $Dom(f) \cap Dom(\Psi_Y) \neq \emptyset$, therefore,

$$\partial(f + \Psi_Y)(y^*) = \partial f(y^*) + \partial \Psi_Y(y^*)$$

it follows that y^* is a solution of (8) if and only if $0 \in \partial f(y^*) + \partial \Psi_Y(y^*)$.

Moreover, f is Frechet-differentiable, therefore,

$$\partial f(y^*) = \{\nabla f(y^*)\} = \{2K^*(Ky^* - z)\}$$

and y^* is a solution of (8) if and only if $-2K^*(Ky^* - z) \in \partial \Psi_Y(y^*)$ which is equivalent to $y^* \in Y$ and

$$\Psi_Y(y^*) + \Psi_Y^*(-2K^*(Ky^* - z)) = \langle y^*, -2K^*Ky^* + 2K^*z\rangle$$

which gives $y^* \in Y$ and

$$\Psi_Y^*(-2K^*(Ky^* - z)) = -2\|Ky^*\|_{\mathcal{O}}^2 + 2\langle K^* z, y^*\rangle$$

$\square$

4. Lagrangian Approach

Consider the problem (7) when the system (5) is observed by one zone sensor (D, f). Here we propose to solve this problem using the Lagrangian multiplier method (see [9]) and we give an algorithm for the numerical construction of the initial state gradient and finally we illustrate the results by numerical simulation. According to the definition of the exact $[\alpha(.), \beta(.)]$-gradient observability in ω, all states that we are going to build their gradient in ω are of the form $K^*\theta$ such that $\theta \in O$. So to solve the problem (7) it sufficient to solve the following problem

$$\left\{ \begin{array}{l} \min \|KK^*\theta - z\|^2_{\mathcal{O}} \\ \theta \in G = \{\hat{\theta} \in \mathcal{O} \mid \chi_\omega \nabla K^* \hat{\theta} \in [\alpha(.), \beta(.)]\} \end{array} \right. \tag{9}$$

The following result gives useful characterization of the solution of (7) and (9).

Proposition 11. *If the system (5) together with the output (6) is observable in Ω and $[\alpha(.), \beta(.)]$-gradient observable in ω then the solution of (9) is given by*

$$\theta^* = (KK^*KK^*)^{-1}KK^*z - \frac{1}{2}(KK^*KK^*)^{-1}(\chi_\omega \nabla K^*)^* \lambda^* \tag{10}$$

and the gradient of the solution in ω of the problem (7) for all the states which are of the form $K^*\theta$ with $\theta \in \mathcal{O}$ is given by

$$y^* = \chi_\omega \nabla K^* \theta^* = R_\omega K^* z - \frac{1}{2} R_\omega \nabla^* \chi^*_\omega \lambda^* \tag{11}$$

where λ^* is the solution of

$$\left\{ \begin{array}{l} \frac{1}{2} R_\omega \nabla^* \chi^*_\omega \lambda^* = -y^* + R_\omega K^* z \\ y^* = P_{[\alpha(.),\beta(.)]}(\rho\lambda^* + y^*) \end{array} \right. \tag{12}$$

while $P_{[\alpha(.),\beta(.)]} : (L^2(\omega))^n \to [\alpha(.), \beta(.)]$, denotes the projection operator, $\rho > 0$ and $R_\omega = \chi_\omega \nabla K^*(KK^*KK^*)^{-1}K$

Proof. If the system (5) together with the output (6) is exactly $[\alpha(.), \beta(.)]$-gradient observable in ω, then $G \neq \emptyset$ and the problem (9) has a solution. The problem (9) is equivalent to the saddle point problem

$$\min_{(\theta, y) \in W} \|KK^*\theta - z\|^2_{\mathcal{O}} \tag{13}$$

where

$$W = \{(\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)] \mid \chi_\omega \nabla K^*\theta - y = 0\}$$

We associate with problem (13) the Lagrangian functional L defined by the formula

$$L(\theta, y, \lambda) = \|KK^*\theta - z\|^2_{\mathcal{O}} + \langle \lambda, \chi_\omega \nabla K^*\theta - y\rangle_{(L^2(\omega))^n}$$

for all $(\theta, y, \lambda) \in \mathcal{O} \times [\alpha(.), \beta(.)] \times (L^2(\omega))^n$.

Let us recall that $(\theta^*, y^*, \lambda^*)$ is a saddle point of L if

$$\max_{\lambda \in (L^2(\omega))^n} L(\theta^*, y^*, \lambda) = L(\theta^*, y^*, \lambda^*) = \min_{\substack{\theta \in \mathcal{O} \\ y \in [\alpha(.), \beta(.)]}} L(\theta, y, \lambda^*)$$

The set $\mathcal{O} \times [\alpha(.), \beta(.)]$ is nonempty, closed and convex, moreover the function $\lambda \to L(\theta, y, \lambda)$ is concave, upper semi continuous, and differentiable. The function $(\theta, y) \to L(\theta, y, \lambda)$ is convex, lower semi-continuous, and differentiable.

Moreover,

$$\exists \lambda_0 \in (L^2(\omega))^n \text{ such that } \lim_{\|(\theta,y)\| \to +\infty} L(\theta, y, \lambda_0) = +\infty$$

and

$$\exists (\theta_0, y_0) \in \mathcal{O} \times [\alpha(.), \beta(.)] \text{ such that } \lim_{\|\lambda\| \to +\infty} L(\theta_0, y_0, \lambda) = -\infty$$

Then L admits a saddle point. Let $(\theta^*, y^*, \lambda^*)$ be a saddle point of L and show that $y^* = \chi_\omega \nabla K^*\theta^*$ is the restriction gradient in ω of the solution of (7) for all the states which are of the form $K^*\theta$ with $\theta \in \mathcal{O}$. We have

$$L(\theta^*, y^*, \lambda) \leq L(\theta^*, y^*, \lambda^*) \leq L(\theta, y, \lambda^*)$$

for all $(\theta, y, \lambda) \in \mathcal{O} \times [\alpha(.), \beta(.)] \times (L^2(\omega))^n$
From the first inequality we have

$$\langle \lambda, \chi_\omega \nabla K^*\theta^* - y^*\rangle \leq \langle \lambda^*, \chi_\omega \nabla K^*\theta^* - y^*\rangle, \quad \forall \lambda \in (L^2(\omega))^n$$

which implies that $\chi_\omega \nabla K^*\theta^* = y^*$ and hence $\chi_\omega \nabla K^*\theta^* \in [\alpha(.), \beta(.)]$.

The second inequality implies that

$$L(\theta^*, y^*, \lambda^*) \leq L(\theta, y, \lambda^*), \forall (\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)]$$

This means that for all $(\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)]$ we have

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega \nabla K^*\theta^* - y^*\rangle \leq \|KK^*\theta - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega \nabla K^*\theta - y\rangle$$

Since $y^* = \chi_\omega \nabla K^* \theta^*$ we have

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} \leq \|KK^*\theta - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega \nabla K^* \theta - y \rangle$$

for all $(\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)]$.

Taking $y = \chi_\omega \nabla K^* \theta \in [\alpha(.), \beta(.)]$, we obtain

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} \leq \|KK^*\theta - z\|^2_{\mathcal{O}}$$

which implies that θ^* is a solution of (9) and so $y_0^* = K^*\theta^*$ whose the restriction of the gradient in ω is $y^* = \chi_\omega \nabla K^* \theta^*$ is a solution of (7) for all the states which are of the form $K^*\theta$ with $\theta \in \mathcal{O}$. Now $(\theta^*, y^*, \lambda^*)$ is a saddle point of L if the following assumptions hold

$$2\langle KK^*\theta^* - z, KK^*(\theta - \theta^*)\rangle + \langle \lambda^*, \chi_\omega \nabla K^*(\theta - \theta^*)\rangle = 0, \forall \theta \in \mathcal{O} \tag{14}$$

$$-\langle \lambda^*, y - y^* \rangle \geq 0, \quad \forall y \in [\alpha(.), \beta(.)] \tag{15}$$

$$\langle \lambda - \lambda^*, \chi_\omega \nabla K^* \theta^* - y^* \rangle = 0, \quad \forall \lambda \in (L^2(\omega))^n \tag{16}$$

From (14) we have

$$2\langle KK^*\theta^* - z, KK^*(\theta - \theta^*)\rangle + \langle \lambda^*, \chi_\omega \nabla K^*(\theta - \theta^*)\rangle = 0, \forall \theta \in \mathcal{O}$$

so

$$\langle 2(KK^*)^*(KK^*\theta^* - z), (\theta - \theta^*)\rangle + \langle (\chi_\omega \nabla K^*)^* \lambda^*, (\theta - \theta^*)\rangle = 0, \forall \theta \in \mathcal{O}$$

then

$$-2(KK^*)^* KK^* \theta^* + 2(KK^*)^* z = (\chi_\omega \nabla K^*)^* \lambda^*$$

we assume that the system is observable in Ω, then KK^*KK^* is invertible, and

$$\theta^* = (KK^*KK^*)^{-1} KK^* z - \frac{1}{2}(KK^*KK^*)^{-1}(\chi_\omega \nabla K^*)^* \lambda^*$$

so y^* is given by

$$y^* = \chi_\omega \nabla K^* (KK^*KK^*)^{-1} KK^* z - \frac{1}{2}\chi_\omega \nabla K^* (KK^*KK^*)^{-1}(\chi_\omega \nabla K^*)^* \lambda^*$$

then

$$y^* = R_\omega K^* z - \frac{1}{2} R_\omega \nabla^* \chi_\omega^* \lambda^*$$

with $R_\omega = \chi_\omega \nabla K^* (KK^*KK^*)^{-1} K$ by (15) we have $-\langle \lambda^*, y - y^* \rangle \geq 0, \forall y \in [\alpha(.), \beta(.)]$ so $\langle (\rho\lambda^* + y^*) - y^*, y - y^* \rangle \leq 0, \forall y \in [\alpha(.), \beta(.)]$ and $\forall \rho > 0$ then

$$y^* = P_{[\alpha(.),\beta(.)]}(\rho\lambda^* + y^*)$$

$\square$

Corollary 12. *If the system (5) together with the output (6) is observable in Ω, gradient observable in ω and the function*

$$L_\omega = [(\chi_\omega \nabla K^*)(\chi_\omega \nabla K^*)^*]^{-1}(\chi_\omega \nabla K^*)(KK^*KK^*) \\ [(\chi_\omega \nabla K^*)^*(\chi_\omega \nabla K^*)]^{-1}(\chi_\omega \nabla K^*)^*$$

is coercive, then for ρ convenably chosen, the system (12) has a unique solution (λ^*, y^*).

Proof. We have

$$y^* = \chi_\omega \nabla K^*(KK^*KK^*)^{-1}KK^*z - \frac{1}{2}\chi_\omega \nabla K^*(KK^*KK^*)^{-1}(\chi_\omega \nabla K^*)^*\lambda^*$$

then

$$\lambda^* = -2L_\omega y^* + 2[(\chi_\omega \nabla K^*)(\chi_\omega \nabla K^*)^*]^{-1}(\chi_\omega \nabla K^*)KK^*z$$

so if $(\theta^*, y^*, \lambda^*)$ is a saddle point of L then the system (12) is equivalent to

$$\begin{cases} \lambda^* = -2L_\omega y^* + 2[(\chi_\omega \nabla K^*)(\chi_\omega \nabla K^*)^*]^{-1}(\chi_\omega \nabla K^*)KK^*z \\ y^* = P_{[\alpha(.),\beta(.)]}(-2\rho L_\omega y^* + 2\rho[(\chi_\omega \nabla K^*)(\chi_\omega \nabla K^*)^*]^{-1}(\chi_\omega \nabla K^*)KK^*z + y^*) \end{cases}$$

It follows that y^* is a fixed point of the function

$$\begin{array}{rcl} F_\rho : [\alpha(.), \beta(.)] & \to & [\alpha(.), \beta(.)] \\ y & \to & P_{[\alpha(.),\beta(.)]}(-2\rho L_\omega y + 2\rho[(\chi_\omega \nabla K^*)(\chi_\omega \nabla K^*)^*]^{-1} \\ & & (\chi_\omega \nabla K^*)KK^*z + y) \end{array}$$

The operator L_ω is coercive, i.e.

$$\exists m > 0 \text{ such that } \langle L_\omega y, y\rangle \geq m\|y\|^2 \quad \forall y \in (L^2(\omega))^n$$

It follows that $\forall t_1, t_2 \in (L^2(\omega))^n$

$$\begin{array}{rcl} \|F_\rho(t_1) - F_\rho(t_2)\|^2 & \leq & \| -2\rho L_\omega(t_1 - t_2) + (t_1 - t_2)\|^2 \\ & \leq & 4\rho^2\|L_\omega\|^2\|t_1 - t_2\|^2 + \|t_1 - t_2\|^2 \\ & & -4\rho\langle L_\omega(t_1 - t_2), (t_1 - t_2)\rangle \\ & \leq & 4\rho^2\|L_\omega\|^2\|t_1 - t_2\|^2 + \|t_1 - t_2\|^2 \\ & & -4\rho m\|t_1 - t_2\|^2 \\ & \leq & (4\rho^2\|L_\omega\|^2 + 1 - 4\rho m)\|t_1 - t_2\|^2 \end{array}$$

and we deduce that if

$$\rho < \frac{m}{\|L_\omega\|^2}$$

then F_ρ is a contractor map, which implies the uniqueness of z^* and λ^*. □

4.1. Numerical Approach

We have seen in the previous paragraph the relationship between saddle points of the Lagrangian L and the estimator solution of (7). Solving equations (10), (11) and (12) is done by applying the algorithm of Uzawa (see [9]) type.

Let $T = KK^*KK^*$, if we choose two functions $(y_0^*, \lambda_1^*) \in [\alpha(.), \beta(.)] \times (L^2(\omega))^n$ and

$$\theta_n^* = T^{-1}KK^*z - \frac{1}{2}T^{-1}K\nabla^*\chi_\omega^*\lambda_n^*, \quad y_n^* = P_{[\alpha(.),\beta(.)]}(\rho\lambda_n^* + y_{n-1}^*)$$

and

$$\lambda_{n+1}^* = \lambda_n^* + (\chi_\omega \nabla K^* \theta_n^* - y_n^*)$$

then we obtain the following algorithm

Step 1: • The subregion ω, the location of the sensor D, the function of measure distribution f.
• Choose the function $y_0 \in (L^2(\omega))^n$ and $\lambda_1 \in (L^2(\omega))^n$.
• Threshold accuracy ε.

Step 2: **Repeat**
- $\triangleright$ Solve $T(\theta_n) = KK^*z - \frac{1}{2}(\chi_\omega \nabla K^*)^*\lambda_n$, $n \geq 1$.
- $\triangleright$ Calculate $y_n = P_{[\alpha(.),\beta(.)]}(\rho\lambda_n + y_{n-1})$, $n \geq 1$.
- $\triangleright$ Calculate $\lambda_{n+1} = \lambda_n + (\chi_\omega \nabla K^* \theta_n - y_n)$, $n \geq 1$.

Until $\left(||y_{n+1} - y_n||_{L^2(\omega)} \leq \varepsilon\right)$

Step 3: The functions θ_n and y_n lead to the initial gradient y^* to be reconstructed in ω.

If $(\theta^*, y^*, \lambda^*)$ is a saddle point of L, then the sequence (θ_n^*) converges to the solution θ^* of the problem (9) and the sequence (y_n^*) converges to y^* (see [9]).

4.2. Simulation Results

In this subsection we give a numerical example that leads to results related to the choice of the subregion, the initial condition and the sensor location. In $\Omega =]0, 1[$, consider the one dimensional system

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = 0.01\dfrac{\partial^2 y(x,t)}{\partial x^2} & \text{in} \quad \Omega\times]0,T[\\ y(0,t) = y(1,t) = 0 & \text{in} \quad]0,T[\\ y(x,0) = y_0(x) & \text{in} \quad \Omega \end{cases} \tag{17}$$

augmented with the output function

$$z(t) = y(b,t), \quad b \in \Omega \tag{18}$$

The initial state gradient to be reconstructed is $y_0(x) = (x-1)\sin(x)$.

Let $\alpha(x) = (x-1)\cos(x) + \sin(x) - 0.2$ and $\beta(x) = (x-1)\cos(x) + \sin(x) + 0.2$. Applying the previous algorithm, we obtain

Global case $\omega = \Omega$.

- If the sensor is located in $b = 0.23$, we obtain the following result

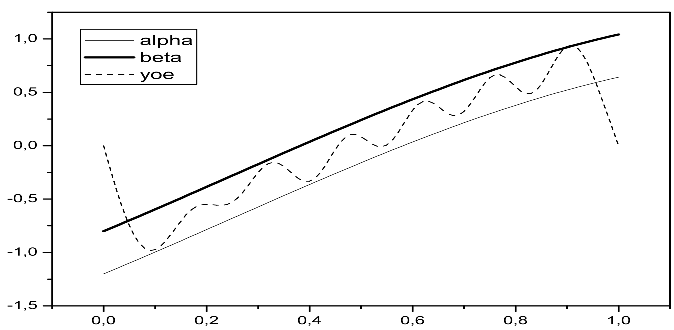

Figure 1: The estimated initial state gradient is between $\alpha(.)$ and $\beta(.)$ in Ω

We note that the estimated initial gradient is between $\alpha(.)$ and $\beta(.)$ and then the sensor is $[\alpha(.), \beta(.)]$-gradient strategic in Ω.

- If the sensor is located in $b = 0.42$ then we have

  The estimated initial gradient is not between $\alpha(.)$ and $\beta(.)$ in Ω and therefore the sensor $(b, \delta(b - \cdot))$ is not $[\alpha(.), \beta(.)]$-gradient strategic in Ω.

Figures 1 and 2 show that if the sensor is $[\alpha(.), \beta(.)]$-gradient strategic in Ω then the estimated initial state gradient is between $\alpha(.)$ and $\beta(.)$.

Regional Case $\omega =]0.30, 0.70[$.

- If the sensor is located in $b = 0.23$ then we obtain

- If the sensor is located in $b = 0.53$ then we obtain

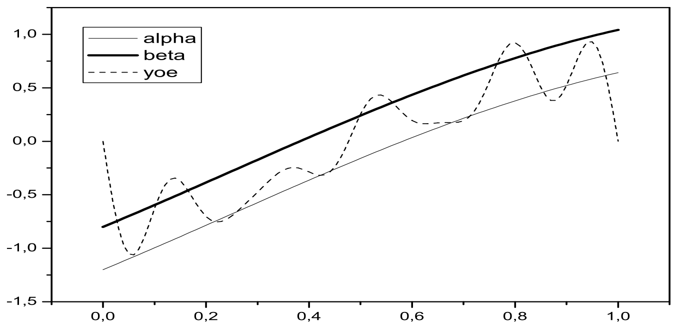

Figure 2: The estimated initial state gradient and $\alpha(.)$, $\beta(.)$ in Ω

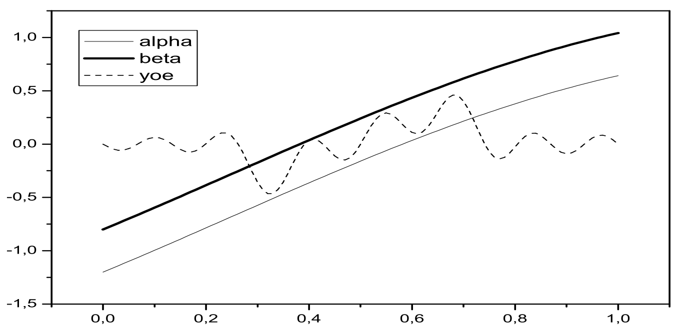

Figure 3: The estimated initial state gradient is between $\alpha(.)$ and $\beta(.)$ in ω

Figures 3 and 4 show that if the sensor $(b, \delta(b-\cdot))$ is $[\alpha(.), \beta(.)]$-gradient strategic in ω then the estimated initial state gradient is between $\alpha(.)$ and $\beta(.)$ in the subregion ω. The estimated initial state gradient is obtained with reconstruction error $\mathcal{E} = 2.08 \times 10^{-5}$. We note that there exists the best location of the sensor allowing a good reconstruction of the initial state gradient.

5. Conclusion

The problem of $[\alpha(.), \beta(.)]$-gradient observability in subregion ω of the system evolution domain Ω is considered and the initial state gradient is characterized by two approaches. This problem motivated by numerous realistic situations en-

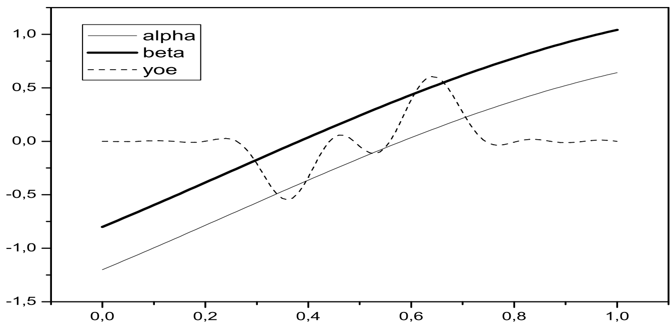

Figure 4: The estimated initial state gradient and $\alpha(.)$, $\beta(.)$ in ω

countered in various applications where one must follow the gradient evolution in a desired subregion of the geometric domain where the system is considered. Moreover, we are explored a useful numerical approach which allows the computation of the obtained algorithm and which is illustrated by numerical example and simulations. The case where ω is a part of the boundary of the system evolution domain, is under consideration and the work will be the subject of the future paper.

References

[1] A. El Jai, M.C. Simon, E. Zerrik, Regional observability and sensor structures, *Int. J. Sensors and Actuators*, **39**, No. 2 (1993), 95-102.

[2] A. El Jai, M. Amouroux, E. Zerrik, Regional observability of distributed systems, *Int. J. Syst. Sci.*, **25** (1994), 301-313.

[3] E. Zerrik, L. Badraoui, A. El Jai, Sensors and regional boundary state reconstruction of parabolic system, *Sensors and Actuators Jornal*, **75** (1999), 102-117.

[4] A. El Jai, A.J. Pritchard, *Sensors and Actuators in Distributed Systems Analysis*, Ellis Horwood Series in Applied Mathematics, J. Wiley (1988).

[5] R.F. Curtain, A.J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Springer-Verlag, Berlin (1978).

[6] R.F. Curtain, H. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Texts in Applied Mathematics, Springer-Verlag, **21**, New York (1995).

[7] E. Zerrik, H. Bourray, Gradient observability for diffustion system, *Int. J. Appl. Math. Comput. Sci.*, **13** (2003), 139-150.

[8] E. Zerrik, F. Ghafrani, Regional gradient-constrainted control problem. Approches and simulations, *Journal of Dynamical and Control Systems*, **9** (2003), 558-599.

[9] M. Fortin, R. Glowinski, *Méthodes de Lagrangien Augmenté. Applications à la Résolution Numérique des Problèmes aux Limites*, Dunod, Paris (1982).

254