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ON WEAK FORMS OF CONTRA-OPEN AND CONTRA-CLOSED FUNCTIONS

C.W. Baker
Department of Mathematics
Indiana University Southeast
New Albany, IN 47150-6405, USA

Abstract: New forms of contra-open and contra-closed functions are introduced. These new forms are used to extend several results in the literature. In particular, theorems stating conditions under which contra-open functions are closed and contra-closed functions are open are improved. Relationships between these functions and other related classes of functions are established.

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1. Introduction

Contra-continuous functions were introduced by Dontchev [4] in 1996. Since then many variations of contra-continuity have been investigated. Contra-open functions and contra-closed functions were introduced by Baker [1] in 1997. In this note we introduce new forms of contra-open and contra-closed functions, called weakly contra-open and weakly contra-closed. It is shown that several results for contra-open and contra-closed functions can be extended to weakly contra-open and weakly contra-closed functions. In particular, we show that a weakly contra-open function with closed fibers is closed and that a weakly

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contra-closed function with a T_1-domain in open. Also a closed graph theorem due to Fuller [5] is strengthened slightly. Relationships between weakly contra-open and weakly contra-closed functions and various forms of openness and closedness are established. For example, we show that weakly contra-open is strictly between closed and slightly closed and that weakly contra-closed is strictly between open and slightly open. Finally, relationships between weakly contra-open functions and weakly contra-closed functions are investigated.

2. Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is preopen if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. A set A is preclosed provided its complement is preopen. A point x of a space X is called a θ-cluster point of A if $\mathrm{Cl}(U) \cap A \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. A set A is θ-closed provided that $A = \mathrm{Cl}_\theta(A)$ and A is θ-open if its complement is θ-closed.

Definition 2.1. A function $f : X \to Y$ is said to be contra-open (see [1]) (respectively, contra-closed (see [1])) if $f(U)$ is closed (respectively, open) for every open (respectively, closed) subset U of X.

Definition 2.2. A function $f : X \to Y$ is said to be slightly open (respectively, slightly closed) provided that, whenever A is clopen in X, $f(A)$ is open (respectively, closed) in Y.

Definition 2.3. A function $f : X \to Y$ is said to be weakly closed (see [6]) if $\mathrm{Cl}(f(U)) \subseteq f(\mathrm{Cl}(U))$ for every open set U in X.

3. Weakly Contra-Open Functions

We define a function $f : X \to Y$ to be weakly contra-open provided that, for every open subset U of X and every closed subset A of X with $A \subseteq U$, we have $\mathrm{Cl}(f(A)) \subseteq f(U)$.

Theorem 3.1. *If $f : X \to Y$ is contra-open, then f is weakly contra-open.*

Proof. Assume $f : X \to Y$ is contra-open and let $A \subseteq U \subseteq X$, where A is closed in X and U is open in X. Then, since $f(U)$ is closed, $\mathrm{Cl}(f(A)) \subseteq \mathrm{Cl}(f(U)) = f(U)$, which proves that f is weakly contra-open. □

Theorem 3.2. *If the function $f : X \to Y$ is closed, then f is weakly contra-open.*

Proof. Assume $f : X \to Y$ is closed and let $A \subseteq U \subseteq X$, where A is closed in X and U is open in X. Since $f(A)$ is closed, $\mathrm{Cl}(f(A)) = f(A) \subseteq f(U)$ and hence f is weakly contra-open. □

Theorem 3.3. *If $f : X \to Y$ is weakly contra-open, then f is slightly closed.*

Proof. Assume U is a clopen subset of X. Then, since f is weakly contra-open, $\mathrm{Cl}(f(U)) \subseteq f(U)$, which proves that $f(U)$ is closed and that f is slightly closed. □

Example 3.4. The identity mapping on the real numbers with the usual topology is closed, hence also weakly contra-open, but not contra-open. Therefore weakly contra-open does not imply contra-open.

Example 3.5. Let $X = \{a, b\}$ have the Sierpenski topology $\tau = \{X, \emptyset, \{a\}\}$ and define $f : X \to X$ by $f(a) = b$ and $f(b) = a$. Then f is contra-open and hence weakly contra-open, but not closed. Thus weakly contra-open does not imply closed.

Example 3.6. Let X denote the real numbers with the usual topology and let Y be the real numbers with the indiscrete topology. Since X is connected, the identity mapping $f : X \to Y$ is slightly closed. However, since $\{0\} \subseteq (-1, 1)$ in X, but in Y $\mathrm{Cl}(f(\{0\})) \not\subseteq f((-1, 1))$, f is not weakly contra-open. Hence slightly closed does not imply weakly contra-open.

We have the following implications, none of which are reversible:

$$\begin{array}{ccccc} \text{contra-open} & \Rightarrow & \text{weakly contra-open} & \Rightarrow & \text{slightly closed} \\ & & \Uparrow & & \\ & & \text{closed} & & \end{array}$$

We now investigate conditions under which weakly contra-open functions are closed.

Theorem 3.7. *If the function $f : X \to Y$ is weakly contra-open and, if for every closed subset F of X and every $y \in Y$ such that $f^{-1}(y) \subseteq X - F$ there exists an open set U in X such that $F \subseteq U$ and $f^{-1}(y) \cap U = \emptyset$, then f is closed.*

Proof. Let F be a closed set in X and suppose $y \notin f(F)$. Then we have $f^{-1}(y) \subseteq X - F$. Hence there exists an open set U in X such that $F \subseteq U$ and $f^{-1}(y) \cap U = \emptyset$. Since f is weakly contra-open, $\mathrm{Cl}(f(F)) \subseteq f(U)$. Since $f^{-1}(y) \cap U = \emptyset$, $y \notin f(U)$ and hence $y \notin \mathrm{Cl}(f(F))$. Hence $\mathrm{Cl}(f(F)) = f(F)$ and $f(F)$ is closed. □

Corollary 3.8. (see [1], Theorem 4) *If the function $f : X \to Y$ is contra-open and, if for every closed subset F of X and every fiber $f^{-1}(y) \subseteq X - F$ there exists an open set U in X such that $F \subseteq U$ and $f^{-1}(y) \cap U = \emptyset$, then f is closed.*

In Theorem 3.7, if f has closed fibers, then $U = X - f^{-1}(y)$ satisfies the hypothesis. Hence we have the following result.

Corollary 3.9. *If the function $f : X \to Y$ is weakly contra-open and has closed fibers, then f is closed.*

Rose and Janković (see [6], Corollary 3.11) proved that a weakly closed function with closed fibers and a normal domain is closed. The above result replaces weak closure with weak contra-openness and does not require the domain to be normal.

Corollary 3.10. (see [1], Corollary 1) *If the function $f : X \to Y$ is contra-open and has closed fibers, then f is closed.*

Definition 3.11. A space X is said to be a C-space (see [2]) if for every open set U in X and every $x \in U$, there exists a closed set F such that $x \in F \subseteq U$.

Corollary 3.12. *If the function $f : X \to Y$ is weakly contra-open with finite fibers and X is a C-space, then f is closed.*

Proof. Let F be a closed set in X and let $y \in Y$ such that $f^{-1}(y) \subset X - F$. Since X is a C-space and $f^{-1}(y)$ is finite, there exists a closed set E such that $f^{-1}(y) \subseteq E \subseteq X - F$. If we take $U = X - E$, then $F \subseteq U$ and $f^{-1}(y) \cap U = \emptyset$ and the desired result follows from Theorem 3.7. □

Corollary 3.13. *If the function $f : X \to Y$ is a weakly contra-open injection and X is a C-space, then f is closed.*

Corollary 3.14. *If the function $f : X \to Y$ is a weakly contra-open injection and X is either T_1 or regular, then f is closed.*

Fuller (see [5], Corollary 3.9) proved that a closed function with closed fibers and a regular domain has a closed graph. Since by Corollary 3.9 a weakly contra-open function with closed fibers is closed, we have the following slight improvement of Fuller's result.

Theorem 3.15. *If the function $f : X \to Y$ is weakly contra-open with closed fibers and X is regular, then f has a closed graph.*

The fact that a weakly contra-open function with closed fibers is closed also yields the following result.

Theorem 3.16. *If the function $f : X \to Y$ is weakly contra-open with θ-closed fibers, then f has a closed graph.*

Finally, we show that a weakly contra-open function shares a property concerning dense sets with closed functions.

Theorem 3.17. *If the function $f : X \to Y$ is weakly contra-open and there exists a subset A of X such that $f(A)$ is dense in Y, then f is surjective.*

Proof. Let $f : X \to Y$ be weakly contra-open and let $A \subseteq X$ such that $f(A)$ is dense in Y. Then $f(\mathrm{Cl}(A))$ is also dense in Y and, since f is weakly contra-open, $Y = \mathrm{Cl}(f(\mathrm{Cl}(A))) \subseteq f(X)$, which proves that f is surjective. □

4. Weakly Contra-Closed Functions

We define a function $f : X \to Y$ to be weakly contra-closed provided that, for every open subset U of X and every closed subset A of X with $A \subseteq U$, we have $f(A) \subseteq \mathrm{Int}(f(U))$.

Theorem 4.1. *If $f : X \to Y$ is contra-closed, then f is weakly contra-closed.*

Proof. Assume $f : X \to Y$ is contra-closed and let $A \subseteq U \subseteq X$, where A is closed in X and U is open in X. Since $f(A)$ is open in Y, $f(A) = \mathrm{Int}(f(A)) \subseteq \mathrm{Int}(f(U))$, which proves that f is weakly contra-closed. □

Theorem 4.2. *If the function $f : X \to Y$ is open, then f is weakly contra-closed.*

Proof. Assume $f : X \to Y$ is open and let $A \subseteq U \subseteq X$, where A is closed in X and U is open in X. Since $f(U)$ is open, $f(A) \subseteq f(U) = \mathrm{Int}(f(U))$ and thus f is weakly contra-closed. □

Theorem 4.3. *If $f : X \to Y$ is weakly contra-closed, then f is slightly open.*

Proof. Assume U is a clopen subset of X. Then, since f is weakly contra-closed, $f(U) \subseteq \mathrm{Int}(f(U))$. Therefore $f(U)$ is open and hence f is slightly open. □

The identity mapping on the real numbers with the usual topology is open, hence also weakly contra-closed but not contra closed. Therefore weakly contra-closed does not imply contra-closed. The function in Example 3.5 is contra-closed and hence weakly contra-closed, but it is not open. Hence weakly contra-closed does not imply open. The function in Example 3.6 is slightly open, but it is not weakly contra-closed. Thus slightly open does not imply weakly contra-closed.

The following implications, none of which are reversible, hold:

contra-closed ⇒ weakly contra-closed ⇒ slightly open
⇑
open

Next conditions under which weakly contra-closed functions are open are investigated.

Theorem 4.4. *If the function $f : X \to Y$ is weakly contra-closed and U is an open subset of X that is a union of closed sets, then $f(U)$ is open.*

Proof. Assume $f : X \to Y$ is weakly contra-closed. Let U be an open subset of X such that $U = \cup_{\alpha \in \mathcal{A}} F_\alpha$, where for every $\alpha \in \mathcal{A}$ F_α is a closed subset of X. Since f is weakly contra-closed $f(F_\alpha) \subseteq \text{Int}(f(U))$ for every $\alpha \in \mathcal{A}$. Thus $f(U) = \cup_{\alpha \in \mathcal{A}} f(F_\alpha) \subseteq \text{Int}(f(U))$, which proves that $f(U)$ is open. □

Corollary 4.5. (see [1], Theorem 11) *If the function $f : X \to Y$ is contra-closed and U is an open subset of X that is a union of closed sets, then $f(U)$ is open.*

Corollary 4.6. *If the function $f : X \to Y$ is weakly contra-open and X is a C-space, then f is open.*

Corollary 4.7. *If the function $f : X \to Y$ is weakly contra-closed and X is regular or T_1, then f is open.*

Corollary 4.8. *If the function $f : X \to Y$ is weakly contra-closed and U is θ-open, then $f(U)$ is open.*

5. Relationships between Weakly Contra-Open and Weakly Contra-Closed Functions

First we establish that weak contra-openness and weak contra-closedness are independent properties.

Example 5.1. Let $X = \{a, b\}$ have the topology $\tau = \{X, \emptyset, \{a\}\}$ and define $f : X \to X$ by $f(a) = f(b) = a$. Then f is open and hence weakly contra-closed, but f is not weakly contra-open. Note that $\{b\} \subseteq X$, but $\text{Cl}(f(\{b\})) \nsubseteq f(X)$. Therefore weakly contra-closed does not imply weakly contra-open.

Example 5.2. Let $X = \{a, b\}$ have the topology $\tau = \{X, \emptyset, \{a\}\}$ and define $f : X \to X$ by $f(a) = f(b) = b$. Then f is closed and hence weakly contra-open but, since $f(\{b\}) \nsubseteq \text{Int}(f(X))$, f is not weakly contra-closed. Hence weakly contra-open does not imply weakly contra-closed.

Therefore weakly contra-open and weakly contra-closed are independent.

Definition 5.3. A function $f : X \to Y$ is said to be contra-preopen (respectively, contra-preclosed (see [3])), if for every open (respectively, closed) set A in X, $f(A)$ is preclosed (respectively, preopen).

Theorem 5.4. *If the function $f : X \to Y$ is weakly contra-open and contra-preclosed, then f is weakly contra-closed.*

Proof. Assume $f : X \to Y$ is weakly contra-open and contra-preclosed and let $A \subseteq U \subseteq X$, where A is closed in X and U is open in X. Since f is weakly contra-open, $\mathrm{Cl}(f(A)) \subseteq f(U)$ and hence $\mathrm{Int}(\mathrm{Cl}(f(A))) \subseteq \mathrm{Int}(f(U))$. Since f is contra-preclosed, $f(A)$ is preopen and hence $f(A) \subseteq \mathrm{Int}(\mathrm{Cl}(f(A)))$. Thus $f(A) \subseteq \mathrm{Int}(f(U))$, which proves that f is weakly contra-closed. □

Theorem 5.5. *If the function $f : X \to Y$ is weakly contra-closed and contra-preopen, then f is weakly contra-open.*

Proof. Assume $f : X \to Y$ is weakly contra-closed and contra-preopen and let $A \subseteq U \subseteq X$, where A is closed in X and U is open in X. Since f is weakly contra-closed, $f(A) \subseteq \mathrm{Int}(f(U))$ and therefore $\mathrm{Cl}(f(A)) \subseteq \mathrm{Cl}(\mathrm{Int}(f(U)))$. Since f is contra-preopen, $f(U)$ is preclosed and thus $\mathrm{Cl}(\mathrm{Int}(f(U))) \subseteq f(U)$. Therefore $\mathrm{Cl}(f(A)) \subseteq f(U)$, which proves that f is weakly contra-open. □

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