HOMOGENIZATION OF SOME SINGULAR NONLINEAR ELLIPTIC PROBLEMS

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Abstract: The paper deals with the asymptotic behaviour of the solutions of quasilinear elliptic Dirichlet problems when the (linear) principal parts H-converge and the lower order terms exhibit a natural growth in the gradient variable and a singular behaviour in the u variable. The lower order term in the limit problem presents the same singularity in the u variable and the same growth in the gradient variable. The quadratic nonlinearity in the gradient is constructed through the correctors associated to the principal parts, on which natural integrability hypotheses are assumed.

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1. Introduction

In this paper we study the homogenization of the following nonlinear elliptic
Dirichlet problem:

\[
\begin{cases}
-\text{div} \left( A_\varepsilon \nabla u_\varepsilon \right) + \lambda u_\varepsilon = \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} + f(x) & \text{in } \Omega, \\
|u_\varepsilon| = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \) is an open bounded set of \( \mathbb{R}^N, N \geq 2 \), \( f \) is a nonnegative function in \( L^m(\Omega) \) with \( m > \frac{N}{2} \), \( 0 < k < 1 \), \( \lambda \geq 0 \) and \( \varepsilon \) is a positive parameter which tends to zero.

The functions \( b_\varepsilon(x, \xi) \) are nonnegative Carathéodory functions on \( \Omega \times \mathbb{R}^N \), quadratically growing in the \( \xi \) variable. We are interested in the asymptotic behaviour, as \( \varepsilon \to 0 \), of (bounded) solutions \( u_\varepsilon \) to (1.1), when \( A_\varepsilon(x) \) is a matrix field in \( M(\alpha, \beta, \Omega) \), which H-converges to a matrix field \( A_0(x) \) (see next section for the definition of the class \( M(\alpha, \beta, \Omega) \) and the notion of H-convergence).

The existence of bounded solutions to (1.1), also for growth \( k \geq 1 \), has originally been proved in [13] for a bounded data \( f \) and a strictly positive \( \lambda \), and extended in [12] to \( f \in L^m(\Omega), m > \frac{N}{2}, \) and \( \lambda \geq 0 \). For other existence results for similar problems, we refer to [7], [3], [2], [1], [4].

A first motivation for studying existence for this kind of problems is the connection with existence of boundary blow-up solutions (see [2]).

A second one is the study of simple functional like

\[
\int_{\Omega} (\sqrt{u} |Du|^2 - fu) \, dx
\]

and their Euler equations, which look like the previous equations.

Moreover, let us consider the following model of growth in porous media:

\[
u_t - \Delta (u^m) = |Du|^q + f
\]

with \( m > 1 \) and \( 1 < q \leq 2 \). If we consider steady states solutions and we perform a change of unknown \( u^m = v \), we get an equation with singular behaviour in \( v \).

Let us also mention that problems like (1.1), depending on a small parameter, appear when the phenomena described above are related to composite materials, where the heterogeneities are very small compared with the global dimension of the sample. Smaller are the heterogeneities, better is the mixture, which appears then, at a first glance, as a homogeneous material. The parameter \( \varepsilon \) describes the heterogeneities of the material as, for instance, in the periodic framework where it describes the periodicity of their distribution. From mathematical point of view, this justify the interest in finding the limit problem, as \( \varepsilon \) goes to zero, which will be the model for the homogenized material.
In the linear case, the theory of G-convergence of E. De Giorgi and S. Spagnolo (cf. [20]) for symmetric matrix fields, and that of H-convergence of F. Murat and L. Tartar for non symmetric matrices (cf. [17], [22] and [19]), allow to deal with general uniformly elliptic second order partial differential operators with oscillatory coefficients (non necessarily periodic). In this framework, the gradient $\nabla u_\varepsilon$ of the solution is (only) weakly converging in $L^2$ to the gradient of the solution $u^0$ of the limit problem. This convergence has been improved in [17], [22] by introducing the notion of corrector, which is a suitable family of matrix fields $C_\varepsilon$ such that $\nabla u_\varepsilon - C_\varepsilon \nabla u_0$ strongly converges to zero in $L^1$.

Homogenization results for problems with a principal part which H-converges and a lower order terms $H_\varepsilon(x,u,\nabla u)$ which has a quadratic growth with respect to $\nabla u$ and is continuous with respect to $u$, have been obtained in several paper (see for example [9], [8], [11]). In these papers, the main point is showing that the corrector for the linear problem is also a corrector for the nonlinear one.

The additional difficulty in the present paper, is due to the singular behaviour of the lower order term with respect to the $u$ variable.

We prove that, up to a subsequence, any sequence $\{u_\varepsilon\}$ of solutions of (1.1) weakly converges in $H^1_0(\Omega)$ to a function $u_0$, solution to

$$ \begin{cases} 
-\text{div} (A_0 \nabla u_0) + \lambda u_0 = \frac{b_0(x,\nabla u_0)}{(u_0)^k} + f(x) & \text{in } \Omega, \\
 u_0 = 0 & \text{on } \partial \Omega,
\end{cases} $$

(1.2)

where $b_0(x,\xi)$ is a Carathéodory function with quadratic growth in the $\xi$ variable, constructed from the corresponding terms $b_\varepsilon(x,\xi)$ by using the corrector $C_\varepsilon$ associated to the H-converging sequence $\{A_\varepsilon\}$. We also show in Section 7, that this result is still true for a singularity with a more general behaviour at the infinity.

As in the papers quoted above, we prove that the corrector for our nonlinear singular problem is again that associated to the linear one. We only assume that the correctors $C_\varepsilon$ are equi-integrable in $(L^2(\Omega))^N$, which is the natural hypothesis, as done in [8].

The proof of this corrector result present some technical difficulties, due to the presence of the singularity, and it requires to assume $b_\varepsilon(x,\xi) \geq 0$. We refer to Remark 5.6 for detailed comments about this assumption.

Finally, let us point out that we need to make precise the definition, of a solution to (1.1) and (1.2), since, in principle, the terms $\frac{b_\varepsilon(x,\nabla u_\varepsilon)}{(u_\varepsilon)^k}$ and $\frac{b_0(x,\nabla u_0)}{(u_0)^k}$ can be singular also inside $\Omega$.

This is done by introducing suitable variational formulations in (3.9) and (3.15), respectively (see also Remark 3.2 for more details).
The plan of the paper is the following:
In Section 2 we recall the definition and the main properties of the H-convergence.
In Section 3 we present the problem and we state the main results.
In Section 4 we establish some a priori estimates.
In Section 5 we prove the corrector result for the nonlinear problem.
In Section 6 we give the proof of the main result.
In Section 7 we consider a more general singularity.

2. About the H-Convergence

In this section we recall the definition and some important facts about the H-convergence. The H-convergence was introduced by F. Murat and L. Tartar (see [17], [22], see also [19]) to treat second order differential operators with non-symmetric oscillating coefficients. It generalizes the G-convergence, introduced by S. Spagnolo (see [20]) for symmetric matrices.

Let $\Omega$ be an open bounded set of $\mathbb{R}^N$ and $\{\varepsilon\}$ a sequence of positive real numbers that converges to zero.

For $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$, consider the class $M(\alpha, \beta, \Omega)$ of the $N \times N$ matrix field $A$ in $(L^\infty(\Omega))^N$, such that $\forall \xi \in \mathbb{R}^N$ and a.e. in $\Omega$,

\[
\begin{align*}
(i) \quad & A(x)\xi \xi \geq \alpha |\xi|^2, \\
(ii) \quad & A^{-1}(x)\xi \xi \geq \frac{1}{\beta} |\xi|^2.
\end{align*}
\]

Observe that ii) implies that $|A(x)\xi| \leq \beta \xi$.

Let us first recall the definition of the H-convergence:

**Definition 2.1.** Let $\{A^\varepsilon\}$ be a sequence in $\mathcal{M}(\alpha, \beta, \Omega)$. We say that $\{A^\varepsilon\}$ H-converges to $A^0 \in \mathcal{M}(\alpha, \beta, \Omega)$ if for any $g$ in $H^{-1}(\Omega)$, the solution $v_\varepsilon$ of

\[
\begin{cases}
-\text{div}(A^\varepsilon \nabla v_\varepsilon) = g \text{ on } \Omega, \\
v_\varepsilon = 0 \text{ on } \partial \Omega,
\end{cases}
\]

satisfies the weak convergences

\[
\begin{cases}
v_\varepsilon \rightharpoonup v_0 \text{ weakly in } H^1_0(\Omega), \\
A^\varepsilon \nabla v_\varepsilon \rightharpoonup A^0 \nabla v_0 \text{ weakly in } (L^2(\Omega))^N,
\end{cases}
\]

(2.1)
where $v_0$ is the unique solution in $H^1_0(\Omega)$ of the following problem:

$$
\begin{cases}
- \text{div} (A^0 \nabla v_0) = g \text{ on } \Omega, \\
v_0 = 0 \text{ on } \partial \Omega.
\end{cases}
$$

(2.3)

We recall here two main properties of the H-convergence. The first one is the following compactness result:

**Theorem 2.2.** (see [17], [22]) Every sequence $\{A^\varepsilon\}$ in $\mathcal{M}(\alpha, \beta, \Omega)$ admits a subsequence (still denoted here $A^\varepsilon$) such that for some $A^0 \in \mathcal{M}(\alpha, \beta, \Omega)$

$$
\{A^\varepsilon\} \text{ H-converges to } A^0.
$$

(2.4)

The second one is the corrector result below.

**Theorem 2.3.** (see [17], [22]) Under assumption (2.4), there exists a sequence of $N \times N$ matrix fields $C^\varepsilon$ (the so-called correctors) such that

$$
\begin{cases}
i) \quad C^\varepsilon \rightharpoonup I \text{ weakly in } (L^2(\Omega))^N, \\
ii) \quad A^\varepsilon C^\varepsilon \rightharpoonup A^0 \text{ weakly in } (L^2(\Omega))^N
\end{cases}
$$

(2.5)

and, for every $\xi \in \mathbb{R}^N$,

$$
\begin{cases}
i) \quad \text{curl}(C^\varepsilon \xi) = 0 \text{ in } (\mathcal{D}'(\Omega))^N, \\
ii) \quad \text{div}(A^\varepsilon C^\varepsilon \xi) = \text{div}(A^0 \xi) \text{ in } \mathcal{D}'(\Omega).
\end{cases}
$$

(2.6)

Moreover, if $v_\varepsilon$ is the solution of (2.1) associated with $g$ and $v_0$ the solution of (2.3) one has

$$
\nabla v_\varepsilon - C^\varepsilon \nabla v_0 \rightarrow 0, \quad \text{strongly in } (L^1(\Omega))^N.
$$

**Remark 2.4.** The model case of H-convergence is the classical periodic case (see [6]), where

$$
A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \quad \text{a.e. in } \Omega,
$$

with

$$
A \in \mathcal{M}(\alpha, \beta, Y) \text{ and } Y\text{-periodic},
$$

$Y$ being the reference period. In this case, the sequence $\{A^\varepsilon\}$ H-converges to the constant positive definite matrix fields $A^0$ defined by

$$
A^0 \lambda = \frac{1}{|Y|} \int_Y A \nabla w_\lambda \, dy,
$$

(2.7)
with $w_\lambda \in H^1(Y)$ solution, for any $\lambda \in \mathbb{R}^n$, of

$$
\begin{cases}
-\operatorname{div}(A\nabla w_\lambda) = 0 & \text{in } Y, \\
w_\lambda - \lambda \cdot y & \text{Y-periodic,} \\
m_Y(w_\lambda - \lambda \cdot y) = 0.
\end{cases}
$$

(2.8)

The corrector matrix $C^\varepsilon = (C^\varepsilon_{ij})_{1 \leq i,j \leq n}$ is given by (for a proof, see for instance [10])

$$
\begin{align*}
C^\varepsilon_{ij}(x) &= C_{ij} \left(\frac{x}{\varepsilon}\right) & \text{a.e. on } \Omega, \\
C_{ij}(y) &= \frac{\partial w_{\varepsilon j}}{\partial y_i}(y), & i,j = 1, \ldots, n \quad \text{a.e. on } Y,
\end{align*}
$$

(2.9)

where $\{e_j\}_{j=1}^N$ is the canonical basis of $\mathbb{R}^N$.

Using the Meyers regularity result (see [16]), it can be showed (see Appendix in [18]) that the functions $|C^\varepsilon|^2$ are equi-integrable.

This, together with the corrector results of [17], [22] (see also the proofs therein) implies that, under the notations above, for every $\delta > 0$, if $\Psi : \Omega \to \mathbb{R}^N$ is a simple function such that

$$
\|\nabla v_0 - \Psi\|_{(L^2(\Omega))^N} \leq \delta,
$$

(2.10)

then

$$
\limsup_{\varepsilon \to 0} \|\nabla v_\varepsilon - C^\varepsilon \Psi\|_{(L^2(\Omega))^N} \leq c \delta,
$$

(2.11)

where $c$ depends only on $\alpha$ and $\beta$.

We will use also the following two results proved in [9]:

**Lemma 2.5.** (see [9]) Let $\{g_\varepsilon\}_\varepsilon$ be a sequence of functions weakly converging in $L^1(\Omega)$ to a function $g_0$ and let $\{t_\varepsilon\}_\varepsilon$ be a sequence of equi-bounded and measurable functions which converge almost pointwise in $\Omega$ to a function $t_0$. Then

$$
\lim_{\varepsilon \to 0} \int_{\Omega} g_\varepsilon t_\varepsilon \, dx = \int_{\Omega} g_0 t_0 \, dx.
$$

**Proposition 2.6.** (see [9]) Under assumption (2.4), let $v_\varepsilon$ be the solution of (2.1), for a given $g$ in $H^{-1}(\Omega)$. Then, for every sequence $\{\varphi_\varepsilon\}$ in $L^\infty(\Omega)$ such that

$$
\begin{align*}
\{\varphi_\varepsilon\} & \text{ is bounded in } L^\infty(\Omega), \\
\varphi_\varepsilon & \to \varphi_0, \quad \text{a.e. in } \Omega,
\end{align*}
$$

(2.12)
one has
\[ \int_{\Omega} A^{\varepsilon} \nabla v^{\varepsilon} \nabla v^{\varepsilon} \varphi^{\varepsilon} \, dx \to \int_{\Omega} A^{0} \nabla v_{0} \nabla v_{0} \varphi_{0} \, dx. \]

3. Position of the Problem and Main Result

Our aim is to study the asymptotic behaviour, as \( \varepsilon \to 0 \), of the following problem:

\[
\begin{cases}
- \text{div} \left( A^{\varepsilon} \nabla u^{\varepsilon} \right) + \lambda u^{\varepsilon} = f(x) & \text{in } \Omega, \\
 u^{\varepsilon} = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(3.1)

where

\[ 0 < k < 1, \]  

(3.2)

\[ \lambda \geq 0, \quad f(x) \in L^{m}(\Omega), \quad m > N/2, \quad f(x) \geq 0, \quad f \not\equiv 0, \]  

(3.3)

and

\[ \{ A^{\varepsilon} \} \text{ is a sequence in } \mathcal{M}(\alpha, \beta, \Omega). \]  

(3.4)

We suppose that for every \( \varepsilon \), the function \( b^{\varepsilon} \) is a nonnegative Carathéodory function on \( \Omega \times \mathbb{R}^{N} \) such that for some positive constant \( c_{1} \) (independent of \( \varepsilon \)), and for almost every \( x \in \Omega \), any \( \xi \) and \( \xi_{1} \) in \( \mathbb{R}^{N} \) one has

\[ \begin{align*}
& \text{i)} \quad b^{\varepsilon}(x, 0) = 0, \\
& \text{ii)} \quad |b^{\varepsilon}(x, \xi) - b^{\varepsilon}(x, \xi_{1})| \leq c_{1}(|\xi| + |\xi_{1}|)|\xi - \xi_{1}|. 
\end{align*} \]  

(3.5)

Remark 3.1. Observe that this assumption means that \( b^{\varepsilon} \) has a quadratic growth. Indeed, if (3.5)ii) holds, then assumption (3.5)i) is equivalent to suppose that for any \( x \in \Omega \) and \( \xi \) in \( \mathbb{R}^{N} \) one has

\[ b^{\varepsilon}(x, \xi) \leq c_{1}|\xi|^{2}. \]  

(3.6)

For \( s \geq 0 \), we will use along the paper the function \( \gamma(s) \) defined by

\[ \gamma(s) = \frac{c_{1}}{\alpha} s^{1-k}, \]  

(3.7)

which is a primitive function of \( \frac{c_{1}}{\alpha s^{k}} \) and the function \( \psi \) given by

\[ \psi(s) = \int_{0}^{s} e^{\gamma(\sigma)} d\sigma. \]  

(3.8)
It is known that there exists at least a solution of problem (3.1), i.e. a function $u_\varepsilon$ such that

$$u_\varepsilon \in H^1_0(\Omega) \cap L^\infty(\Omega), \quad u_\varepsilon > 0,$$

$$\int_{\Omega} A^\varepsilon \nabla u_\varepsilon \nabla \Phi \, dx + \lambda \int_{\Omega} u_\varepsilon \Phi \, dx = \int_{\Omega} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \Phi \, dx + \int_{\Omega} f \Phi \, dx,$$

$$\forall \Phi \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

The existence of such a solution (for $b_\varepsilon$ of any sign) has been proved by D. Giachetti and F. Murat in [13], for $\lambda$ strictly positive and a bounded data $f$ and by the authors under assumption (3.3) in [12].

**Remark 3.2.** 1) In [13] the test functions in the definition of solution are taken in $D(\Omega)$. Actually, it is easy to show that we can take test functions in $H^1_0(\Omega) \cap L^\infty(\Omega)$ as in (3.9). Indeed, a function $\Phi$ in $H^1_0(\Omega) \cap L^\infty(\Omega)$ can be approximated in the $H^1_0$-norm by a sequence $\{\Phi_n\}$ in $D(\Omega)$, which is bounded in $L^\infty(\Omega)$, then weakly * converging to $\Phi$ in $L^\infty(\Omega)$.

2) Let us mention that the definition of solution of problem (1.1) given in [13] is a little bit different, namely (forgetting the parameter $\varepsilon$):

$$u \in H^1_0(\Omega) \cap L^\infty(\Omega), \quad u \geq 0,$$

$$\int_{\Omega} A \nabla u \nabla \Phi \, dx + \lambda \int_{\Omega} u \Phi \, dx = \int_{\Omega} \frac{b(x, \nabla u)}{u^k} \chi_{u > 0} \Phi \, dx + \int_{\Omega} f \Phi \, dx,$$

$$\forall \Phi \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

Indeed, when $b$ is nonnegative as assumed in the present paper, due to the strong maximum principle (see [15], Theorem 8.19) the solution is strictly positive almost everywhere. Then, the characteristic function $\chi_{u > 0}$ which appears in the previous definition is equal to 1 almost everywhere.

We can state now the main results of this paper.

**Theorem 3.3.** Assume that (3.2) and (3.3) hold. Let $\{A^\varepsilon\}_{\varepsilon}$ be a sequence in $M(\alpha, \beta, \Omega)$ and $\{b_\varepsilon\}$ a sequence of nonnegative Carathéodory functions on $\Omega \times \mathbb{R}^N$ satisfying (3.5). Suppose that $u_\varepsilon$ satisfies (3.9). Then, there exist a subsequence (still denoted by $\{\varepsilon\}$), a matrix $A^0$ in $M(\alpha, \beta, \Omega)$, a Carathéodory function $b_0(x, \xi)$ on $\Omega \times \mathbb{R}^N$ satisfying (3.5) up to a multiplicative constant, and...
a strictly positive function \( u_0 \) such that, as \( \varepsilon \to 0 \),

\[
\{ A^\varepsilon \} \ H\text{-converges to } A^0, \quad (3.10)
\]

\[
u_\varepsilon \to u_0 \quad \text{weakly in } H^1_0(\Omega), \quad (3.11)
\]

\[
u_\varepsilon \to u_0 \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \quad (3.12)
\]

\[
\frac{b_\varepsilon(x, \nabla \nu_\varepsilon)}{(u_\varepsilon)^k} \rightharpoonup \frac{b_0(x, \nabla u_0)}{(u_0)^k} \quad \text{in } D'(\Omega), \quad (3.13)
\]

\[
-\text{div}(A^\varepsilon \nabla \nu_\varepsilon) \to -\text{div}(A^0 \nabla u_0) \quad \text{strongly in } H^{-1}(\omega), \quad (3.14)
\]

for every open set \( \omega \subset \subset \Omega \) and

\[
\begin{array}{ll}
\int_\Omega A^0 \nabla u_0 \nabla \Phi \, dx + \lambda \int_\Omega u_0 \Phi \, dx &= \int_\Omega \frac{b_0(x, \nabla u_0)}{(u_0)^k} \Phi \, dx + \int_\Omega f \Phi \, dx, \\
&\forall \Phi \in H^1_0(\Omega) \cap L^\infty(\Omega).
\end{array} \tag{3.15}
\]

**Remark 3.4.**

1. Following [9], the function \( b_0(x, \xi) \) is defined as a weak limit in \( L^1(\Omega) \), using the fact that (up to a subsequence) the following convergence holds true:

\[
b_\varepsilon(x, C^\varepsilon \xi) \rightharpoonup b_0(x, \xi) \quad \text{weakly in } L^1(\Omega), \quad \text{for every } \xi \in \mathbb{R}^N.
\]

In particular, in the periodic case the results reads as follows. Let \( A^\varepsilon \) be as in Remark 2.4 and assume that \( b_\varepsilon \) is of the form

\[
b_\varepsilon(x, \xi) = b\left(\frac{x}{\varepsilon}, \xi\right)
\]

where \( b(y, \xi) \) is a Carathéodory function on \( Y \times \mathbb{R}^N \), \( Y \)-periodic with respect to the first variable and verifying

\[
i) \quad b(y, 0) = 0,
\]
\[
ii) \quad |b(y, \xi) - b(y, \xi_1)| \leq c_1(|\xi| + |\xi_1|)|\xi - \xi_1|,
\]

for some positive constant \( c_1 \), almost every \( y \in Y \), any \( \xi \) and \( \xi_1 \) in \( \mathbb{R}^N \). Then, in equation (3.15) the homogenized matrix \( A^0 \) is given by (2.7), and the nonlinear limit function \( b_0(x, \xi) = b_0(\xi) \) is independent on \( x \) and given by

\[
b_0(\xi) = \frac{1}{|Y|} \int_Y b(y, C(y)\xi) \, dy, \quad \text{for every } \xi \in \mathbb{R}^N.
\]
2. Note that the functions $b_\varepsilon(x,\xi)$ are assumed nonnegative in the previous theorem. It would be very interesting to allow a general sign for $b_\varepsilon(x,\xi)$. The point where this assumption is used is in the proof of Theorem 5.1. For further details, we refer to the Remark 5.6, where we point out the technical difficulty.

4. A Priori Estimates

We give here uniform estimates for the sequence \{${u_\varepsilon}$\} of the solutions of problem (3.9). Their proofs follow the outlines of the analogous estimates for the approximating solutions, showed in [13] (Section 4, steps 1-3) in order to prove the existence result. We prove them here for the reader’s convenience.

Along this paper, we will denote by $c$ different constants independent of $\varepsilon$.

For any function $v$ in $H^1_0(\Omega)$, we define

$$u_+ = \max\{u, 0\}, \quad u_- = -\min\{u, 0\}, \quad \text{a.e. on } \Omega,$$

which, by known results, still belong to $H^1_0(\Omega)$. Clearly, $u = u^+ - u^-$.  

**Proposition 4.1.** Under the hypotheses of Theorem 3.3, there exists a constant $c$, independent of $\varepsilon$, such that

$$\|\psi(u_\varepsilon)\|_{H^1_0(\Omega)} \leq c,$$  \hspace{1cm} (4.1)

and

$$\|u_\varepsilon\|_{H^1_0(\Omega)} \leq c,$$  \hspace{1cm} (4.2)

where $\psi$ is defined by (3.7)-(3.8). If $N > 2$ the results is still true for $f \in L^N_\infty(\Omega)$.

**Proof.** We begin by proving (4.1) and choose as test function in (3.9) the function

$$v_\varepsilon = \psi(u_\varepsilon)e^{\gamma(u_\varepsilon)}.$$

where $\gamma$ and $\psi$ are defined by (3.7) and (3.8), respectively.

Let us first show that $v_\varepsilon$ is an admissible test. It clearly belongs to $L^\infty(\Omega)$. On the other hand, the real function $\psi(s)e^{\gamma(s)}$ is Lipschitz continuous on $[0, a]$ for every $a > 0$, since

$$(\psi(s)e^{\gamma(s)})' = \psi'(s)e^{\gamma(s)} + \psi(s)e^{\gamma(s)}\frac{1}{s^k}$$
and by the de l’Hôpital rule, \( \lim_{s \to 0^+} \frac{\psi(s)}{s^k} = 0 \). Then, the Stampacchia Theorem implies that \( u_\varepsilon \in H^1_0(\Omega) \).

Using (3.4), (3.5) and (3.8), we get

\[
\begin{align*}
\alpha \int_{\Omega} |\nabla \psi(u_\varepsilon)|^2 2^\gamma(u_\varepsilon) dx + c_1 \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} e^{\gamma(u_\varepsilon)} dx + \lambda \int_{\Omega} u_\varepsilon \psi(u_\varepsilon) e^{\gamma(u_\varepsilon)} dx \\
\leq c_1 \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} e^{\gamma(u_\varepsilon)} dx + \int_{\Omega} f e^{\gamma(u_\varepsilon)} \psi(u_\varepsilon) dx.
\end{align*}
\]

Observe that one also has

\[
\lim_{s \to +\infty} \frac{e^{\gamma(s)}}{\psi(s)} = \lim_{s \to +\infty} \frac{1}{s^k} = 0, \tag{4.3}
\]

which implies that for every \( \eta > 0 \) there exists a constant \( c \) such that

\[ e^{\gamma(s)} \leq \eta \psi(s) + c. \]

Hence, if \( m' = m/m - 1 \),

\[
\begin{align*}
\int_{\Omega} |\nabla \psi(u_\varepsilon)|^2 dx &\leq \eta \int_{\Omega} f(\psi(u_\varepsilon))^2 dx + c \int_{\Omega} f \psi(u_\varepsilon) dx \\
&\leq \eta ||f||_{L^m(\Omega)} ||\psi(u_\varepsilon)||_{L^{2m}(\Omega)}^2 + c ||f||_{L^m(\Omega)} + \eta ||f||_{L^m(\Omega)} ||\psi(u_\varepsilon)||_{L^{m'}(\Omega)}^2, \tag{4.4}
\end{align*}
\]

where we used the Sobolev embeddings and, for \( N > 2 \), the fact that \( m' < 2m' < 2^* \).

Estimate (4.2) follows immediately from (4.1), since \( \gamma \) is nonnegative, so that \( |\nabla u_\varepsilon| \leq |\nabla \psi(u_\varepsilon)| \).

To show the last result observe that if \( N > 2 \), we can take \( m = \frac{N}{2} \) in (4.4).

\[ \square \]

**Proposition 4.2.** Under the hypotheses of Theorem 3.3, there exists a positive constant \( M \) such that

\[ ||u_\varepsilon||_{L^\infty(\Omega)} \leq M, \tag{4.5} \]

for every \( \varepsilon \).
**Proof.** From (4.2) it follows that setting

\[ A_{h,\varepsilon} = \{ x \in \Omega : u_\varepsilon > h \} \]

one has

\[ \lim_{h \to \infty} \text{meas} (A_{h,\varepsilon}) = 0, \quad \text{uniformly with respect to } \varepsilon. \]

This allows to adapt the arguments introduced in the proof of Lemma 3.1 of [4] for proving an existence result (see also [12]). It is done in two steps. The first one proves that, if \( f \in L^2_N(\Omega) \), the sequence \( \{ e^{u_\varepsilon} \} \) is bounded in \( L^p(\Omega) \), for every \( p > 1 \). The second one shows estimate (4.5) by Stampacchia’s methods, using the fact that \( f \in L^m(\Omega) \) for \( m > \frac{N}{2} \). \( \square \)

**Proposition 4.3.** Under the hypotheses of Theorem 3.3, there exists a constant \( c \), independent of \( \varepsilon \), such that

\[ \left\| \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \right\|_{L^1(\Omega)} \leq c. \quad (4.6) \]

We set for every \( \varepsilon \) and \( n \in \mathbb{N} \),

\[ \nu_\varepsilon^{n} = \frac{1}{n} \| u_\varepsilon \|_{L^\infty(\Omega)} \]

and use as test function in (3.9) the function

\[ v_\varepsilon^{n} = e^{[\gamma(u_\varepsilon) - \gamma(\nu_\varepsilon^{n})] + 1}, \]

which belongs to \( H^1_0(\Omega) \cap L^\infty(\Omega) \), since it vanishes on the set \( \{ u_\varepsilon \leq \nu_\varepsilon^{n} \} \).

From (3.5) we get

\[
\frac{c_1}{\alpha} \int_{\Omega \cap \{ u_\varepsilon \geq \nu_\varepsilon^{n} \}} A^{\varepsilon} \nabla u_\varepsilon \nabla u_\varepsilon \frac{e^{[\gamma(u_\varepsilon) - \gamma(\nu_\varepsilon^{n})]}}{(u_\varepsilon)^k} \, dx + \lambda \int_{\Omega} u_\varepsilon (e^{[\gamma(u_\varepsilon) - \gamma(\nu_\varepsilon^{n})] + 1}) \, dx \\
\leq c_1 \int_{\Omega \cap \{ u_\varepsilon \geq \nu_\varepsilon^{n} \}} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} (e^{[\gamma(u_\varepsilon) - \gamma(\nu_\varepsilon^{n})]} - 1) \, dx + \int_{\Omega} f(e^{[\gamma(u_\varepsilon) - \gamma(\nu_\varepsilon^{n})] + 1}) \, dx.
\]

Using (3.4) and the fact that the last term of the left-hand side is nonnegative, after cancellation of the two equal terms we obtain

\[ c_1 \int_{\Omega \cap \{ u_\varepsilon \geq \nu_\varepsilon^{n} \}} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} \, dx \leq \int_{\Omega} f(e^{[\gamma(u_\varepsilon)]} - 1) \, dx \leq c, \]
where by Proposition (4.2), the constant $c$ is independent of $\varepsilon$. Passing to the limit, for fixed $\varepsilon$, as $n \to \infty$, gives
\[
\int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} \, dx \leq c.
\]  
(4.7)

Using (3.6), we conclude the proof. $\square$

**Remark 4.4.**
1) Let us point out that the estimates given in this section are straightforward if for each $\varepsilon$ the function $u_\varepsilon$ is the SOLA (Solution Obtained as Limit of Approximations) obtained in [13]. In this case, the corresponding estimates for the approximating sequences $\{u^n_\varepsilon\}_{n \in \mathbb{N}}$ are uniform (with respect to $\varepsilon$ and $n$), so that they hold true for the limit $u_\varepsilon$, uniformly in $\varepsilon$.

2) If $\lambda > 0$ and $f \in L^\infty(\Omega)$ estimate (4.5) can be proved as a first estimate, choosing as test function $\varepsilon \gamma(u_\varepsilon)(u_\varepsilon - \frac{1}{\lambda} \|f\|_{L^\infty(\Omega)})_+$ and obtaining $M = \frac{1}{\lambda}\|f\|_{L^\infty(\Omega)}$. Then, in view of estimate (4.5), one can prove the uniform $H^{1,0}$-estimate without using (4.3).

**Corollary 4.5.** Under the hypotheses of Theorem 3.3, there exists a subsequence (still denoted $\{\varepsilon\}$), a matrix $A^0$ in $\mathcal{M}(\alpha, \beta, \Omega)$, a Carathéodory function $b_0(x, \xi)$ on $\Omega \times \mathbb{R}^N$ satisfying (3.5) up to a multiplicative constant, and a function $u_0 \in H^{1,0}_0(\Omega) \cap L^\infty(\Omega)$ such that
\[
\begin{align*}
\{A^\varepsilon\} &\ H\text{-converges to } A^0, \\
i &u_\varepsilon \rightharpoonup u_0 \ \text{weakly in } H^{1,0}_0(\Omega), \\
n &u_\varepsilon \rightarrow u_0 \ \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \\
iv &b_\varepsilon(x, C^\varepsilon \Psi) \rightharpoonup b_0(x, \Psi) \ \text{weakly in } L^1(\Omega),
\end{align*}
\]
(4.8)

where $\{C^\varepsilon\}$ is the corrector sequence given by Theorem 2.3 for the subsequence verifying convergence i).

**Proof.** Convergences i) follows by compactness from Theorem 2.2. Convergences ii) and iii) and the fact that $u_0 \in L^\infty(\Omega)$ follow from the a priori estimates given above.

The last convergence and the properties of $b_0$ have been proved in [9] and [8]. $\square$

**N.B.** From now on, we will deal with this subsequence.
5. A Corrector for the Nonlinear Problem

In this section, following the ideas introduced in [9] for the quadratic (non singular) case, we define a suitable linear problem associated to a weak cluster point of the sequence of the solutions of problem (3.9) and we prove that the corrector for this linear problem (see Section 2) is also a corrector for our nonlinear problem.

Related to the subsequence given by Corollary 4.5, we consider the following linear problem:

\[
\begin{align*}
-\text{div} (A^\varepsilon \nabla v_\varepsilon) + \lambda v_\varepsilon &= -\text{div} (A^0 \nabla u_0) + \lambda u_0 & \text{on } \Omega, \\
v_\varepsilon &= 0 & \text{on } \partial\Omega.
\end{align*}
\]

(5.1)

where \(0 \leq u_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)\) is defined in (4.8).

From (4.8)i) by H-convergence we have, as \(\varepsilon\) tends to zero,

\[
\begin{align*}
i) &\quad v_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega), \\
ii) &\quad A^\varepsilon \nabla v_\varepsilon \rightharpoonup A^0 \nabla u_0 \text{ weakly in } (L^2(\Omega))^N.
\end{align*}
\]

(5.2)

We state now the following result, which is proved at the end of this section.

**Theorem 5.1.** Under the assumptions of Theorem 3.3, we have

\[
\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{(L^2(\omega))^N} = 0,
\]

(5.3)

for every open set \(\omega \subset \subset \Omega\).

A straightforward and important consequence of this proposition and inequality (2.11) is the following corrector result for the nonlinear problem (3.9):

**Corollary 5.2.** Under the assumptions of Theorem 3.3, for every \(\delta > 0\), let \(\Psi : \Omega \to \mathbb{R}^N\) be a simple function such that

\[
\|\nabla u^0 - \Psi\|_{(L^2(\Omega))^N} \leq \delta.
\]

(5.4)

Then,

\[
\limsup_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \Psi\|_{(L^2(\omega))^N} \leq c \delta,
\]

for every open set \(\omega \subset \subset \Omega\) and where \(c\) depends only on \(\alpha\) and \(\beta\).

To prove Theorem 5.1 we give first some technical lemmas.
Lemma 5.3. Let \( \mu \in ]0,1] \). Under the hypotheses of Theorem 3.3, we have
\[
\int_{\Omega \cap \{ u_\varepsilon \leq \mu \}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} |\varphi| \, dx \leq c \mu, \quad \text{for every } \varphi \in \mathcal{D}(\Omega),
\]
where \( c = c(\varphi) \) is independent of \( \varepsilon \).

Proof. Let us take as test function in (3.9)
\[
\Phi = -|\varphi| \varphi_n(u_\varepsilon) \quad \varphi \in \mathcal{D}(\Omega),
\]
where for \( s \geq 0 \) and every \( n \in \mathbb{N} \),
\[
\varphi_n(s) = \begin{cases} 
(e^{\gamma(\mu)} - \gamma(s) - 1) & \text{if } \frac{\mu}{n} < s, \\
(e^{\gamma(\mu)} - \gamma(s) - 1) & \text{if } s \leq \frac{\mu}{n}.
\end{cases}
\]
This test function is admissible since it is Lipschitz continuous, \( \varphi_n \) vanishing for \( s \geq \mu \).

By (3.4), (3.5) and (3.7) one has
\[
- \int_{\Omega \cap \{ u_\varepsilon \leq \mu \}} A^\varepsilon \nabla u_\varepsilon \nabla (|\varphi|) \varphi_n(u_\varepsilon) \, dx \\
+ c_1 \int_{\Omega \cap \{ \frac{\mu}{n} < u_\varepsilon \leq \mu \}} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} \, |\varphi| e^{\gamma(\mu)} - \gamma(u_\varepsilon) \, dx \leq \lambda \int_{\Omega \cap \{ u_\varepsilon \leq \mu \}} u_\varepsilon \varphi_n(u_\varepsilon) |\varphi| \, dx
\]
\[
\leq - \int_{\Omega \cap \{ u_\varepsilon \leq \mu \}} \left( \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} + f(\varphi) \right) \varphi_n(u_\varepsilon) |\varphi| \, dx \leq 0,
\]
since \( f \) and \( b \) are nonnegative.

Since \( \mu \leq 1 \) and \( \gamma \) is increasing, \( |\varphi_n| \leq \gamma(1) \). Consequently, by the Hölder inequality it comes
\[
c_1 \int_{\Omega \cap \{ \frac{\mu}{n} < u_\varepsilon \leq \mu \}} |\nabla u_\varepsilon|^2 \, |\varphi| \, dx \leq e^{\gamma(1)} \lambda \int_{\Omega \cap \{ u_\varepsilon \leq \mu \}} u_\varepsilon |\varphi| \, dx
\]
\[
+ \beta e^{\gamma(1)} \int_{\{ u_\varepsilon \leq \mu \} \cap \text{supp } \varphi} |\nabla u_\varepsilon|^2 \, dx \frac{1}{2} \left( \int_{\Omega} |\nabla \varphi|^2 \, dx \frac{1}{2} \right)^{\frac{1}{2}} = B_1 + B_2.
\]
We have
\[
B_1 \leq e^{\gamma(1)} \lambda \| \varphi \|_{L^1(\Omega)} \mu. \quad (5.7)
\]
We prove now that we have also

\[ B_2 \leq c\mu, \tag{5.8} \]

where \( c \) depends on \( \varphi \) and on the constants in Proposition 4.2 and Proposition 4.3.

Let us take

\[ \Phi = -\varphi_1^2(u_\varepsilon - \mu) \]

as test function in (3.9), where \( \varphi_1 \) is a function in \( D(\Omega) \) such that

\[ 0 \leq \varphi_1 \leq 1, \quad \varphi_1 = 1 \text{ on } \text{supp } \varphi. \]

A similar computation as above, using again the fact that \( f \geq 0 \), gives

\[
\begin{align*}
\alpha \int_{\Omega \cap \{u_\varepsilon \leq \mu\}} |\nabla u_\varepsilon|^2 \varphi_1^2 \, dx &+ 2 \int_{\Omega \cap \{u_\varepsilon \leq \mu\}} \varphi_1 A^\varepsilon \nabla u_\varepsilon \nabla \varphi_1 (u_\varepsilon - \mu) \, dx \\
&\leq \lambda \int_{\Omega \cap \{u_\varepsilon \leq \mu\}} u_\varepsilon (\mu - u_\varepsilon) \varphi_1^2 \, dx \\
&+ \int_{\Omega \cap \{u_\varepsilon \leq \mu\}} b_\varepsilon(x, \nabla u_\varepsilon) (\mu - u_\varepsilon) \varphi_1^2 \, dx + \int_{\Omega \cap \{u_\varepsilon \leq \mu\}} f(u_\varepsilon - \mu) \varphi_1^2 \, dx \\
&\leq \mu \lambda \int_{\Omega} u_\varepsilon \, dx + \mu \int_{\Omega} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \, dx \leq c\mu,
\end{align*}
\]

where \( c \) depend on the constants in Proposition 4.2 and Proposition 4.3.

This implies (5.8), since in view of (4.2) one has also

\[
2 \int_{\Omega \cap \{u_\varepsilon \leq \mu\}} \varphi_1 A^\varepsilon \nabla u_\varepsilon \nabla \varphi_1 (\mu - u_\varepsilon) \, dx \leq c\mu,
\]

so that, using (5.6)-(5.7) we have for every \( n \in \mathbb{N} \),

\[
\int_{\Omega \cap \{\frac{n}{2} < u_\varepsilon \leq \mu\}} |\nabla u_\varepsilon|^2 \, dx \leq c\mu.
\]

Passing to the limit as \( n \to \infty \) and using (3.6) we conclude the proof. \( \square \)

**Remark 5.4.** In the spirit of Remark 4.4, the estimate given by Lemma 5.3 is straightforward if, for each \( \varepsilon \), the function \( u_\varepsilon \) is the SOLA (Solution Obtained as Limit of Approximations) obtained in [13].
Observe now that, since the function \( u_0 \geq 0 \) is in \( H^1_0(\Omega) \cap L^\infty(\Omega) \), there exists a sequence \( \{u_n\} \subset D(\Omega) \) such that
\[
\begin{cases}
 u_n \rightarrow u_0 & \text{strongly in } H^1_0(\Omega) \quad \text{as } n \rightarrow +\infty, \\
 \|u_n\|_{L^\infty(\Omega)} \leq c, \quad u_n \geq 0,
\end{cases}
\]
for any \( n \), where \( c \) is a constant independent of \( n \).

Following the ideas of [9] let us introduce, for any \( n \in \mathbb{N} \), the sequence \( \{v_{n,\varepsilon}\}_\varepsilon \) of the solutions of
\[
\begin{cases}
 -\text{div}(A^\varepsilon(x)\nabla v_{n,\varepsilon}) + \lambda v_{n,\varepsilon} = -\text{div}(A^0\nabla u_n) + \lambda u_n & \text{in } \Omega, \\
 v_\varepsilon = 0 & \text{on } \partial\Omega,
\end{cases}
\]
whose variational formulation is
\[
\int_\Omega A^\varepsilon(x)\nabla v_{n,\varepsilon}\nabla \Phi \, dx + \lambda \int_\Omega v_{n,\varepsilon} \Phi \, dx = \int_\Omega A^0(x)\nabla u_n\nabla \Phi \, dx + \lambda \int_\Omega u_n \Phi \, dx
\]
for every \( \Phi \in H^1_0(\Omega) \).

Then, from (5.9) there exists a constant \( c \) independent of \( \varepsilon \) and \( n \), such that
\[
\|v_{n,\varepsilon}\|_{H^1_0(\Omega)} \leq c,
\]
and by H-convergence, for any \( n \),
\[
v_{n,\varepsilon} \rightharpoonup u_n \quad \text{weakly in } H^1_0(\Omega)
\]
as \( \varepsilon \) tends to zero.

Also, from classical results from G. Stampacchia (see [21]), for any fixed \( n \) we have
\[
\|v_{n,\varepsilon}\|_{L^\infty(\Omega)} \leq c_n,
\]
for every \( \varepsilon \).

**Lemma 5.5.** Under the hypotheses of Theorem 3.3, for each fixed \( n \) we have
\[
\lim_{\varepsilon \rightarrow 0} \int_\Omega |\nabla v_{n,\varepsilon}|^2 \, dx = 0,
\]
where \( v_{n,\varepsilon} \) is the solution of (5.10).
Proof. We take \(-v_{n,\varepsilon}\) as test function in (5.11) and for any \(\eta > 0\) we get, using Young inequality,

\[
\alpha \int_{\Omega} |\nabla v_{n,\varepsilon}^-|^2 dx + \lambda \int_{\Omega} |v_{n,\varepsilon}^-|^2 dx \leq - \int_{\Omega} A^0 \nabla u_n \nabla v_{n,\varepsilon}^- dx - \lambda \int_{\Omega} u_n v_{n,\varepsilon}^- dx
\]

\[
\leq c(\eta) \int_{\Omega} |\nabla u_n|^2 \chi_{\{v_{n,\varepsilon} \leq 0\}} dx + \eta \int_{\Omega} |\nabla v_{n,\varepsilon}^-|^2 dx,
\]

since \(u_n v_{n,\varepsilon}^- \geq 0\). Choosing \(\eta\) sufficiently small, we obtain

\[
0 \leq \int_{\Omega} |\nabla v_{n,\varepsilon}^-|^2 dx \leq c \int_{\Omega} |\nabla u_n|^2 \chi_{\{v_{n,\varepsilon} \leq 0\}} dx,
\]

which implies (5.15), since from (5.9),

\[
\chi_{\{v_{n,\varepsilon} \leq 0\}} \chi_{\{u_n \neq 0\}} \rightarrow \chi_{\{u_n < 0\}} = 0 \text{ a.e. in } \Omega \quad (5.16)
\]

and \(\nabla u_n = 0\) where \(u_n = 0\). \(\square\)

Proof of Theorem 5.1. We choose as test function in (3.9) and (5.11) the function

\[
\Phi = \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 \in H^1_0(\Omega) \cap L^\infty(\Omega),
\]

where \(\varphi\) is in \(D(\Omega)\) and

\[
\xi_{\nu}(s) = se^{\nu s^2}, \quad \forall s \in \mathbb{R},
\]

which is an increasing function on \(\mathbb{R}\), \(\nu\) being a positive constant to be chosen.

This gives, after substraction of the two identities

\[
\int_{\Omega} A^\varepsilon \nabla (u_\varepsilon - v_{n,\varepsilon}) \nabla (u_\varepsilon - v^+_{n,\varepsilon}) \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 dx
\]

\[
+ 2 \int_{\Omega} A^\varepsilon \nabla (u_\varepsilon - v_{n,\varepsilon}) \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi \nabla \varphi dx
\]

\[
+ \lambda \int_{\Omega} (u_\varepsilon - v_{n,\varepsilon}) \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 dx
\]

\[
= \int_{\Omega} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)_k} \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 dx + \int_{\Omega} f \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 dx
\]

\[
- \int_{\Omega} A^0 \nabla u_n \xi_{\nu}'(u_\varepsilon - v^+_{n,\varepsilon}) \nabla (u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 dx
\]

\[
- 2 \int_{\Omega} A^0 \nabla u_n \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi \nabla \varphi dx - \lambda \int_{\Omega} u_n \xi_{\nu}(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 dx.
\]

(5.17)
Due to (3.4) we have
\[
\alpha \int_\Omega |\nabla (u_\varepsilon - v^+_{n,\varepsilon})|^2 \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 \, dx \leq \int_\Omega \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi^2 \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \, dx
\]
\[- \int_\Omega A^\varepsilon \nabla v^-_{n,\varepsilon}, \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \varphi \nabla (u_\varepsilon - v^+_{n,\varepsilon}) \, dx
\]
\[-2 \int_\Omega A^\varepsilon \nabla v^-_{n,\varepsilon} \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \varphi \nabla \varphi \, dx
\]
+ \int f \varphi^2 \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \, dx - 2 \int_\Omega A^0 \nabla u_n \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \varphi \nabla \varphi \, dx
\]
\[-2 \int_\Omega A^0 \nabla u_n \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \nabla \varphi \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \, dx
\]
\[- \int_\Omega \lambda (u_n + v^-_{n,\varepsilon}) \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 \, dx.
\]
(5.18)

For every fixed \( \mu \in [0, 1] \), we estimate the first integral at the right-hand side of this inequality as
\[
\int_\Omega \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi^2 \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \, dx \leq \int_{\{u_\varepsilon > \mu\}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi^2 \xi_\nu (u_\varepsilon - v^+_{n,\varepsilon}) \, dx
\]
+ \mu e^\nu \mu^2 \int_{\{u_\varepsilon \leq \mu\}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi^2 \, dx \doteq I_1 + I_2,
(5.19)

where we used Proposition 4.2 and the fact that \( \xi_\nu \) is increasing.

In view of Lemma 5.3,
\[
I_2 \leq c \mu^2 e^\nu \mu^2,
(5.20)
\]
where \( c = c(\varphi) \) is independent on \( \varepsilon \).

As far as the term \( I_1 \) is concerned, using (3.5) we can write
\[
I_1 \leq \frac{2c_1}{\mu^k} \int_\Omega \left( |\nabla (u_\varepsilon - v^+_{n,\varepsilon})|^2 + |\nabla v^+_{n,\varepsilon}|^2 \right) \varphi^2 |\xi_\nu (u_\varepsilon - v^+_{n,\varepsilon})| \, dx.
(5.21)
\]

Choosing
\[
\nu = \left( \frac{c_1}{\alpha \mu^k} \right)^2
(5.22)
\]
(which depends on \( \mu \)), from the definition of \( \xi_\nu (s) \) we get for any \( s \in \mathbb{R} \),
\[
\alpha \xi_\nu (s) - \frac{2c_1}{\mu^k} |\xi_\nu (s)| = e^{\nu s^2} \left( \alpha (1 + 2s^2 \nu) - \frac{2c_1}{\mu^k} |s| \right) \geq \frac{\alpha}{2}.
\]
Moreover, by (5.20)
\[ I_2 \leq c \mu^2 \left( e^{\mu^2(1-k)} \right)^{\frac{2}{\alpha}}. \]
This, together with (3.4) and (5.18)-(5.21) gives
\[
\frac{\alpha}{2} \int_\Omega |\nabla (u_\varepsilon - v_{n,\varepsilon}^+)|^2 \varphi^2 \, dx
\leq \int_\Omega |\nabla (u_\varepsilon - v_{n,\varepsilon}^+)|^2 \left( \alpha \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) - \frac{c_1}{\mu k} |\xi\nu (u_\varepsilon - v_{n,\varepsilon}^+)| \right) \varphi^2 \, dx
\leq \frac{2c_1}{\mu k} \int_\Omega |\nabla v_{n,\varepsilon}^+|^2 |\xi\nu (u_\varepsilon - v_{n,\varepsilon}^+)| \varphi^2 \, dx + c \mu^2 \frac{c_1 \mu^{2(1-k)}}{\alpha}
- \int_\Omega A^\varepsilon \nabla v_{n,\varepsilon}^+, \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \varphi^2 \nabla (u_\varepsilon - v_{n,\varepsilon}^+) \, dx
- 2 \int_\Omega A^\varepsilon \nabla v_{n,\varepsilon}^+, \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \varphi \nabla \varphi \, dx
+ \int_\Omega f \varphi^2 \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \, dx - 2 \int_\Omega A^0 \nabla u_n \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \varphi \nabla \varphi \, dx
- 2 \int_\Omega A^\varepsilon \nabla (u_\varepsilon - v_{n,\varepsilon}^+) \varphi \nabla \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \, dx
- \int_\Omega A^0 \nabla u_n \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \varphi^2 \, dx
- \int_\Omega \lambda (u_n + v_{n,\varepsilon}^-) \xi\nu (u_\varepsilon - v_{n,\varepsilon}^+) \varphi^2 \, dx.
\]
Let us fix $n$ and $\mu$ (which implies also $\nu$ fixed).

Observe first that, as $\varepsilon \to 0$, from (4.8) and (5.13) the sequence $\{u_\varepsilon - v_{n,\varepsilon}^+\}$ converges (up to a subsequence) almost everywhere to $u_0 - u_n$. Then, from Proposition 4.2 and (5.14) the sequence $\{\xi\nu (u_\varepsilon - v_{n,\varepsilon}^+)\}$ strongly converges in $L^2(\Omega)$ to $\xi\nu (u_0 - u_n)$, as $\varepsilon \to 0$ and, up to a subsequence, almost everywhere.

Hence, by (3.4), Proposition 4.2, Lemma 5.5 and Proposition 2.6 we have
\[
\lim_{\varepsilon \to 0} \frac{2c_1}{\mu k} \int_{\Omega} |\nabla v_{n,\varepsilon}^+|^2 \varphi^2 |\xi\nu (u_\varepsilon - v_{n,\varepsilon}^+)| \, dx
\leq \lim_{\varepsilon \to 0} \left( \frac{c}{\mu k} \int_{\Omega} |\nabla v_{n,\varepsilon}^-|^2 \, dx + \frac{4c_1}{\mu k} \int_{\Omega} |\nabla v_{n,\varepsilon}^+|^2 \varphi^2 |\xi\nu (u_\varepsilon - v_{n,\varepsilon}^+)| \, dx \right)
\leq \lim_{\varepsilon \to 0} \frac{4c_1}{\alpha \mu k} \int_{\Omega} A^\varepsilon \nabla v_{n,\varepsilon} \nabla v_{n,\varepsilon} |\xi\nu (u_\varepsilon - v_{n,\varepsilon}^+)| \varphi^2 \, dx
= \frac{4c_1}{\alpha \mu k} \int_{\Omega} A^0 \nabla u_n \nabla u_n |\xi\nu (u_0 - u_n)| \varphi^2 \, dx.
\]
On the other hand, the sequence \( \{ \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \} \) is bounded (with respect to \( \varepsilon \)) in \( L^\infty(\Omega) \), due to (4.5) and (5.14). Then, by (3.4), (4.2), (5.12) and (5.15) we have

\[
\lim_{\varepsilon \to 0} \int_\Omega A^\varepsilon \nabla v^-_{n,\varepsilon} \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 \nabla (u_\varepsilon - v^+_{n,\varepsilon}) \, dx \\
\leq c(n, \nu) \lim_{\varepsilon \to 0} \| \nabla v^-_{n,\varepsilon} \|_{L^2(\Omega)} \| \nabla (u_\varepsilon - v^+_{n,\varepsilon}) \|_{L^2(\Omega)} = 0
\]  
(5.25)

and

\[
\lim_{\varepsilon \to 0} \left| \int_\Omega A^\varepsilon \nabla v^-_{n,\varepsilon} \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) 2\varphi \nabla \varphi \, dx \right| \\
\leq c(n, \nu) \lim_{\varepsilon \to 0} \| \nabla v^-_{n,\varepsilon} \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^\infty(\Omega)} = 0.
\]  
(5.26)

Moreover, using again (4.8), (5.13) and (5.15) we obtain

\[
\lim_{\varepsilon \to 0} \left( \int_\Omega f \varphi^2 \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \, dx - 2 \int_\Omega A^0 \nabla u_n \nabla \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \varphi \, dx \right) \\
- \int_\Omega A^0 \nabla u_n \nabla (\xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon})) \varphi^2 \, dx - \int_\Omega \lambda(u_n + v^-_{n,\varepsilon}) \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \varphi^2 \, dx \\
= \int_\Omega f \varphi^2 \xi'_\nu(u_0 - u_n) \, dx - 2 \int_\Omega A^0 \nabla u_n \nabla \xi'_\nu(u_0 - u_n) \varphi \, dx \\
- \int_\Omega A^0 \nabla u_n \nabla \xi'_\nu(u_0 - u_n) \varphi^2 \, dx - \int_\Omega \lambda u_n \xi'_\nu(u_0 - u_n) \varphi^2 \, dx,
\]  
(5.27)

while by Proposition 4.1 and (5.12)

\[
\left| \int_\Omega A^\varepsilon \nabla (u_\varepsilon - v^+_{n,\varepsilon}) \nabla \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \varphi \right| \, dx \leq c \| \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \|_{L^2(\Omega)},
\]

where \( c \) is independent of \( \varepsilon \) and \( n \), which implies that

\[
\limsup_{\varepsilon \to 0} \left( \int_\Omega A^\varepsilon \nabla (u_\varepsilon - v_{n,\varepsilon}) \nabla \xi'_\nu(u_\varepsilon - v^+_{n,\varepsilon}) \varphi \right) \, dx \leq c \| \xi'_\nu(u_0 - u_n) \|_{L^2(\Omega)}.
\]  
(5.28)

Consequently, using (5.24)-(5.28), we can pass to the upper limit in (5.20)
Observe now that from (5.9) the sequence \( \{ \xi_\nu(u_0 - u_n) \} \) is bounded in \( L^\infty(\Omega) \) and (up to a subsequence) converges to 0 a.e. in \( \Omega \). Also, \( \nabla u_n \) converges (up to a subsequence) to \( \nabla u_0 \) a.e. in \( \Omega \) and the integrals \( \int_\Omega |\nabla u_n|^2 \, dx \) are equi-integrable.

Then, since

\[
\int_\Omega A_0 \nabla u_n \nabla u_n |\xi_\nu(u_0 - u_n)| \varphi^2 \, dx \leq c \int_\Omega |\nabla u_n|^2 |\xi_\nu(u_0 - u_n)| \, dx,
\]

from the Vitali Theorem we have

\[
\lim_{n \to \infty} \int_\Omega A_0 \nabla u_n \nabla u_n |\xi_\nu(u_0 - u_n)| \varphi^2 \, dx = 0
\]

so that, passing to the limit in (5.29) as \( n \to \infty \), for fixed \( \mu \) gives

\[
\limsup_{n \to \infty} \limsup_{\epsilon \to 0} \int_\Omega |\nabla (u_\epsilon - v_\nu^n,\epsilon)|^2 \varphi^2 \, dx \leq c \mu^2 \left( e^{\mu (2(1-k))} \right) \frac{\epsilon^2}{\alpha^2}.
\]

By the last inequality and Lemma 5.5 it follows that

\[
\limsup_{n \to \infty} \limsup_{\epsilon \to 0} \int_\Omega |\nabla (u_\epsilon - v_n,\epsilon)|^2 \varphi^2 \, dx \leq c \mu^2 \left( e^{\mu (2(1-k))} \right) \frac{\epsilon^2}{\alpha^2},
\]

for every \( \varphi \) in \( D(\Omega) \).

Now, it is easy to check that taking \( v_\epsilon - v_n,\epsilon \) as test function in (5.1) and (5.10) respectively and substracting, gives

\[
\| \nabla (v_\epsilon - v_n,\epsilon) \|_{L^2(\Omega)} \leq c \| u_0 - u_n \|_{H^1_0(\Omega)},
\]

(5.31)
where $c$ is independent of $\varepsilon$ and $n$.

Hence, using (5.30), (5.31) and (5.9) we get
\[
\limsup_{\varepsilon \to 0} \int_{\Omega} |\nabla (u_\varepsilon - v_\varepsilon)|^2 \varphi^2 \, dx \leq 2 \limsup_{n \to \infty} \limsup_{\varepsilon \to 0} \int_{\Omega} |\nabla (u_\varepsilon - v_{n,\varepsilon})|^2 \varphi^2 \, dx \\
+ 2 \|\varphi\|_{L^\infty(\Omega)}^2 \limsup_{n \to \infty} \limsup_{\varepsilon \to 0} \int_{\Omega} |\nabla (v_\varepsilon - v_{n,\varepsilon})|^2 \, dx \\
\leq 2c\mu^2 \left( e^{\mu^2(1-k)} \right) \frac{\varepsilon^2}{\alpha^2} + c \lim_{n \to \infty} \int_{\Omega} |\nabla (u_0 - u_n)|^2 \, dx = 2c\mu^2 \left( e^{\mu^2(1-k)} \right) \frac{\varepsilon^2}{\alpha^2}.
\]

Since $\mu$ is arbitrary, this gives the claimed result for a subsequence. On the other hand, one can repeat the same proof for any subsequence of the initial one and still obtain (5.3) for a subsequence. This shows that actually convergence (5.3) holds for the whole sequence given by 4.5. \qed

Remark 5.6. We note that the only point in the proof of Theorem 5.1 where the assumption $b_\varepsilon(x, \xi) \geq 0$ is used is in (5.19). Actually, we need it to get rid of the dependence on $n$ (through the function $v_{n,\varepsilon}^+$) in $I_2$. In order to do this, we use the monotonicity of the function $\xi_\nu(s)$, but we are obliged to assume $b_\varepsilon(x, \xi) \geq 0$.

6. Proof of Theorem 3.3

Along this section $\{\varepsilon\}$ denotes, as before, the subsequence for which (4.8) holds true. As already mentioned, Theorem 5.1 and in particular, its immediate consequence stated in Corollary 5.2, plays a central role when proving Theorem 3.3.

We give first some lemmas.

Lemma 6.1. Under the hypotheses of Theorem 3.3, let $u_0$ be given by (4.8). Then,
\[
\frac{|\nabla u_0|^2}{(u_0)^k} \chi_{u_0 > 0} \in L^1(\Omega),
\]
where we define $\frac{|\nabla u_0|^2}{(u_0)^k} = 0$ on the set $\{u_0 = 0\}$. 

Proof. Since
\[
\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^k} = \frac{4}{(2-k)^2}|\nabla (u_1 - k\varepsilon)|^2,
\]
setting \(z_\varepsilon = u_1 - k\varepsilon\), from (4.7) we obtain
\[
\int_\Omega |\nabla z_\varepsilon|^2 \, dx \leq c.
\]
Hence, up to a subsequence, \(z_\varepsilon\) converges weakly in \(H^1_0(\Omega)\) and almost everywhere in \(\Omega\) to some function \(z^0\). By (4.8), \(z_0 = u_0^{1 - \frac{k}{2}}\). Therefore, \(u_0^{1 - \frac{k}{2}}\) belongs to \(H^1_0(\Omega)\) and
\[
\frac{|\nabla u_0|^2}{(u_0)^k} = \frac{4}{(2-k)^2}|\nabla (u_1 - k\varepsilon)|^2
\]
on \(\{u_0 > 0\}\). □

Lemma 6.2. Under the hypotheses of Theorem 3.3, we have
\[
b_\varepsilon(x, \nabla u_\varepsilon) \rightharpoonup b_0(x, \nabla u_0) \quad \text{weakly in } L^1(\Omega),
\]
where \(b_0\) satisfy (3.5).

Proof. In view of Corollary 5.2, the result follows by applying to the function \(H_\varepsilon(x, s, \xi) = b_\varepsilon(x, \xi)\), the same arguments used in Step 2 of the proof of Theorem 3.1 of \cite{8}. □

Lemma 6.3. Under the hypotheses of Theorem 3.3, we have for every \(\mu \in [0, 1]\),
\[
\lim_{\varepsilon \to 0} \int_{\Omega \cap \{u_\varepsilon > \mu\}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi \, dx = \int_{\Omega \cap \{u_0 > \mu\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} \varphi \, dx,
\]
for every \(\varphi \in D(\Omega)\).

Proof. Let us mention first that since \(u_\varepsilon\) converges to \(u_0\) a.e. in \(\Omega\), we have
\[
\chi_{\{u_\varepsilon > \mu\}} \chi_{\{u_0 \neq \mu\}} \to \chi_{\{u_0 > \mu\}} \quad \text{a.e. in } \Omega. \quad (6.1)
\]
Then,
\[
\int_{\Omega \cap \{u_\varepsilon > \mu\}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi \, dx = \int_{\Omega \cap \{u_\varepsilon > \mu, u_0 = \mu\}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi \, dx
\]
\[
+ \int_{\Omega \cap \{u_\varepsilon > \mu, u_0 \neq \mu\}} \frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \varphi \, dx = J_1^\varepsilon + J_2^\varepsilon.
\]
We have, using Lemma 6.2,
\[ |J_1^ε| ≤ \int_Ω \frac{b_ε(x, \nabla u_ε)}{μ^k} \chi_{\{u_0=μ\}} |φ| dx \to \int_Ω \frac{b_0(x, \nabla u_0)}{μ^k} \chi_{\{u_0=μ\}} |φ| dx = 0, \]
since \(b_0(x, 0) = 0\). Concerning \(J_2^ε\), we write
\[ J_2^ε = \int_Ω \frac{b_ε(x, \nabla u_ε)}{(u_ε)^k} \chi_{\{u_0>μ\}} \chi_{\{u_0≠μ\}} φ dx. \]

Since, from (6.1),
\[ t_ε = \frac{1}{(u_ε)^k} \chi_{\{u_0>μ\}} \chi_{\{u_0≠μ\}} \to \frac{1}{(u_0)^k} \chi_{\{u_0>μ\}} \text{ a.e. in } Ω \]
and
\[ \|t_ε\|_{L^∞(Ω)} ≤ \frac{1}{μ^k}, \]
by Lemma 6.2 and Lemma 2.5,
\[ J_2^ε \to \int_{Ω\cap\{u_0>μ\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} φ dx, \text{ for every } φ \in D(Ω). \]

This gives the result. \(\square\)

*Proof of Theorem 3.3.* It remains to prove convergences (3.13) and (3.14), and pass to the limit in (3.9) in order to show that \(u_0\) is solution of (3.15).

Let us prove first that
\[ \frac{b_ε(x, \nabla u_ε)}{(u_ε)^k} \to \frac{b_0(x, \nabla u_0)}{(u_0)^k} \chi_{u_0>0} \text{ in } D'(Ω), \] (6.2)
Let \(φ \in D(Ω)\). Using Lemma 5.3, for \(0 < μ ≤ 1\) we have
\[ \left| \int_Ω \frac{b_ε(x, \nabla u_ε)}{(u_ε)^k} φ dx - \int_Ω \frac{b_0(x, \nabla u_0)}{(u_0)^k} φ \chi_{u_0>0} dx \right| \]
\[ ≤ \left| \int_{Ω∩\{u_0>μ\}} \frac{b_ε(x, \nabla u_ε)}{(u_ε)^k} φ dx - \int_{Ω∩\{u_0>μ\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} φ dx \right| \]
\[ + \left| \int_{Ω∩\{u_0≤μ\}} \frac{b_ε(x, \nabla u_ε)}{(u_ε)^k} φ dx \right| + \left| \int_{Ω∩\{u_0≤μ\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} φ \chi_{u_0>0} dx \right| \] (6.3)
\[ ≤ \left| \int_{Ω∩\{u_0>μ\}} \frac{b_ε(x, \nabla u_ε)}{(u_ε)^k} φ dx - \int_{Ω∩\{u_0>μ\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} φ dx \right| \]
\[ + cμ + \left| \int_{Ω∩\{u_0≤μ\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} φ \chi_{u_0>0} dx \right|. \]
This gives, passing to the upper limit and using Lemma 6.3
\[
0 \leq \lim_{\varepsilon \to 0} \sup \left| \int_{\Omega} \frac{b_{\varepsilon}(x, \nabla u_{\varepsilon})}{(u_{\varepsilon})^k} \varphi \, dx - \int_{\Omega} \frac{b_0(x, \nabla u_0)}{(u_0)^k} \varphi \, u_{\varepsilon} > 0 \, dx \right|
\leq c \mu + \left| \int_{\Omega \cap \{u_0 \leq \mu\}} \frac{b_0(x, \nabla u_0)}{(u_0)^k} \varphi \, u_{\varepsilon} > 0 \, dx \right|.
\]

Passing now to the the limit as \( \mu \) goes to zero we have (3.13), since by Lemma 6.1 the function \( \frac{b_0(x, \nabla u_0)}{(u_0)^k} \chi_{u_0 > 0} \) is in \( L^1(\Omega) \).

To prove (3.14) observe first that by Theorem 5.1, for any open set \( \omega \subset \subset \Omega \) and for any sequence \( \varphi_{\varepsilon} \) weakly converging in \( H^1_0(\omega) \) we have
\[
< - \text{div} (A^\varepsilon (\nabla u_{\varepsilon} - \nabla v_{\varepsilon})), \varphi_{\varepsilon} >_{H^{-1}(\omega), H^1_0(\omega)} = \int_{\Omega} A^\varepsilon (\nabla u_{\varepsilon} - \nabla v_{\varepsilon}) \nabla \varphi_{\varepsilon} \, dx \to 0.
\]

In view of (5.2) i), this implies (3.14), since from (5.1)
\[
- \text{div} (A^\varepsilon (\nabla u_{\varepsilon})) = - \text{div} (A^\varepsilon (\nabla u_{\varepsilon} - \nabla v_{\varepsilon})) - \lambda (u_0 - v_{\varepsilon}) - \text{div} (A^0 \nabla u_0).
\]

Finally, in order to prove that \( u_0 \) satisfies the equation, first, using (3.12), (6.2) and (3.14) we pass to the limit in the equation in (3.9), for \( \varphi \in D(\Omega) \) to obtain
\[
\int_{\Omega} A^0 \nabla u_0 \nabla \Phi \, dx + \lambda \int_{\Omega} u_0 \, \Phi \, dx = \int_{\Omega} \frac{b_0(x, \nabla u_0)}{(u_0)^k} \, \Phi \, u_{\varepsilon} > 0 \, dx + \int_{\Omega} f \, \Phi \, dx.
\]

Then, we use the maximum principle as in Remark 3.2, to deduce that \( u_0 > 0 \). This gives (3.13) and (3.15). □

7. Generalizations and Comments

Let us show that in problem (3.1) the singularity can have at the infinity a more general behaviour than \( \frac{1}{s^k} \).

Let us consider the problem
\[
\begin{cases}
- \text{div} (A^\varepsilon (\nabla u_{\varepsilon})) + \lambda u_{\varepsilon} = b_{\varepsilon}(x, \nabla u_{\varepsilon})g(u_{\varepsilon}) + f(x) & \text{in } \Omega, \\
u_{\varepsilon} = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(7.1)

where \( g \) satisfies
\[
g \in C^0([0, +\infty[), \quad \exists \, s_0, \text{ such that } g(s) = \frac{1}{s^k} \text{ on } [0, s_0], \quad \lim_{s \to +\infty} g(s) = 0.
\]

(7.2)
The existence (in the sense given in Section 3) of a solution of this problem has been proved in [12].

**Theorem 7.1.** Under the assumptions (3.2)-(3.5) and (7.2), let \( u_\varepsilon \) be a solution of (7.1). Then, there exist a subsequence (still denoted by \( \{ \varepsilon \} \)), a matrix \( A^0 \) in \( \mathcal{M}(\alpha, \beta, \Omega) \), a Carathéodory function \( b_0(x, \xi) \) on \( \Omega \times \mathbb{R}^N \) satisfying (3.5) up to a multiplicative constant, and a strictly positive function \( u_0 \) such that one has convergences (3.10)-(3.12), (3.12), together with the following convergence:

\[
\frac{b_\varepsilon(x, \nabla u_\varepsilon)}{(u_\varepsilon)^k} \rightharpoonup b_0(x, \nabla u_0)g(u_0)^k \quad \text{in} \ D'(\Omega),
\]

(7.3)

The function \( u^0 \) is a solution of the following homogenized problem:

\[
\begin{cases}
-\text{div} \left( A^0(\nabla u_0) \right) + \lambda u_0 = b_0(x, \nabla u_0)g(u_0) + f(x) & \text{in } \Omega, \\
u_0 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(7.4)

We prove first the following lemma

**Lemma 7.2.** Suppose that \( g : ]0, +\infty[ \rightarrow \mathbb{R} \) is a nonnegative function such that

\[
g \in C^0(]0, +\infty[), \quad \exists s_0, \text{ such that } g(s) = \frac{1}{s^k} \text{ on } ]0, s_0], \quad g \in L^\infty(]s_0, +\infty[).
\]

(7.5)

Then, there exists \( s_1 < s_0 \) such that

\[
\frac{1}{s^k} = \max_{[s, +\infty[} g, \quad \text{for every } s < s_1.
\]

(7.6)

**Proof.** Since \( \lim_{s \rightarrow +\infty} g(s) = +\infty \), there exists \( s_1 < s_0 \) such that \( g(s) > \max_{]s_0, +\infty[} g \) for every \( s \in ]0, s_1] \). This prove the results, since \( g(s) = \frac{1}{s^k} \) is decreasing on \( ]0, s_0] \). \( \square \)

**Proof of Theorem 7.1.** The proof follows the same outlines of that of Theorem 3.3, replacing \( \frac{1}{s^k} \) by \( g(s) \) and the function \( \gamma \) given in (3.7) by

\[
\gamma(s) = \int_0^s g(t)dt.
\]

We detail here only the points where the proof differs. The first point is (4.3), which is replaced by

\[
\lim_{s \rightarrow +\infty} e^{\gamma(s)} = \lim_{s \rightarrow +\infty} g(s) = 0.
\]
The second one concerns the proofs of (5.19) in Theorem 5.1 and Lemma 6.3, where we use Lemma 7.2 and choose $\mu \in [0, s_1]$.

The last point is to prove, as in Lemma 6.1, that

$$|\nabla u_0|^2 g(u_0) \chi_{u_0 > 0} \in L^1(\Omega).$$

To do that, we write

$$|\nabla u_\varepsilon|^2 g(u_\varepsilon) = |\nabla (G(u_\varepsilon))|^2,$$

where $G(s) = \int_0^s g^{1/2}(t) \, dt$ and set $z_\varepsilon = G(u_\varepsilon)$. Then, we conclude as in the proof of Lemma 6.1, with $z_0 = G(u_0)$. $\square$

**Remark 7.3.** Observe that in the proof of Theorem 7.1, the fact that $\lim_{s \to +\infty} g(s) = 0$ is only needed to prove (4.3). According with 2) of Remark 4.4, if $\lambda > 0$ and $f \in L^\infty(\Omega)$, then in Theorem 7.1 the assumption $\lim_{s \to +\infty} g(s) = 0$ can be replaced by the fact that $g \in L^\infty([s_0, +\infty[)$.

**Remark 7.4.** It could be interesting to extend the homogenization result proved in this paper to problems with different assumptions on the data and other behaviours of the nonlinear term, as for instance

- nonlinear functions $b_\varepsilon(x, \xi)$ of any sign (not necessarily nonnegative),
- a general structure $b_\varepsilon(x, s, \xi)$ for the lower term,
- a datum $f$ of any sign (not necessarily nonnegative),
- a singularity $1/s^k$ with $k \geq 1$,
- a subquadratic growth in the gradient.

**References**


