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ON $(m,(p,q),n)$-QUASI-IDEALS IN TERNARY SEMIRINGS

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Abstract: In this paper, we introduce the notion of $(m,(p,q),n)$-quasi-ideals, m-right ideals, (p,q)-lateral ideals and n-left ideals of ternary semirings and prove some standard results related to the same. We also prove some results on minimal $(m,(p,q),n)$-quasi-ideals of ternary semirings.

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1. Introduction and Preliminaries

O. Steinfeld [1] introduce the notion of quasi-ideal in rings and semigroups. S. Kar [5] introduced the concept of quasi-ideals in ternary semirings and generalise the results in many ways. R. Chinram [2] studied the concept of (m,n)-quasi-ideals in semirings. Throughout this paper S will always denote a ternary semiring with zero. Recall [5] the following:

A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a+b)cd = acd + bcd$,

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(iii) $a(b+c)d = abd + acd$,

(iv) $ab(c+d) = abc + abd$, for all $a, b, c, d, e \in S$.

Definition 1.1. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then "0" is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Definition 1.2. An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$, for all $t_1, t_2, t_3 \in T$.

Definition 1.3. An additive subsemigroup I of S is called a left (right, lateral) ideal of S if s_1s_2i (respectively is_1s_2, s_1is_2) $\in I$ for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, a right and a lateral ideal of S then I is called an ideal of S.

Definition 1.4. An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$.

Definition 1.5. An element a in a ternary semiring S is called regular if there exists an element x in S such that $axa = a$. A ternary semiring is called regular if all of its elements are regular.

2. $(m, (p, q), n)$-Quasi-Ideals

In this section, we introduce the concept of $(m, (p, q), n)$-quasi-ideals in ternary semirings and prove some results related to the same.

Definition 2.1. A sub-semiring Q of a ternary semiring S is called an $(m, (p, q), n)$-quasi-ideal of S if $Q(SS)^m \cap S^p(Q + SQS)S^q \cap (SS)^nQ \subseteq Q$, where m, n, p, q are positive integers greater than 0 and $p + q =$ even.

Remark. Every quasi-ideal of a ternary semiring S is $(1, (1, 1), 1)$-quasi-ideal of S. But $(m, (p, q), n)$-quasi-ideal of a ternary semiring S need not be a quasi-ideal of S.

Example 2.2. Let $Z^- \setminus \{-1\}$ be the set of all negative integers excluding $\{0\}$. Then $Z^- \setminus \{-1\}$ is a ternary semiring with usual binary addition and ternary multiplication. Consider $Q = \{-3\} \cup \{k \in Z^- : k \leq -72\}$. Clearly Q is non-empty subsemiring of S and also Q is $(2, (1, 1), 3)$-quasi-ideal of S. Now, $\{-60\} \in QSS \cap S(Q + SQS)S \cap SSQ$. But $\{-60\} \notin Q$. Therefore $QSS \cap S(Q + SQS)S \cap SSQ \nsubseteq Q$. Hence Q is not quasi-ideal of $Z^- \setminus \{-1\}$.

Lemma 2.3. *Non-empty intersection of arbitrary collection of ternary subsemirings of a ternary semiring S is a ternary subsemiring of S.*

Proof. Let T_i be a ternary subsemiring of S for all $i \in I$ such that $\bigcap_{i\in I} T_i \neq \emptyset$. Let $t_1, t_2, t_3 \in \bigcap_{i\in I} T_i$. Then $t_1, t_2, t_3 \in T_i$ for all $i \in I$. Since T_i is a ternary subsemiring of S for all $i \in I$, therefore $t_1t_2t_3 \in T_i$ for all $i \in I$. Therefore $t_1t_2t_3 \in \bigcap_{i\in I} T_i$. Similarly, $t_1 + t_2 \in \bigcap_{i\in I} T_i$ for all $t_1, t_2 \in \bigcap_{i\in I} T_i$. Hence $\bigcap_{i\in I} T_i$ is a ternary subsemiring of S. □

Theorem 2.4. *Let S be a ternary semiring and Q_i be an $(m,(p,q),n)$-quasi-ideal of S such that $\bigcap_{i\in I} Q_i \neq \emptyset$. Then $\bigcap_{i\in I} Q_i$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Clearly $\bigcap_{i\in I} Q_i$ is a subsemiring of S (by Lemma 2.3). Let

$$x \in \left[\bigcap_{i\in I} Q_i(SS)^m\right] \cap \left[S^p \bigcap_{i\in I} Q_i S^q + S^p S \bigcap_{i\in I} Q_i S S^q\right] \cap \left[(SS)^n \bigcap_{i\in I} Q_i\right].$$

Then $x \in \bigcap_{i\in I} Q_i(SS)^m$, $x \in S^p \bigcap_{i\in I} Q_i S^q + S^p S \bigcap_{i\in I} Q_i S S^q$ and $x \in (SS)^n \bigcap_{i\in I} Q_i$. This implies $x \in Q_i(SS)^m$, $x \in [S^p Q_i S^q + S^p S Q_i S S^q]$ and $x \in (SS)^n Q_i$ for all $i \in I$. Therefore $x \in [Q_i(SS)^m] \cap [S^p Q_i S^q + S^p S Q_i S S^q] \cap [(SS)^n Q_i] \subseteq Q_i$ for all $i \in I$, since Q_i is an $(m,(p,q),n)$-quasi-ideal of S. Thus $x \in Q_i$ for all $i \in I$. Therefore $x \in \bigcap_{i\in I} Q_i$. Hence $\bigcap_{i\in I} Q_i$ is an $(m,(p,q),n)$-quasi-ideal of S. □

Remark. Let Z^- be the set of all negative integers under ternary multiplication and $Q_i = \{k \in Z^- : k \leq -i\}$ for all $i \in I$. Then Q_i is an $(2,(1,1),3)$-quasi-ideal of Z^- for all $i \in I$. But $\bigcap_{i\in I} Q_i = \emptyset$. So condition $\bigcap_{i\in I} Q_i \neq \emptyset$ is necessary.

Definition 2.5. Let S be a ternary semiring. Then a subsemiring

(i) R of S is called an m-right ideal of S if $R(SS)^m \subseteq R$.

(ii) M of S is called an (p,q)-lateral ideal of S if $S^p M S^q + S^p S M S S^q \subseteq M$,

(iii) L of S is called an n-left ideal of S if $(SS)^n L \subseteq L$,

where m, n are positive integers and $p + q$ is an even positive integer.

Theorem 2.6. *Every m-right, (p,q)-lateral and n-left ideal of a ternary semiring S is an $(m,(p,q),n)$-quasi-ideal of S. But converse need not be true.*

Proof. One way is straight forward. Conversely, let $S = M_2(Z_0^-)$ be the ternary semiring of 2×2 square matrices over Z_0^-. Consider

$$Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^- \right\}.$$

Then Q is an $(2,(1,1),3)$-quasi-ideal of S. But it is not 2-right ideal, $(1,1)$-lateral ideal and 3-left ideal of S. □

Theorem 2.7. *Let S be a ternary semiring. Then the following holds:*

(i) *Let R_i be an m-right ideal of S such that $\bigcap_{i \in I} R_i \neq \emptyset$. Then $\bigcap_{i \in I} R_i$ is an m-right ideal of S.*

(ii) *Let M_i be an (p,q)-lateral ideal of S such that $\bigcap_{i \in I} M_i \neq \emptyset$. Then $\bigcap_{i \in I} M_i$ is an (p,q)-lateral ideal of S.*

(iii) *Let L_i be an n-left ideal of S such that $\bigcap_{i \in I} L_i \neq \emptyset$. Then $\bigcap_{i \in I} L_i$ is an n-left ideal of S.*

Proof. Similar to the proof of Theorem 2.4 □

Theorem 2.8. *Let R be an m-right ideal, M be an (p,q)-lateral ideal and L be an n-left ideal of a ternary semiring S. Then $R \cap M \cap L$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Suppose $Q = R \cap M \cap L$. Since every m-right, (p,q)-lateral and n-left ideal of ternary semiring S is an $(m,(p,q),n)$-quasi-ideal of S, therefore R, M and L are $(m,(p,q),n)$-quasi-ideals of S. Clearly, $R \cap M \cap L$ is non-empty. By Theorem 2.4, we have $Q = R \cap M \cap L$ is an $(m,(p,q),n)$-quasi-ideal of S. □

Lemma 2.9. *Let Q be an $(m,(p,q),n)$-quasi-ideal of a ternary semiring S. Then*

(i) *$R = Q + Q(SS)^m$ is an m-right ideal of S.*

(ii) *$M = Q + (S^pQS^q + S^pSQSS^q)$ is an (p,q)-lateral ideal of S.*

(iii) *$L = Q + (SS)^nQ$ is an n-left ideal of S.*

Proof. It is easy to show that R is ternary subsemiring of S. Now to show that R is an m-right ideal of S.

$$R(SS)^m = [(Q + Q(SS)^m](SS)^m$$

$$\begin{aligned}&= Q(SS)^m + Q(SS)^m(SS)^m\\&= Q(SS)^m + Q(SSSS)^m\\&= Q(SS)^m + Q(SS)^m\\&\subseteq Q(SS)^m \subseteq R.\end{aligned}$$

Therefore R is an m-right ideal of S. Similarly, we can show that M is an (p,q)-lateral ideal of S and L is an n-left ideal of S. □

Theorem 2.10. *Every $(m,(p,q),n)$-quasi-ideal in a regular ternary semiring S is the intersection of m-right, (p,q)-lateral and n-left ideal of S.*

Proof. Let S be regular ternary semiring and Q be an $(m,(p,q),n)$-quasi-ideal of S. Then $R = Q + Q(SS)^m$, $M = Q + (S^pQS^q + S^pSQSS^q)$ and $L = Q + (SS)^nQ$ are m-right, (p,q)- lateral and n-left ideal of S respectively. Clearly $Q \subseteq R$, $Q \subseteq M$ and $Q \subseteq L$ implies $Q \subseteq R \cap M \cap L$. Since S is regular therefore $Q \subseteq Q(SS)^m$, $Q \subseteq S^pQS^q + S^pSQSS^q$ and $Q \subseteq (SS)^nQ$.
Thus $R = Q(SS)^m$, $M = S^pQS^q + S^pSQSS^q$ and $L = (SS)^nQ$. Now

$$R \cap M \cap L = Q(SS)^m \cap (S^pQS^q + S^pSQSS^q) \cap (SS)^nQ \subseteq Q$$

Hence, $Q = R \cap M \cap L$. □

3. Minimal $(m,(p,q),n)$-Quasi-Ideals

In this section, we study the concept of minimal $(m,(p,q),n)$-quasi-ideals of ternary semiring S.

An $(m,(p,q),n)$-quasi-ideal Q of a ternary semiring S is called minimal $(m,(p,q),n)$-quasi-ideal of S if Q does not properly contain any $(m,(p,q),n)$-quasi-ideal of S. Similarly, we can define minimal m-right ideals, minimal (p,q)-lateral ideals and minimal n-left ideals of a ternary semiring.

Lemma 3.1. *Let S be a ternary semiring and $a \in S$. Then the following holds:*

(i) *$a(SS)^m$ is an m-right ideal of S.*

(ii) *$(S^paS^q + S^pSaSS^q)$ is an (p,q)-lateral ideal of S.*

(iii) *$(SS)^na$ is an n-left ideal of S.*

(iv) *$a(SS)^m \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. (i), (ii) and (iii) is obvious. (iv) follows from (i), (ii), (iii) and Theorem 2.8. □

Theorem 3.2. *Let S be a ternary semiring and Q an $(m,(p,q),n)$-quasi-ideal of S. Then Q is minimal iff Q is the intersection of some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L of S.*

Proof. Suppose Q is minimal $(m,(p,q),n)$-quasi-ideal of S. Let $a \in Q$. Then by above Lemma, we have $a(SS)^m$ is an m-right ideal, $(S^paS^q+S^pSaSS^q)$ is an (p,q)-lateral ideal, $(SS)^na$ is an n-left ideal and $a(SS)^m \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na$ is an $(m,(p,q),n)$-quasi-ideal of S. Now,

$$\begin{aligned} &a(SS)^m \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na \\ &\quad \subseteq Q(SS)^m \cap (S^pQS^q + S^pSQSS^q) \cap (SS)^nQ \\ &\quad \subseteq Q. \end{aligned}$$

Since Q is minimal therefore $a(SS)^m \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na = Q$. Now, to show that $a(SS)^m$ is minimal m-right ideal of S. Let R be an m-right ideal of S contained in $a(SS)^m$. Then

$$\begin{aligned} &R \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na \\ &\quad \subseteq a(SS)^m \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na \\ &\quad = Q\,. \end{aligned}$$

Since $R \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na$ is an $(m,(p,q),n)$-quasi-ideal of S and Q is minimal, therefore $R \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na = Q$. This implies $Q \subseteq R$ and therefore

$$a(SS)^m \subseteq Q(SS)^m \subseteq R(SS)^m \subseteq R$$

implies $R = a(SS)^m$. Thus m-right ideal $a(SS)^m$ is minimal. Similarly, we can prove that $(S^paS^q + S^pSaSS^q)$ is minimal (p,q)-lateral ideal of S and $(SS)^na$ is minimal n-left ideal of S.

Conversely, assume that $Q = R \cap M \cap L$ for some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L. So, $Q \subseteq R, Q \subseteq M$ and $Q \subseteq L$. Let Q' be an $(m,(p,q),n)$-quasi-ideal of S contained in Q. Then $Q'(SS)^m \subseteq Q(SS)^m \subseteq R(SS)^m \subseteq R$. Similarly, $(S^pQ'S^q + S^pSQ'SS^q) \subseteq M$ and $(SS)^nQ' \subseteq (SS)^nQ \subseteq L$.

Now $Q'(SS)^m$ is an m-right ideal of S, as $Q'(SS)^m(SS)^m \subseteq Q'(SS)^m$. Similarly, $(S^pQ'S^q + S^pSQ'SS^q)$ is an (p,q)-lateral ideal of S and $(SS)^nQ'$ is an

n-left ideal of S. Since R, M and L are minimal m-right ideal, minimal (p,q)-lateral ideal and minimal n-left ideal of S respectively, therefore $Q'(SS)^m = R, S^pQ'S^q + S^pSQ'SS^q = M$ and $(SS)^nQ' = L$.
Thus $Q = R \cap M \cap L = Q'(SS)^m \cap (S^pQ'S^q + S^pSQ'SS^q) \cap (SS)^nQ' \subseteq Q'$.
Hence $Q = Q'$. Thus Q is minimal $(m,(p,q),n)$-quasi-ideal of S. □

Note. A ternary semiring S need not contains a minimal $(m,(p,q),n)$-quasi-ideal of S.
For example, let Z^- be the set of all negative integers. Then Z^- is a ternary semiring with usual binary addition and ternary multiplication. Let $Q = \{-1,-2,-3,-4,\ldots\}$. Then Q is an $(2,(1,1),3)$-quasi-ideal of Z^-. Suppose Q is minimal $(2,(1,1),3)$-quasi-ideal of Z^-. Let $Q' = Q \setminus \{-1\}$. Then we can easily show that Q' is an $(2,(1,1),3)$-quasi-ideal of Z^-. But Q' is proper subset of Q. This is contradiction. Hence, Z^- does not contain a minimal $(m,(p,q),n)$-quasi-ideal.

Theorem 3.3. *Let S be a ternary semiring. Then the following holds:*

(i) *An m-right ideal R is minimal iff $a(SS)^m = R$ for all $a \in R$.*
(ii) *An (p,q)-lateral ideal M is minimal iff $(S^paS^q + S^pSaSS^q) = M$ for all $a \in M$.*
(iii) *An n-left ideal L is minimal iff $(SS)^na = L$ for all $a \in L$.*
(iv) *An $(m,(p,q),n)$-quasi-ideal Q is minimal iff $a(SS)^m \cap (S^paS^q + S^pSaSS^q) \cap (SS)^na = Q$ for all $a \in Q$.*

Proof. (i) Suppose m-right ideal R is minimal. Let $a \in R$. Then $a(SS)^m \subseteq R(SS)^m \subseteq R$. By Lemma 3.1, we have $a(SS)^m$ is an m-right ideal of S. Since R is minimal m-right ideal of S therefore $a(SS)^m = R$.
Conversely, Suppose that $a(SS)^m = R$ for all $a \in R$. Let R' be an m-right ideal of S contained in R. Let $x \in R'$. Then $x \in R$. By assumption, we have $x(SS)^m = R$ for all $x \in R$. $R = x(SS)^m \subseteq R'(SS)^m \subseteq R'$. This implies $R \subseteq R'$. Thus, $R = R'$.
Hence, R is minimal m-right ideal. (ii), (iii) and (iv) can be proved similarly to (i). □

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References

[1] O. Steinfeld, *Quasi-Ideals in Rings and Semigroups*, Akademiai Kiado, Budapest (1978).

[2] R. Chinram, A note on (m, n) Quasi-ideals in semirings, *International Journal of Pure and Applied Mathematics*, **49**, No. 1 (2008), 45-50.

[3] R. Chinram, Generalized quasi-ideals of semigroups, *KKU Sci. J.*, **37**, No. 2 (2009), 213-210.

[4] T.K. Dutta, S. Kar, On regular ternary semirings, In: *Advances in Algebra, Proceedings of the ICM Satellite conference in Algebra and Related Topics*, World Scientific, New Jersey (2003), 343-355.

[5] S. Kar, On quasi-ideal and bi-ideals in ternary semirings, *Int. J. Math. & Math. Sci.*, **18** (2005), 3015-3023.

[6] V.N. Dixit, S. Dewan, A note on quasi and bi-ideals in ternary semigroups, *Int. J. Math. & Math. Sci.* **18**, No. 3 (1995), 501-508.