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ON RADIUS PROBLEMS IN THE CLASS OF UNIVALENT FUNCTIONS

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Abstract: Let $SL^*(\beta)$ denote the class of all analytic functions f in the unit disc $\mathbb{U}$ with the normalization $f(0) = f'(0)-1 = 0$, and satisfying the condition

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - (1-\beta)\right| < 1-\beta \quad ,(z \in \mathbb{U})\}.$$

Thus, $\frac{zf'(z)}{f(z)}$ is the interior of the right half of the lemniscate of Bernoulli γ : $(x^2+y^2)^2 - 2(1-\beta)(x^2-y^2) = 0$. The radii of $\beta-$convexity, $\beta-$starlikeness (and some of others) for $f \in SL^*(\beta)$ are determined.

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1. Introduction

Let H denote the class of analytic functions in the unit disc $\mathbb{U}$ on the complex plane $\mathbb{C}$. Let A denote the subclass of H consisting of functions normalized by $f(0) = f'(0) - 1 = 0$. The set of all functions $f \in A$ that are starlike univalent in $\mathbb{U}$ will be denoted by S^*. The set of all functions $f \in A$ that are convex univalent in $\mathbb{U}$ will be denoted by C^*.

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Robertson [1] introduced the classes $S^*(\beta)$ and $C^*(\beta)$ of starlike and convex functions of order $\beta \leq 1$, which is defined by

$$S^*(\beta) = \left\{ f \in A : \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \qquad ,(z \in \mathbb{U}) \right\}, \tag{1.1}$$

and

$$C^*(\beta) = \left\{ f \in A : \Re\left\{\frac{zf''(z)}{f(z)} + 1\right\} > \beta \quad , \quad (z \in \mathbb{U}) \right\}. \tag{1.2}$$

If $(0 \leq \beta < 1)$, then a function in either of these sets is univalent. If $\beta < 0$ it may fail to be univalent.

If f and g are analytic functions in $\mathbb{U}$. Then the function f is said to be subordinate to g, and can be written as

$$f \prec g \quad \text{and} \quad f(z) \prec g(z), \quad (z \in \mathbb{U}),$$

if and only if there exists the Schwarz function w, analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. Furthermore, if g is univalent in $\mathbb{U}$ we have the following equivalence

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } \ f(u) \subseteq g(u) \quad .$$

Let us consider the class $SL^*(\beta)$ as follows:

$$SL^*(\beta) = \{f \in A : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - (1-\beta)\right| < 1 - \beta \quad , (z \in \mathbb{U})\}.$$

In this way, many interesting classes of analytic functions can be defined (see for instance [2]).

It is easy to see that $f \in SL^*(\beta)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec q_0(z) = \sqrt{(1-\beta)(1+z)}, \quad q_0(0) = 1 - \beta.$$

Notice that $L := \left\{w \in \mathbb{C}, \Re(w) > 0 : |w^2 - (1-\beta)| < 1 - \beta\right\}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma : (x^2+y^2)^2 - 2(1-\beta)(x^2-y^2) = 0$. It can be verified that $L \subset \left\{w : \left|w^2 - \sqrt{2}/2\right| < \sqrt{2}/2\right\} for (M \geq \sqrt{2}/2)$. Thus we have $SL^*(\beta) \subset S^*_M$ where

$$S^*_M := \left\{ f \in A : \left|\frac{zf'(z)}{f(z)} - M\right| < M \right\}, \quad M > 1/2. \tag{1.3}$$

Note that the class S^*_M has been investigated in [3].

Moreover, it is easy to see $L \subset \left\{w : |Argw| < \frac{\beta\pi}{2}\right\}$. Thus $SL^*(\beta) \subset SS^*(\beta) \subset S^*$, where $SS^*(\beta)$ denotes the class of strongly starlike functions of order β.

$$SS^*(\beta) = \left\{f \in A : \left|Arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2}\right\}, \qquad 0 < \beta \leq 1, \tag{1.4}$$

which was introduced in [4].

Also, $k - ST \subset SL^*$ for $k = 2 + \sqrt{2}$, where $k - ST$ is the class of k-starlike functions introduced in [5].

$$k - ST := \left\{f \in A : \Re\left[\frac{zf'(z)}{f(z)}\right] > k\left|\frac{zf'(z)}{f(z)} - 1\right|\right\}, \qquad k \geq 0. \tag{1.5}$$

Theorem 1.1. *The function f belongs to the class $SL^*(\beta)$ if and only if there exists an analytic function $q \in H, q(0) = 0, q(z) \prec q_0(z) = \sqrt{(1-\beta)(1+z)}$, $q_0(0) = 1 - \beta$ such that*

$$f(z) = z\exp\int_0^z \frac{q(t) - 1}{t}\, dt. \tag{1.6}$$

Let $f \in SL^*(\beta)$ and let $q(z) := \frac{zf'(z)}{f(z)}$. Then $q(\mathbb{U}) \subseteq L$ and by integrating this equation we obtain (1.6). If f is given by (1.6) with an analytic $q, q(0) = 0$, then $q \prec q_0(z) = \sqrt{(1-\beta)(1+z)}$.

Now, logarithmic differentiating (1.6) we obtain $q(z) := \frac{zf'(z)}{f(z)}$. Therefore, $\frac{zf'(z)}{f(z)} \prec q_0(z) = \sqrt{(1-\beta)(1+z)}$.

Set

$$q_1 = \frac{3+2z}{3+z}, \quad q_2 = \frac{5+3z}{5+z}, \quad q_3 = \frac{8+4z}{8+z}, \quad q_4 = \frac{9+5z}{9+z},$$

and since $q_i \prec q_0$ for $i = 1, 2, 3, 4$, then by (1.6) the functions

$$f_1(z) = z + \frac{z^2}{3}, \quad f_2(z) = z(1+\frac{z}{5})^2, \quad f_3(z) = z(1+\frac{z}{8})^3, \quad f_4(z) = z(1+\frac{z}{9})^4$$

are in $SL^*(\beta)$.

Making use of the definitions of (1.1) and (1.2), we say that a function $f \in A$ is said to be $\beta - convex$ in $|z| < r$, whenever $\Re\left\{\frac{zf''(z)}{f(z)} + 1\right\} > \beta$ for $|z| < r$. And is said to be β-starlike in $|z| < r$, whenever $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$ for $|z| < r$. In the same way, making use of the definitions of (1.3)-(1.5), a function

$f \in A$ is said to be M-starlike in $|z| < r$, or is said to be strongly starlike of order b in $|z| < r$, or is said k-starlike in $|z| < r$, whenever, $\left|\frac{zf'(z)}{f(z)} - M\right| < M$ or $\left|Arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2}$ or $\Re\left[\frac{zf'(z)}{f(z)}\right] > k\left|\frac{zf'(z)}{f(z)} - 1\right|$, respectively, in $|z| < r$.

Theorem 1.2. *If the function f belongs to the class $SL^*(\beta)$, then:*

(1) f is $\beta - convex$ in $|z| < r(\beta)$ where $r(\beta)$ denote the smallest positive root of the equation

$$4(1-\beta)^3(1-r)^3 = r^2+4\beta^2(1-\beta)^2(1-r)^2+4r\beta(1-\beta)(1-r), \quad \beta \in (0,1). \tag{1.7}$$

(2) f is β-starlike in $|z| < 1 - \frac{\beta^2}{1-\beta}$, $\quad \beta \in (0,1)$.

(3) f is M-starlike in $|z| < r(M)$, where

$$r(M) = \begin{cases} \frac{4M^2}{1-\beta} - 1 & for \quad \frac{1}{2} \le M \le \frac{\sqrt{2}}{2}, \\ 1 & for \quad M \ge \frac{\sqrt{2}}{2}. \end{cases} \tag{1.8}$$

(4) f is strongly starlike of order β in $|z| < \sin\beta\pi$, where $0 < \beta \le 1$.

(5) f is k-starlike in $|z| < r(k), k \ge 0$, where

$$r(k) = 1 - \frac{k^2}{(1-\beta)(k+1)^2}. \tag{1.9}$$

All of these inequalities are sharp.

Proof. Let $f \in SL^*(\beta)$ then by (1.6) we obtain $\frac{zf'(z)}{f(z)} \prec \sqrt{(1-\beta)(1+z)}$, so $\frac{zf'(z)}{f(z)} = \sqrt{(1-\beta)(1+\omega(z))}$, where ω satisfies $\omega(0) = 0, |\omega(z)| < 1$ for $|z| < 1$.

By the Schwarz lemma, ω satisfies $|\omega(re^{i\theta})| < r$. The proof falls naturally into five parts.

Part 1. Let us recall that (see [2], vol. 2, p. 77)

$$|\omega'(z)| \le \frac{1 - |\omega(z)|^2}{1 - |z|^2}. \tag{1.10}$$

Differentiating $\frac{zf'(z)}{f(z)} = \sqrt{(1-\beta)(1+\omega(z))}$, and using (1.10) we obtain

$$\begin{aligned} \Re\left\{\frac{zf''(z)}{f(z)} + 1\right\} =& \Re\left\{\sqrt{(1-\beta)(1+\omega(z))} + \frac{z\omega'(z)}{2(1-\beta)(1+\omega(z))}\right\} \\ \ge& \Re\left\{\sqrt{(1-\beta)(1+\omega(z))}\right\} - \left|\frac{z}{2}\right| \frac{1 - |\omega(z)|^2}{(1-\beta)(1-\omega(z))(1-|z|^2)} \end{aligned}$$

$$\geq \Re\left\{\sqrt{(1-\beta)(1+\omega(z))}\right\} - \left|\frac{z}{2}\right| \frac{1+|z|}{(1-\beta)(1-|z|^2)}$$
$$= \Re\left\{\sqrt{(1-\beta)(1+\omega(z))}\right\} - \frac{|z|}{2(1-\beta)(1-|z|)}$$
$$\geq \sqrt{(1-\beta)(1-|z|)} - \frac{|z|}{2(1-\beta)(1-|z|)}$$
$$= \sqrt{(1-\beta)(1-r)} - \frac{r}{2(1-\beta)(1-r)}.$$

The function $h(r) = \sqrt{(1-\beta)(1-r)} - \frac{r}{2(1-\beta)(1-r)}$ is decreasing in $[0,1)$ and $h(0) = \sqrt{1-\beta}$. The equation $h(r) = \beta$ is equivalent to (1.7). Therefore it has the smallest positive root $r(\beta)$ in $[0,1)$ and f is $\beta-$convex in $|z| < r \leq r(\beta)$.

Part 2. If $f \in SL^*(\beta)$, then by the Schwarz lemma ω satisfies $|\omega(z)| < |z|$. So for $|z| < 1 - \frac{\beta^2}{1-\beta}$, we have

$$\Re\sqrt{(1-\beta)(1+\omega(z))} > \sqrt{(1-\beta)(1-|z|)} = \sqrt{(1-\beta)\left(1-(1-\frac{\beta^2}{1-\beta})\right)} = \beta.$$

Therefore, f is β-starlike in $|z| < 1 - \frac{\beta^2}{1-\beta}$.

Part 3. If $f \in SL^*(\beta)$, then it is M-starlike in $|z| < r$ whenever

$$\left|\sqrt{(1-\beta)(1+\omega(re^{i\theta}))} - M\right| < M, \quad (0 \leq \theta < 2\pi). \tag{1.11}$$

A simple geometric observation gives that $\sqrt{(1-\beta)(1+r)} < 2M$ is sufficient for (1.11). From this we obtain the inequality (1.8).

Part 4. If $f \in SL^*(\beta)$, then it is strongly starlike of order β in $|z| < r$, whenever,

$$\left|Arg(1-\beta)(1+\omega(re^{i\theta}))\right| < \beta\pi/2, \quad (0 \leq \theta < 2\pi),$$

or equivalently

$$\left|Arg\sqrt{(1-\beta)(1+\omega(re^{i\theta}))}\right| < \beta\pi, \quad (0 \leq \theta < 2\pi). \tag{1.12}$$

A simple geometric observation gives that $|z| < \sin\beta\pi$, is sufficient for (1.12).

Part 5. If $f \in SL^*(\beta)$, then it is k-starlike in $|z| < r$ whenever

$$\Re\sqrt{(1-\beta)(1+\omega(z))} > k\left|\sqrt{(1-\beta)(1+\omega(z))} - 1\right|. \qquad (1.13)$$

By the Schwarz lemma, $|\omega(z)| < |z|$ and we get

$$\Re\sqrt{(1-\beta)(1+\omega(z))} \geq \sqrt{(1-\beta)(1-r))}$$

$$\left|\sqrt{(1-\beta)(1+\omega(z))} - 1\right| \leq 1 - \sqrt{(1-\beta)(1-r))}.$$

Thus the inequality

$$\sqrt{(1-\beta)(1-r))} > k(1 - \sqrt{(1-\beta)(1-r))}), \qquad (1.14)$$

follows the inequality (1.13). The condition (1.14) is equivalent to $r < r(k)$ where $r(k)$ is given in (1.9).

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