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A STUDY OF DIFFERENTIAL OPERATORS FOR PARTICULAR COMPLETE ORTHONORMAL SYSTEMS ON A 3D BALL

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Abstract: In this article, we introduce a class of differential operators for two complete orthonormal systems in $\mathrm{L}^2(\mathcal{B})$, where $\mathcal{B}$ is a ball in $\mathbb{R}^3$, such that these orthonormal systems are eigenfunctions. We study further properties of these operators. It turns out, for instance, that the Sobolev norm, which is used in geomathematics for a spline interpolation and approximation method on $\mathcal{B}$, can be interpreted as the $\mathrm{L}^2(\mathcal{B})$-norm of the image of a (pseudo-)differential operator. This result justifies an analogy of the splines on the ball to their counterparts on the real line and the sphere.

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1. Introduction

The approximation of functions on a 3d-ball has a series of important applications at present. In particular, such methods play an essential role in tomography problems of geophysics and medical imaging, see for example the survey article [27]. Since the structures to be determined usually consist of layers with (almost) spherical boundaries, the use of Euclidean approximation

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methods (restricted from $\mathbb{R}^3$ to the ball) are inappropriate. Therefore, functions based on a radial-angular decomposition are more useful. Classical approximation methods use a truncated singular value decomposition which requires an orthonormal basis for the relevant function space (e.g. $\mathrm{L}^2(\mathcal{B})$, where $\mathcal{B}$ is a ball with center at 0 and radius $R > 0$). Typically, orthogonal/orthonormal polynomials are chosen as such a basis. For intervals on the real line, such polynomials are well known and have already been investigated in detail for many decades (see for example, [31]). For instance, the Legendre polynomials represent a famous system of orthogonal polynomials on $[-1, 1]$. Moreover, for the unit sphere Ω in $\mathbb{R}^3$, the system of spherical harmonics $\{Y_{n,j}\}_{n=0,1,...;j=-n,...n}$ is also an established function system which is widely used and has many known properties (see, for instance, [16], [18], and [30]). Moreover, general theoretical investigations on orthogonal polynomials in several variables including the consideration of domains such as a d-dimensional ball are available, see, for example, [11]. For geophysical applications, such as tomographic problems on the ball $\mathcal{B}$, the following two orthonormal basis systems have, in particular, been used up to now:

$$G^{\mathrm{I}}_{m,n,j}(x) := \sqrt{\frac{4m+2n+3}{R^3}} P_m^{(0,n+1/2)}\left(2\frac{|x|^2}{R^2} - 1\right)\left(\frac{|x|}{R}\right)^n Y_{n,j}\left(\frac{x}{|x|}\right), \tag{1}$$

$x \in \mathcal{B}, m, n \in \mathbb{N}_0;\ j = -n, ..., n$ (see [6], [10], [17], and [22]) as well as

$$G^{\mathrm{II}}_{m,n,j}(x) := \sqrt{\frac{2m+3}{R^3}} P_m^{(0,2)}\left(2\frac{|x|}{R} - 1\right) Y_{n,j}\left(\frac{x}{|x|}\right), \tag{2}$$

$x \in \mathcal{B} \setminus \{0\}, m, n \in \mathbb{N}_0;\ j = -n, ..., n$ (see [32]). Here, $P_m^{(a,b)}$ is a Jacobi polynomial of degree m corresponding to the parameters (a, b). Note that the system of type II is not an algebraic polynomial in x_1, x_2, x_3 and is discontinuous in $x = 0$ for $n > 0$. Both systems have their advantages and disadvantages as discussed in [27]. Furthermore, it should be noted that type I represents a particular case of a basis discussed in [11]. Based on the systems, advanced approximation methods such as spline and wavelet methods were developed in view of different applications, which are inverse gravimetry (see [12], [21], [22], [23], [24], [26], [28], and [29]), travel-time tomography (see [4] and [5]), normal mode tomography (in combination with inverse gravimetry) (see [7], [8], and [9]), MEG-EEG tomography (see [13]) and seismic wave interpolation (see [20] and [25]), where in the latter two cases different orthonormal bases were used. Further, locally supported approximating structures on the unit ball were introduced in [1], [2], and [3].

The mentioned spline method justifies its name due to an analogy to the spherical spline method developed in [14], [15], and [16], where a particular Sobolev norm is minimized by the interpolating spline. In the spherical case, it is known that this Sobolev norm is representable as the L^2-norm of a certain (pseudo-)derivative of the interpolating function. This analogy to the corresponding minimum property (Holladay's Theorem) of natural cubic splines on the real line is one justification of the name "spline". A corresponding interpretation of the Sobolev norm used for the spline method on the ball $\mathcal{B}$ has not been known, yet (note that we restrict our attention here only to the concept of Sobolev spaces used for the spline method).

In this paper, we show that a corresponding result can be proved for the ball, too. For this purpose, we first show in Section 2 that the basis functions (1) and (2) can be represented as eigenfunctions of particular differential operators (note that, in [11], there also exists a differential operator in Cartesian coordinates for type I, where we maintain the use of polar coordinates here). These differential operators provide us with the possibility to construct further (pseudo-)differential operators in Section 3, for which we show in Section 4 that they are associated to Sobolev norms on the ball $\mathcal{B}$. This, finally, yields a previously missing link in the construction of the spline method on the ball out of the spline method on the sphere. The new result can also provide a basis for the derivation of further properties of the discussed orthogonal basis functions in the future.

2. Derivation of Differential Operators for the Orthonormal Systems of Type I and Type II

Jacobi polynomials in general and Legendre polynomials in particular are known to be eigenfunctions of associated univariate differential operators (see, for example, [31]). Analogously, the spherical harmonics are eigenfunctions of the Beltrami operator, which is a spherical differential operator (see, for instance, [16]). This property already proved to be helpful to derive further properties of Jacobi polynomials and spherical harmonics, respectively. In this section, we will determine certain three-dimensional differential operators for which type I and type II, respectively, are eigenfunctions. The differential operators themselves shall be examined further.

First we find the differential operator for which the system of type I is an eigenfunction. The differential equation of the Jacobi polynomials $y = P_m^{(a,b)}$ is given by (see [31])

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_m^{(a,b)}(x)+(b-a-(a+b+2)x)\,\frac{\mathrm{d}}{\mathrm{d}x}P_m^{(a,b)}(x) \\ +m(m+a+b+1)P_m^{(a,b)}(x)=0. \tag{3}$$

Putting $a=0, b=n+\frac{1}{2}, x=u$, in the equation given above, we have

$$\begin{aligned}\left(1-u^2\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}P_m^{(0,n+\frac{1}{2})}(u)+\left(n+\frac{1}{2}-\left(n+\frac{5}{2}\right)u\right)\frac{\mathrm{d}}{\mathrm{d}u}P_m^{(0,n+\frac{1}{2})}(u)& \\ +m\left(m+n+\frac{3}{2}\right)P_m^{(0,n+\frac{1}{2})}(u) &= 0.\end{aligned} \tag{4}$$

Substituting $u=\frac{2r^2}{R^2}-1$, i.e. $r=R\sqrt{\frac{u+1}{2}}$ where $r\in[0,R]$, and using the chain rule for differentiation, equation (4) becomes

$$\begin{aligned}\left(R^2-r^2\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)& \\ +2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)& \\ +4m\left(m+n+\frac{3}{2}\right)P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right) &= 0.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\left[(R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\right. \\ &\left.+2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right) \\ &= -4m\left(m+n+\frac{3}{2}\right)\ P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right).\end{aligned} \tag{5}$$

Now, letting

$$Y(r):=P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n$$

and

$$g(r):=P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right),$$

we get

$$Y=g\frac{r^n}{R^n}. \tag{6}$$

Differentiating with respect to r, we have

$$Y' = g'\frac{r^n}{R^n} + n\frac{r^{n-1}}{R^n}g = g'\frac{r^n}{R^n} + \frac{n}{r}Y. \tag{7}$$

Again differentiating equation (7) with respect to r, we have,

$$Y'' = g''\frac{r^n}{R^n} + 2n\frac{r^{n-1}}{R^n}g' + n(n-1)\frac{r^{n-2}}{R^n}g.$$

Using equations (6) and (7), we reformulate the equation given above as

$$\begin{aligned} Y'' &= g''\frac{r^n}{R^n} + \frac{2n}{r}\left(Y' - \frac{n}{r}Y\right) + \frac{n(n-1)}{r^2}Y \\ &= g''\frac{r^n}{R^n} + \frac{2n}{r}Y' - \frac{n(n+1)}{r^2}Y. \end{aligned}$$

Now multiplying the equation given above by $R^2 - r^2$, we have

$$(R^2 - r^2)Y'' = (R^2 - r^2)g''\frac{r^n}{R^n} + (R^2 - r^2)\frac{2n}{r}Y' - (R^2 - r^2)\frac{n(n+1)}{r^2}Y. \tag{8}$$

Putting the value of $(R^2 - r^2)g''$ taken from (5) in (8) we get

$$\begin{aligned} (R^2 - r^2)Y'' &= -2\left(n\left(1 - \frac{r^2}{R^2}\right) + 1 - \frac{2r^2}{R^2}\right)g'\frac{R^2}{r}\frac{r^n}{R^n} \\ &\quad -4m\left(m + n + \frac{3}{2}\right)g\frac{r^n}{R^n} \\ &\quad +(R^2 - r^2)\frac{2n}{r}Y' - (R^2 - r^2)\frac{n(n+1)}{r^2}Y. \end{aligned}$$

Now using (6) and (7) in the equation above, we have,

$$\begin{aligned} (R^2 - r^2)Y'' &= -2\left(n\left(1 - \frac{r^2}{R^2}\right) + 1 - \frac{2r^2}{R^2}\right)\frac{R^2}{r}\left(Y' - \frac{n}{r}Y\right) \\ &\quad - 4m\left(m + n + \frac{3}{2}\right)Y + (R^2 - r^2)\frac{2n}{r}Y' \\ &\quad - (R^2 - r^2)\frac{n(n+1)}{r^2}Y \\ &= -2\left(n\left(1 - \frac{r^2}{R^2}\right) + 1 - \frac{2r^2}{R^2}\right)\frac{R^2}{r}Y' \\ &\quad + 2\left(n\left(1 - \frac{r^2}{R^2}\right) + 1 - \frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{n}{r}Y \end{aligned}$$

$$
\begin{aligned}
& - 4m\left(m+n+\frac{3}{2}\right)Y + (R^2-r^2)\frac{2n}{r}Y' \\
& - (R^2-r^2)\frac{n(n+1)}{r^2}Y \\
= \quad & \left(-\frac{2R^2}{r}+4r\right)Y' \\
& + \left(\frac{n^2R^2}{r^2}+\frac{nR^2}{r^2}-n^2-3n-4m\left(m+n+\frac{3}{2}\right)\right)Y \\
= \quad & -2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}Y' \\
& + \left(n(n+1)\frac{R^2}{r^2}-n(n+3)-4m\left(m+n+\frac{3}{2}\right)\right)Y.
\end{aligned}
$$

This gives

$$
\begin{aligned}
& (R^2-r^2)Y'' + 2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}Y' - n(n+1)\frac{R^2}{r^2}Y \\
= \quad & -\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)Y.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\left((R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - n(n+1)\frac{R^2}{r^2}\right)Y \\
&\qquad\qquad = -\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)Y. \qquad (9)
\end{aligned}
$$

This shows that

$$
D_r^{\mathrm{I}} := (R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - n(n+1)\frac{R^2}{r^2}
$$

is a differential operator with the eigenfunction

$$
Y(r) = P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n, \quad r=|x|,\, r\in[0,R],
$$

where $-(n(n+3)+4m(m+n+\frac{3}{2}))$ is the corresponding eigenvalue.

Let us define another operator ${}^*\Delta^{\mathrm{I}}$ by

$$
{}^*\Delta_x^{\mathrm{I}} := D_{|x|}^{\mathrm{I}} \circ \Delta^*_{\frac{x}{|x|}},
$$

where Δ^* is the Beltrami operator, for which the spherical harmonics Y_n of degree n are the eigenfunctions corresponding to the eigenvalue $-n(n+1)$ (for further details see [16]), i.e.

$$\Delta^* Y_{n,j} = -n(n+1)Y_{n,j}.$$

Now, it is easy to show that ${}^*\Delta^{\mathrm{I}}$ is a differential operator for which the basis functions $G^{\mathrm{I}}_{m,n,j}$ are eigenfunctions and the corresponding eigenvalues are

$$n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right).$$

Theorem 1. *The basis function $G^{\mathrm{I}}_{m,n,j}$ is an eigenfunction of the differential operator ${}^*\Delta^{\mathrm{I}}$ defined above corresponding to the eigenvalue*

$$n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right),$$

i.e.,

$$\begin{aligned}
{}^*\Delta^{\mathrm{I}}_x\left(G^{\mathrm{I}}_{m,n,j}(x)\right) &= \left(D^{\mathrm{I}}_{|x|}\circ\Delta^*_{\frac{x}{|x|}}\right)\left(G^{\mathrm{I}}_{m,n,j}(x)\right)\\
&= D^{\mathrm{I}}_{|x|}\circ\Delta^*_{\frac{x}{|x|}}\left[\sqrt{\frac{4m+2n+3}{R^3}}P_m^{(0,n+\frac{1}{2})}\left(\frac{2|x|^2}{R^2}-1\right)\right.\\
&\qquad\left.\times\left(\frac{|x|}{R}\right)^n Y_{n,j}\left(\frac{x}{|x|}\right)\right]\\
&= n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)\left(G^{\mathrm{I}}_{m,n,j}(x)\right),
\end{aligned}$$

$x\in\mathcal{B}$.

Note that the differential operator depends on the degree of the used spherical harmonic, such that a reference to n would actually be necessary in the notation. We omit this, however, for the sake of a better readability. We, therefore, do not have one fixed differential operator for which all basis functions are eigenfunctions.

Now, we derive the differential operator for the system of type II. If we put $a=0$, $b=2$ and $x=u$ in equation (3), we get a differential equation for the Jacobi polynomials $P_m^{(0,2)}$ as follows

$$\left((1-u^2)\frac{\mathrm{d}^2}{\mathrm{d}u^2}+(2-4u)\frac{\mathrm{d}}{\mathrm{d}u}\right)P_m^{(0,2)}(u)+m(m+3)P_m^{(0,2)}(u)=0. \tag{10}$$

Now substituting $u = \frac{2r}{R} - 1$, i.e., $r = \frac{R(u+1)}{2}$ in (10) and using the chain rule we get

$$\begin{aligned}\left(rR\left(1-\frac{r}{R}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (3R-4r)\frac{\mathrm{d}}{\mathrm{d}r}\right)P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\\ = \quad -m(m+3)P_m^{(0,2)}\left(\frac{2r}{R}-1\right).\end{aligned}$$

This implies that $rR\left(1-\frac{r}{R}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+(3R-4r)\frac{\mathrm{d}}{\mathrm{d}r}$ is a differential operator for which $P_m^{(0,2)}\left(\frac{2r}{R}-1\right)$ is an eigenfunction with the corresponding eigenvalue $-m(m+3)$, where $r \in [0, R]$.

Let us denote the differential operator of $P_m^{(0,2)}\left(\frac{2r}{R}-1\right)$ by D_r^{II}, i.e.,

$$D_r^{\mathrm{II}} := rR\left(1-\frac{r}{R}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (3R-4r)\frac{\mathrm{d}}{\mathrm{d}r}.$$

Now, we define another operator by ${}^*\Delta_x^{\mathrm{II}} := D_{|x|}^{\mathrm{II}} \circ \Delta^*_{\frac{x}{|x|}}$. Also here it is easy to show that ${}^*\Delta^{\mathrm{II}}$ is a differential operator for which $G_{m,n,j}^{\mathrm{II}}$ is an eigenfunction and the corresponding eigenvalue is

$$n(n+1)m(m+3).$$

Theorem 2. *The basis functions $G_{m,n,j}^{\mathrm{II}}$ are eigenfunctions of the differential operator ${}^*\Delta^{\mathrm{II}}$, where $G_{m,n,j}^{\mathrm{II}}$ corresponds to the eigenvalue $m(m+3)n(n+1)$, i.e.*

$${}^*\Delta_x^{\mathrm{II}}\left(G_{m,n,j}^{\mathrm{II}}(x)\right) = m(m+3)n(n+1)G_{m,n,j}^{\mathrm{II}}(x), \quad x \in \mathcal{B}.$$

3. Some Properties of the Differential Operators ${}^*\Delta_x^{\mathrm{I}}$ and ${}^*\Delta_x^{\mathrm{II}}$

We denote the eigenvalues of $P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n$ corresponding to D_r^{I} by

$$D^{\mathrm{I}\wedge}(m,n) := -\left(n(n+3) + 4m\left(m+n+\frac{3}{2}\right)\right).$$

We can observe that $D^{\mathrm{I}\wedge}(0,0) = 0$. This shows that D^{I} is not invertible. As a consequence, ${}^*\Delta_x^{\mathrm{I}} = D_{|x|}^{\mathrm{I}} \circ \Delta^*_{\frac{x}{|x|}}$ with the eigenvalues

$$\left({}^*\Delta^{\mathrm{I}}\right)^\wedge(m,n) := n(n+1)\left(n(n+3) + 4m\left(m+n+\frac{3}{2}\right)\right)$$

is also not invertible. Similarly, we set

$$\left({}^*\Delta^{\mathrm{II}}\right)^\wedge(m,n) := mn(n+1)(m+3),$$

which also shows that ${}^*\Delta^{\mathrm{II}}$ is not invertible, because

$$\left({}^*\Delta^{\mathrm{II}}\right)^\wedge(m,0) = \left({}^*\Delta^{\mathrm{II}}\right)^\wedge(0,n) = 0.$$

Next, with the help of the differential operators given above, we find two new operators which are invertible. Note that there exist similar results for spherical (pseudo-)differential operators (see [16]).

Theorem 3. *The differential operators* ${}^{**}\Delta^{\mathrm{I}}_x := (-D^{\mathrm{I}}_{|x|} + \frac{9}{4}) \circ (\Delta^*_{\frac{x}{|x|}} + \frac{1}{4})$ *and* ${}^{**}\Delta^{\mathrm{II}}_x := (-D^{\mathrm{II}}_{|x|} + \frac{9}{4}) \circ (\Delta^*_{\frac{x}{|x|}} + \frac{1}{4})$ *as well as the iterated operators* $({}^{**}\Delta^{\mathrm{I}})^l$ *and* $({}^{**}\Delta^{\mathrm{II}})^l$, $l \in \mathbb{N}$, *are invertible. Moreover, for any* $l \in \mathbb{N}$, *their eigenvalues corresponding to* $G^{\mathrm{I}}_{m,n,j}$ *and* $G^{\mathrm{II}}_{m,n,j}$, *respectively, satisfy*

$$\begin{aligned}
\left(\left({}^{**}\Delta^{\mathrm{I}}\right)^l\right)^\wedge(m,n) &= \left(\left(n+\frac{1}{2}\right)\left(n+2m+\frac{3}{2}\right)\right)^{2l}, \\
\left(\left({}^{**}\Delta^{\mathrm{II}}\right)^l\right)^\wedge(m,n) &= \left(\left(n+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\right)^{2l}.
\end{aligned}$$

Proof. The differential operator $(-D^{\mathrm{I}} + \frac{9}{4})$ has the symbol (i.e. the eigenvalues)

$$\left(-D^{\mathrm{I}} + \frac{9}{4}\right)^\wedge(m,n) = \left(n+2m+\frac{3}{2}\right)^2,$$

since equation (9) yields

$$\begin{aligned}
&\left(-D^{\mathrm{I}}_r + \frac{9}{4}\right)\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right) \\
&= -D^{\mathrm{I}}_r\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right) \\
&\quad + \frac{9}{4}\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right) \\
&= \left(n(n+3) + 4m\left(m+n+\frac{3}{2}\right)\right) \\
&\quad \times \left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{9}{4}\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)\\
=&\left(n+2m+\frac{3}{2}\right)^2\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right).
\end{aligned}
$$

Hence, we conclude that $(-D^{\mathrm{I}}+\frac{9}{4})^\wedge(m,n)=\left(n+2m+\frac{3}{2}\right)^2\neq 0$ for all $m,n\in\mathbb{N}_0$. This shows that $(-D^{\mathrm{I}}+\frac{9}{4})$ is invertible. By applying induction, we have

$$
\left(\left(-D^{\mathrm{I}}+\frac{9}{4}\right)^l\right)^\wedge(m,n)=\left(n+2m+\frac{3}{2}\right)^{2l}.
$$

Furthermore, the operator $-\Delta^*+\frac{1}{4}$ has the eigenvalues

$$
\left(-\Delta^*+\frac{1}{4}\right)^\wedge(n)=\left(n+\frac{1}{2}\right)^2
$$

where $n=0,1,\ldots$, and hence has an inverse $(-\Delta^*+\frac{1}{4})^{-1}$ which is a rational pseudo-differential operator of order -2 (for further details see [16]). More generally, $(-\Delta^*+\frac{1}{4})^l$ is a rational pseudo-differential operator of order $2l$ and has the spherical symbol (i.e. the eigenvalues)

$$
\left(\left(-\Delta^*+\frac{1}{4}\right)^l\right)^\wedge(n)=\left(n+\frac{1}{2}\right)^{2l},\quad n=0,1,\ldots. \tag{11}
$$

Therefore, from the discussion given above, we can conclude that

$$
{}^{**}\Delta^{\mathrm{I}}_x=\left(-D^{\mathrm{I}}_{|x|}+\frac{9}{4}\right)\circ\left(-\Delta^*_{\frac{x}{|x|}}+\frac{1}{4}\right)
$$

is invertible and the eigenvalues corresponding to $G^{\mathrm{I}}_{m,n,j}$ are

$$
\begin{aligned}
&\left(\left({}^{**}\Delta^{\mathrm{I}}\right)^l\right)^\wedge(m,n)\\
=&\left(\left(-D^{\mathrm{I}}+\frac{9}{4}\right)^l\right)^\wedge(m,n)\left(\left(-\Delta^*+\frac{1}{4}\right)^l\right)^\wedge(n)\\
=&\left(\left(n+\frac{1}{2}\right)\left(n+2m+\frac{3}{2}\right)\right)^{2l}
\end{aligned}
$$

for all $m,n\in\mathbb{N}_0$ and all $j=-n,...,n$.

Similarly, $-D^{\mathrm{II}}+\frac{9}{4}$ has the eigenvalues

$$\left(-D^{\mathrm{II}}+\frac{9}{4}\right)^{\wedge}(m)=\left(m+\frac{3}{2}\right)^{2},\quad m=0,1,\dots,$$

since

$$\begin{aligned}&\left(-D^{\mathrm{II}}+\frac{9}{4}\right)P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\\ =\;& -D^{\mathrm{II}}\left(P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\right)+\frac{9}{4}P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\\ =\;& m(m+3)\left(P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\right)+\frac{9}{4}P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\\ =\;& \left(m+\frac{3}{2}\right)^{2}P_m^{(0,2)}\left(\frac{2r}{R}-1\right).\end{aligned}$$

By induction, we obtain

$$\left(\left(-D^{\mathrm{II}}+\frac{9}{4}\right)^{l}\right)^{\wedge}(m)=\left(m+\frac{3}{2}\right)^{2l},\quad m=0,1\dots. \tag{12}$$

From (12), one can conclude that $(-D^{\mathrm{II}}+\frac{9}{4})$ is invertible.

As $-\Delta^*+\frac{1}{4}$ and $-D^{\mathrm{II}}+\frac{9}{4}$ both are invertible, the combined operator

$$^{**}\Delta^{\mathrm{II}}=\left(-D^{\mathrm{II}}+\frac{9}{4}\right)\circ\left(\Delta^*+\frac{1}{4}\right)$$

is also invertible. Therefore, using equations (11) and (12), we finally have the eigenvalues of $\left({}^{**}\Delta^{\mathrm{II}}\right)^{l}$ corresponding to $G^{\mathrm{II}}_{m,n,j}$,

$$\begin{aligned}\left(\left({}^{**}\Delta^{\mathrm{II}}\right)^{l}\right)^{\wedge}(m,n) &= \left(\left(\left(-D^{\mathrm{II}}+\frac{9}{4}\right)\circ\left(-\Delta^*+\frac{1}{4}\right)\right)^{l}\right)^{\wedge}(m,n)\\ &= \left(\left(n+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\right)^{2l}.\end{aligned}$$

□

Definition 4. For any $s\in\mathbb{R}$, we define the operators $({}^{**}\Delta^{\mathrm{I}})^s$ and $({}^{**}\Delta^{\mathrm{II}})^s$ by their eigenvalues

$$\left(\left({}^{**}\Delta^{\mathrm{I}}\right)^{s}\right)^{\wedge}(m,n) := \left(\left(n+\frac{1}{2}\right)\left(n+2m+\frac{3}{2}\right)\right)^{2s}$$

$$\left(\left({}^{**}\Delta^{\mathrm{II}}\right)^{s}\right)^{\wedge}(m,n) \quad := \quad \left(\left(n+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\right)^{2s}$$

corresponding to $G^{\mathrm{I}}_{m,n,j}$ and $G^{\mathrm{II}}_{m,n,j}$, respectively.

Note that all eigenvalues are independent of the order j of the chosen spherical harmonic $Y_{n,j}$, i.e. the operators are isotropic.

4. Application to Sobolev Spaces

We first summarize the introduction of Sobolev spaces on $\mathcal{B}$ from [27].

Definition 5. A sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ satisfies the summability condition of type I if

$$\sum_{m,n=0}^{\infty} A^2_{m,n} n(2m+n)\frac{\left(n+m+\frac{1}{2}\right)^{2m}}{(m!)^2} < +\infty,$$

whereas the summability condition of type II is given by

$$\sum_{m,n=0}^{\infty} A^2_{m,n} n m^5 < +\infty.$$

If a sequence satisfies the summability condition of type I or II, respectively, we say that the sequence is I- or II- summable, respectively.

Definition 6. Let the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ be bounded and $\mathrm{X} \in \{\mathrm{I}, \mathrm{II}\}$ be given. Then the space $\mathcal{H} := \mathcal{H}\left((A_{m,n}), \mathrm{X}, \mathcal{B}\right)$ contains all $F \in \mathrm{L}^2(\mathcal{B})$ such that $\langle F, G^{\mathrm{X}}_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})} = 0$ for all (m,n,j) with $A_{m,n} = 0$ and

$$\sum_{\substack{m,n=0\\A_{m,n}\neq 0}}^{\infty} A^{-2}_{m,n} \sum_{j=1}^{2n+1} \left\langle F, G^{\mathrm{X}}_{m,n,j}\right\rangle^2_{\mathrm{L}^2(\mathcal{B})} < +\infty.$$

Moreover, $\mathcal{H}$ is equipped with the inner product

$$\langle F_1, F_2\rangle_{\mathcal{H}} := \sum_{\substack{m,n=0\\A_{m,n}\neq 0}}^{\infty} A^{-2}_{m,n} \sum_{j=1}^{2n+1} \left\langle F_1, G^{\mathrm{X}}_{m,n,j}\right\rangle^2_{\mathrm{L}^2(\mathcal{B})} \left\langle F_2, G^{\mathrm{X}}_{m,n,j}\right\rangle^2_{\mathrm{L}^2(\mathcal{B})}.$$

Theorem 7. *The spaces $\mathcal{H} := \mathcal{H}((A_{m,n}), \mathrm{X}, \mathcal{B})$, which we defined in Definition 6, are reproducing kernel Hilbert spaces with the unique reproducing kernel*

$$K_{\mathcal{H}}(x,y) = \sum_{m,n=0}^{\infty} \sum_{j=1}^{\infty} A_{m,n}^2 G_{m,n,j}^{\mathrm{X}}(x) G_{m,n,j}^{\mathrm{X}}(y); \quad x,y \in \mathcal{B},$$

if the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ is X*-summable.*

Based on these Sobolev spaces, we can now clarify the domain of the operators ${}^{**}\Delta^{\mathrm{I}}$ and ${}^{**}\Delta^{\mathrm{II}}$.

Definition 8. For any $s \in \mathbb{R}_0^+$, we define the spaces

$$\mathcal{H}_s^{\mathrm{I}}(\mathcal{B}) := \mathcal{H}\left(\left(\left(n + 2m + \frac{3}{2}\right)^{-s} \left(n + \frac{1}{2}\right)^{-s}\right), \mathrm{I}, \mathcal{B}\right)$$

and

$$\mathcal{H}_s^{\mathrm{II}}(\mathcal{B}) := \mathcal{H}\left(\left(\left(m + \frac{3}{2}\right)^{-s} \left(n + \frac{1}{2}\right)^{-s}\right), \mathrm{II}, \mathcal{B}\right).$$

Obviously, $\mathcal{H}_{s_1}^{\mathrm{X}}(\mathcal{B}) \subset \mathcal{H}_{s_2}^{\mathrm{X}}(\mathcal{B})$ for $s_1 \geq s_2$ and $\mathrm{X} \in \{\mathrm{I}, \mathrm{II}\}$. Furthermore, $\mathcal{H}_0^{\mathrm{I}}(\mathcal{B}) = \mathcal{H}_0^{\mathrm{II}}(\mathcal{B}) = \mathrm{L}^2(\mathcal{B})$.

We can now reformulate Definition 4 more precisely.

Definition 9. Let $s, t \in \mathbb{R}_0^+$ with $s \geq 2t$. Then we formally define the operators

$$\left({}^{**}\Delta^{\mathrm{I}}\right)^t : \mathcal{H}_s^{\mathrm{I}}(\mathcal{B}) \to \mathcal{H}_{s-2t}^{\mathrm{I}}(\mathcal{B})$$

and

$$\left({}^{**}\Delta^{\mathrm{II}}\right)^t : \mathcal{H}_s^{\mathrm{II}}(\mathcal{B}) \to \mathcal{H}_{s-2t}^{\mathrm{II}}(\mathcal{B})$$

by

$$\begin{aligned}
\left({}^{**}\Delta^{\mathrm{I}}\right)^t F_1 &= \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left(n + \frac{1}{2}\right)\left(n + 2m + \frac{3}{2}\right)\right)^{2t} \\
&\qquad \times \left\langle F_1, G_{m,n,j}^{\mathrm{I}} \right\rangle_{\mathrm{L}^2(\mathcal{B})} G_{m,n,j}^{\mathrm{I}}, \\
\left({}^{**}\Delta^{\mathrm{II}}\right)^t F_2 &= \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left(n + \frac{1}{2}\right)\left(m + \frac{3}{2}\right)\right)^{2t}
\end{aligned}$$

$$\times \left\langle F_2, G^{\mathrm{II}}_{m,n,j} \right\rangle_{\mathrm{L}^2(\mathcal{B})} G^{\mathrm{II}}_{m,n,j}$$

for $F_1 \in \mathcal{H}^{\mathrm{I}}_s(\mathcal{B})$ and $F_2 \in \mathcal{H}^{\mathrm{II}}_s(\mathcal{B})$.

To verify that the operators are well-defined, we show that

$$\begin{aligned}
\left\| \left({}^{**}\Delta^{\mathrm{I}} \right)^t F_1 \right\|^2_{\mathcal{H}^{\mathrm{I}}_{s-2t}(\mathcal{B})} &= \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(n + \frac{1}{2} \right)^{2s-4t} \left(n + 2m + \frac{3}{2} \right)^{2s-4t} \\
&\quad \times \left\langle \left({}^{**}\Delta^{\mathrm{I}} \right)^t F_1, G^{\mathrm{I}}_{m,n,j} \right\rangle^2_{\mathrm{L}^2(\mathcal{B})} \\
&= \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(n + \frac{1}{2} \right)^{2s} \left(n + 2m + \frac{3}{2} \right)^{2s} \\
&\quad \times \left\langle F_1, G^{\mathrm{I}}_{m,n,j} \right\rangle^2_{\mathrm{L}^2(\mathcal{B})} \\
&= \| F_1 \|^2_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})} < +\infty.
\end{aligned}$$

Analogous considerations can be made for type II. This also yields the following result.

Theorem 10. *Let $s, t \in \mathbb{R}^+_0$ with $s \geq 2t$. If $F_1 \in \mathcal{H}^{\mathrm{I}}_s(\mathcal{B})$ and $F_2 \in \mathcal{H}^{\mathrm{II}}_s(\mathcal{B})$, then*

$$\left\| \left({}^{**}\Delta^{\mathrm{I}} \right)^t F_1 \right\|_{\mathcal{H}^{\mathrm{I}}_{s-2t}(\mathcal{B})} = \| F_1 \|_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})},$$

$$\left\| \left({}^{**}\Delta^{\mathrm{II}} \right)^t F_2 \right\|_{\mathcal{H}^{\mathrm{II}}_{s-2t}(\mathcal{B})} = \| F_2 \|_{\mathcal{H}^{\mathrm{II}}_s(\mathcal{B})}.$$

In particular,

$$\left\| \left({}^{**}\Delta^{\mathrm{I}} \right)^{s/2} F_1 \right\|_{\mathrm{L}^2(\mathcal{B})} = \| F_1 \|_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})},$$

$$\left\| \left({}^{**}\Delta^{\mathrm{II}} \right)^{s/2} F_2 \right\|_{\mathrm{L}^2(\mathcal{B})} = \| F_2 \|_{\mathcal{H}^{\mathrm{II}}_s(\mathcal{B})}.$$

Therefore, the Sobolev norms $\|\cdot\|_{\mathcal{H}_s^{\mathrm{I}}(\mathcal{B})}$ and $\|\cdot\|_{\mathcal{H}_s^{\mathrm{II}}(\mathcal{B})}$ can be interpreted as the $\mathrm{L}^2(\mathcal{B})$-norms of certain generalized derivatives (i.e. in the sense of pseudo-differential operators). This is an important new result, since norms of this kind are used to measure the non-smoothness of interpolating functions on $\mathcal{B}$, where it is known that the interpolating spline minimizes the Sobolev norm among all interpolants (see [4], [5], [7], [8], [9], and [27] for further details). Hence, this minimum property can, indeed, be considered as an analogue of corresponding results on the real line (see [19, Theorem 2.3.3]) and the sphere (see [16, Lemma 6.1.4]).

Moreover, since we defined that

$$\left\langle \left({}^{**}\Delta^{\mathrm{X}}\right)^t F, G^{\mathrm{X}}_{m,n,j} \right\rangle_{\mathrm{L}^2(\mathcal{B})} = \left(\left({}^{**}\Delta^{\mathrm{X}}\right)^{\wedge}(m,n)\right)^t \left\langle F, G^{\mathrm{X}}_{m,n,j}\right\rangle_{\mathrm{L}^2(\mathcal{B})}$$

for $\mathrm{X} \in \{\mathrm{I}, \mathrm{II}\}$, $F \in \mathcal{H}_s^{\mathrm{X}}(\mathcal{B})$, $s \geq 2t,\ m, n \in \mathbb{N}_0, j = 1, ..., 2n+1$, we immediately get the following result.

Theorem 11. *The operators* $({}^{**}\Delta^{\mathrm{I}})^t$ *and* $({}^{**}\Delta^{\mathrm{II}})^t$ *are self-adjoint in the sense that*

$$\left\langle \left({}^{**}\Delta^{\mathrm{X}}\right)^t F_1, F_2 \right\rangle_{\mathrm{L}^2(\mathcal{B})} = \left\langle F_1, \left({}^{**}\Delta^{\mathrm{X}}\right)^t F_2 \right\rangle_{\mathrm{L}^2(\mathcal{B})}$$

for all $F_1, F_2 \in \mathcal{H}_s^{\mathrm{X}}(\mathcal{B})$ *with* $\mathrm{X} \in \{\mathrm{I}, \mathrm{II}\}$.

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