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THE MATHEMATICS OF THE STRAW FLUTE

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Abstract: It is possible to make a simple flute-like instrument using a straw. I used an electronic tuner to determine reference tones, and then located the required positions to create finger holes to form a diatonic scale. Pythagoras created Pythagorean tuning as part of his research into the relationship between scales and the length of strings. The oscillations that represent a scale and the length of the strings creating it can be represented by a geometric series, showing us the close association between music and mathematics.

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Key Words: reference tones, scales, Pythagorean tuning, geometric series, modular arithmetic

1. Creating Sounds

One day on an NHK television show called *Necchu Jikan* ("Time for Obsessions") I saw a man named Toru Kamiya playing a flute made from a drinking straw. He flattened the straw's mouth and cut a 5-mm slit down each side, creating two reeds that would vibrate when he blew through it. Watching Mr. Kamiya's interesting performance brought out the music lover in me, and prompted me to try my hand at creating my own straw flute.

I immediately went out and bought some straws. The straws that are commonly available are 210 mm long and 6 mm in diameter, and some of them are bendable to make drinking easier. The bending type is particularly useful when you want to connect several straws.

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Like Mr. Kamiya, I pressed the mouth of the straw together and used scissors to cut 5 mm slits, creating two reeds. Blowing through the thing did make a sound, but it required significant effort because they were made from polypropylene, which is stiff and has a high level of shape retention. Mr. Kamiya says that there are several things one can do to prevent this, such as holding the straw shut with a clip, lightly sanding it, slightly heating it with a lighter, or using an iron to heat the straw while you press it closed.

As to how the thing should be blown, it is impossible to make any sound with a light breath as you can with a recorder. Instead, you must hold your lips tightly shut and blow quite powerfully. When I was in junior high school I played the clarinet in the school band, so I already had the knack for producing a sound with reed instruments. I was able to produce sound by blowing the straw flute in just the same way. The sound was quite lovely, much like that of an oboe.

Once you can make sounds, it is natural to want to play a tune. I knew a melody that used only the do-re-mi of the musical scale, and opening a few holes in the straw allowed me to play it like a simple trumpet.

2. Setting Reference Tones

Encouraged by my success, I next wanted to try creating a straw flute that could play not just three notes, but the seven notes of a full scale. I used a recorder as my guide to placing the finger holes, but the result was not what I had hoped for. There are over 500 years of history behind the positioning and size of the holes on that instrument, so it is no surprise that guesswork will not suffice. I therefore challenged myself to create an instrument with more precise pitch.

The first thing to determine is the base note and the length. As shown in Table 1, there are two predominant ways of naming the notes of a scale, and most instruments use the series as shown in the table, starting and ending with "do." The most common reference tone is A4 ("la"), however, which is 440 hertz. Hertz (abbreviated Hz) is a unit of vibrational frequency, and 440 Hz means 440 vibrations per second. Humans have a specific range of vibrations that they can hear, usually said to be from 20 to somewhere between 15,000 and 20,000 Hz. A standard 88-key piano has a range that goes up to about 4000 Hz using keys labeled A0 up to C8. The A4 key is right about at the center of the keyboard, and so is used as the reference note.

Since wind instruments begin their scales with "do," I took C4 as a reference

Do	Re	Mi	Fa	Sol	La	Ti	Do
C	D	E	F	G	A	B	C

Table 1: The musical scale

note. A longer instrument produces a deeper sound, and a shorter instrument a higher sound, so some tuning is required to create a perfect C4 note. In the past, U-shaped tuning forks were used to tune instruments, but these days there are electronic tuners that can be obtained inexpensively, so I bought one. Playing the straw at its current length displayed its scale, and while playing around with this clever little device I found that I could produce a C4 note with a straw 304 mm long.

Since store-bought straws are only 210 mm long, to lengthen one to 304 mm I had to join two of them. You can do so using cellophane tape, but using another straw with a slightly smaller diameter makes the assembly removable, which is quite convenient.

3. Scales as Geometric Series

A diatonic scale has seven notes that repeat at the octave, do-re-mi-fa-sol-la-ti-do. The transition from "mi" to "fa" and that of "ti" to "do" are considered half steps, so in all there are five whole steps and two half steps, for a total of seven. The interval of pitches from the low "do" to the high "do" is called an octave. Examining an octave as sound frequencies, the high note will be double the frequency of its low note. Actually, it is easier to consider an octave not as seven unequal steps, but rather by converting each of the whole steps into two half steps, for a total of twelve semitones

$$5 \times 2 + 2 \times 1 = 12 \ (semitones).$$

The reference note A4 is 440 Hz, so the A3 note that is one octave below it is 220 Hz. The frequency of the C4 note found between them can be calculated according to the method of equal temperament described below.

Equal temperament divides an octave into twelve equal parts. When doing so it does not use an arithmetic progression, but rather a geometric one, as follows. The A3 note's frequency is 220 Hz and that of the A4 note is 440 Hz, for a difference of 220 Hz. However. the spacing of each note is not determined by dividing this difference by twelve, but rather by multiplying the starting 220

Hz value by a certain value twelve times so that it reaches 440 Hz. Since we want one octave to double the starting frequency, we can find that number by taking the twelfth root of two

$$2^{\frac{1}{12}} \approx 1.059.$$

This number is the common ratio of the geometric series. Moving from A3 to C4 we have one full step and one half step, for a total of three half steps, and cubing the constant we found above we calculate a frequency of 262 Hz

$$220 \times \left(2^{\frac{1}{12}}\right)^3 \approx 262\ (Hertz).$$

Table 2 shows a list of frequencies from C4 to D5 that I calculated using a spreadsheet.

Scale		Temperament	Frequency (Hz)	Tube length (mm)
do	C4	1.00	262	304
re	D4	1.12	294	268
mi	E4	1.26	330	237
fa	F4	1.33	349	223
sol	G4	1.50	392	197
la	A4	1.68	440	173
ti	B4	1.89	494	153
do	C5	2.00	523	143
re	D5	2.24	587	126

Table 2: Scale frequencies and tube lengths

4. End Correction

Using the tuner, I found that I could produce a C4 note with a straw length of 304 mm. I also know the frequencies that represent the C4 through D5 notes. The next step is to use that information to calculate the required hole positions for the D4 through D5 notes.

But before doing that, I did some calculations to confirm that a tube length of 304 mm would produce a frequency of 262 Hz. To do that, I used the following formula, dredged up from my memories of physics class in high school:

Speed of sound (m/s) = Frequency (beats/s)   wavelength (m)

We already know what the speed of sound is. At one standard atmosphere of pressure, it will vary according to the temperature t, as

$$331.5 \times 0.6t\ m/s,$$

so at a temperature of 15 °C, the speed is 340.65 m/s. We can simplify things a little by rounding this off to 340 m/s.

For a stringed instrument, the length of the string is one half that of the wavelength it produces, because both ends of the string are fixed and so determine the position of the nodes. For a wind instrument, the length of the tube will be either 1/2 or 1/4 the wavelength, depending on if the instrument is open or close ended. Recorders and flutes are of the open-ended variety, with both ends open and determining the peaks of the sound waves produced. The clarinet is of the closed-end type, with one end (the one with the mouthpiece) closed, thus determining the position of one node and one peak.

Like the clarinet, our straw flute is a closed instrument, so the length of the tube will be 1/4 that of the sound waves it produces. A recorder will have the same length as a straw flute, about 30 cm, but the recorder will produce a scale one octave higher since it is an open instrument.

Using 340 m/s as the speed of sound, and wanting a frequency of 262 Hz to produce a scale starting at C4, I calculated a desired tube length of 325 mm.

Tube length = wavelength / 4 = speed of sound / frequency / 4 = 325 mm

According to my calculations, my tube should have been 325 mm long to produce a C4 note, but according to my electronic tuner a length of 304 mm was required, a difference of 21 mm. Wondering why, I set out to investigate. As it turns out, in instruments with a closed end, the peaks of the waveform will form slightly outside of the open end, and so the wavelength will be four times the length of the tube plus that added bit. This adjustment is called gend correction.h I assume that the air continues to vibrate for some distance linearly in the direction of the tube, even when the tube is no longer present. We now have the following formula:

Calculated length = Actual length + End correction

So, to create a 262 Hz C4 note, we needed a calculated distance of 325 mm, and taking from that our 21 mm correction we get an actual length of 304 mm.

The values for the tube lengths in Table 2 are the lengths found after making this correction.

The positions for opening holes are determined by the tube length. In Mr. Kamiya's performance, he used a pair of scissors to cut holes in the straw that allowed him to play a do-re-mi-fa-sol-la-ti-do scale. It is a kind of demonstration that can only be done using an inexpensive prop like a drinking straw, and is a good confirmation that the important thing for scales is not the position of the holes but rather the length of the instrument.

So, the length for a C4 note is 304 mm, which means that the holes for the D4 and E4 notes should be 268 mm and 237 mm from the mouthpiece, respectively. It is not necessary to open the holes in a straight line along the tube. As long as the distance from the mouthpiece is the same, they only need to be placed somewhere along the line of concentricity. Indeed, since the low C4 note is played while pressing the little finger of one's right hand over the hole, the hold is often shifted a bit to the right to make that movement easier. The high D5 note is played with the left thumb over its hole, so that hole is placed on the bottom of the tube, a full 180 degree rotation along the line of concentricity from the top.

Because the steps between the E4 and F4 notes and the B4 and C5 notes are half steps, their holes must be placed a bit closer to their neighbors, as compared with the whole step notes. It is therefore best to place the high C5 note a little farther down than the calculated location, but, to make up for that, make the hole a little smaller. They say that the hole positioning and size for recorders has been evolving for 500 years to make them easier to play with human hands.

Two possible ways to create the holes includes cutting them out with scissors or burning them out with a soldering iron. Mr. Kamiya, however, suggests using a simple hole punch like you might find at a stationary store, and I found that to be a very effective method. Figure 1 shows the results of my efforts. I was able to use just this straw to play a number of my favorite folk songs, and even some classical pieces.

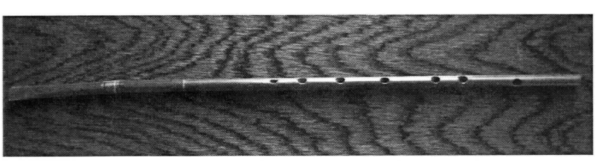

Figure 1: A handmade straw flute

My interest next turned to the relationship between the length of the straw and the scales that it played. I had created a flute that played a single octave out of a 30 cm drinking straw, but what would happen if I made one from a 60 cm straw, instead?

Before we investigate that, however, I need to say a word about timbre. Say that you played a C note on some musical instrument. While you might think that you are hearing a single sound, in truth you are hearing a whole collection of other subtle sounds along with it. Those sounds are called the harmonic overtones of the fundamental tone C. It is the fundamental tones and their overtones that make up the timbre of the instrument. The reason a C note played on a flute sounds different from a C note played on a violin is because the sounds have differing ratios of overtones. Putting this in mathematical terms, we can say that performing a Fourier transform of the sound decomposes it into frequencies with varying sine waves. The power spectrum of the various frequencies found by the Fourier transform is related to the timbre, and performing an inverse Fourier transform on them results in the production of sound. Synthesizers are an application of exactly this principle.

Next, let me explain the relationship between harmonics and octaves. The second harmonic of a fundamental tone is the scale one octave above it. You might then think that the third harmonic would be two octaves above, but that would be the G note one perfect fifth (see below) into the octave above (3/2 of the harmonic)

$$3 = \frac{3}{2} \times 2.$$

Because the frequencies corresponding with scales are represented using a geometric sequence, the octave above is the second harmonic, two octaves above is the fourth harmonic (because $2^2 = 4$), three octaves above is the eighth harmonic ($2^3 = 8$), and so on. The length of the tube is in inverse proportion to the frequency, and so must be 1/2, 1/4, and 1/8 the length, respectively. Conversely, the 1/2 harmonic is the scale one octave below, so in that case the tube must be twice as long.

At least that is how the calculations work out, so I tried connecting straws to see what happened to the scale. Using the tuner again, I found that C4 was at 304 mm, C3 was at 614 mm, and C2 was at 1214 mm. Since my store-bought straws were 210 mm long, producing C2 would require around six of them. Trying it out produced a very low scale, much like that of a bass tuba. Going the other way, shortening the straws produced high scales with C5 at 143 mm and C6 at 69 mm. So, it would seem that the range of seven octaves covered by an orchestra could be done using only drinking straws, but practical considerations such as the width of the human hand that must cover the holes makes me think that a 304 mm straw is the most practical length.

5. Pythagorean Tuning

Pythagoras discovered scales and chords experimentally. Dividing the length of a string into twelve equal parts and taking the sound made by the full length as "do," then the ninth length produces "fa," the eighth length "sol," and the sixth length a high "do." Pythagoras showed that this combination of "do," "sol," and (high) "do" produces a beautiful chord.

Shortening a string to 2/3 its original length produces a pitch 3/2 as high as it did before. This is the relationship between "do" and "sol," and is called a perfect fifth. A perfect fifth is an interval of three full tones and one semitone. Similarly, a string shortened to 3/4 its original length will produce a pitch 4/3 higher than before. This is the relationship between "do" and "fa," and is called a perfect fourth. A perfect fourth is an interval of two full tones and one semitone. Starting with "do" and applying the relationship of perfect fifths twelve times produces twelve semitones. Stated as an equation, we have:

$$a_n = a_1 + (n-1)d \; mod \; p,$$

$$d = 7, \; p = 12.$$

This forms an arithmetic series with an initial term of 1 and common difference of 7, using modular arithmetic to subtract 12 from any values that exceed 12. Starting with "do"($a_1 = 1$), we progress through all of the values until the 13th repetition, where we get back to "do"($a_{13} = 1$). Figure 2 shows a scale of perfect fifths with n on the horizontal axis and a_n on the vertical:

$a_1 = 1,$
$a_2 = 1 + 7 = 8,$
$a_3 = 1 + 14 = 15 \; mod \; 12 = 3,$
$\cdots$
$a_{13} = 1 + 12 \times 7 = 85 \; mod \; 12 = 1.$

We have discussed how to create a scale of twelve semitones using just the relationship of perfect fifths between "do" and "sol," so let us next use that to find the frequency ratios between each of the scales. For example, since moving 3/2 up the scale from "re" takes us to the "re" one octave higher, to move one octave down we divide this by 2, giving us 9/8

$$\left(\frac{3}{2}\right)^2 \times \frac{1}{2} = \frac{9}{8}.$$

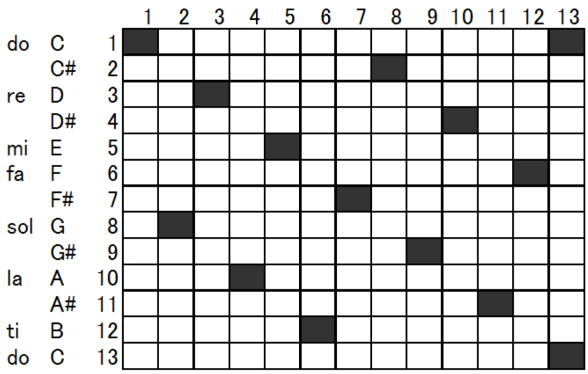

Figure 2: A scale of perfect fifths

Repeating this gives the Pythagorean tuning, as shown in Table 3. Pythagorean tuning uses $9/8 = 1.125$ as the spacing between the full tones between "do" and "re," "re" and "mi," "fa" and "sol," "sol" and "la," and "la" and "ti," and $256/243 = 1.053$ as the spacing between the semitones between "mi" and "fa," and between "ti" and "do." Note that in this case the full tones are not exactly twice the size of the semitones, but multiplying all of these values together gives 2

$$1 \times \frac{9}{8} \times \frac{9}{8} \times \frac{256}{243} \times \frac{9}{8} \times \frac{9}{8} \times \frac{9}{8} \times \frac{256}{243} = 2.$$

The tuning system that Pythagoras found is less related to music than to integer mathematics. Since the sounds produced by a string or a tube are related to the length of those objects, and since playing any scale will require an integer as its frequency ratio, one might say that Pythagorean tuning makes some sense.

However, the last frequency calculated by the twelfth multiplication of $3/2$ gives us

$$\left(\frac{3}{2}\right)^{12} \approx 129.7 \approx 128 = 2^7,$$

which is slightly higher than the $2^7 = 128$ value of the seventh octave. This gap is called the Pythagorean comma, and is not a negligible value. This issue led to the idea of temperament, which we addressed at the beginning of this paper.

Moving from Pythagorean tuning to temperament, however, required waiting for the mathematical development of rational numbers. There are also other improvements to Pythagorean tuning such as just intonation. Each method has its advantages and disadvantages.

I would like to close with a thank you to Mr. Toru Kamiya for kindly explaining how to make a straw flute.

	Temperament	Pythagorean tuning		Just intonation	
do	1.00	1	1.00	1	1.00
re	1.12	9/8	1.13	9/8	1.13
mi	1.26	81/64	1.27	5/4	1.25
fa	1.33	4/3	1.33	4/3	1.33
sol	1.50	3/2	1.50	3/2	1.50
la	1.68	27/16	1.69	5/3	1.67
ti	1.89	243/128	1.90	15/8	1.88
do	2.00	2	2.00	2	2.00

Table 3: Pythagorean tuning and just intonation

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