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SOME GENERATING FUNCTIONS FOR MEASURES OF PROBABILISTIC ENTROPY AND DIRECTED DIVERGENCE

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Abstract: Some generating functions are introduced which generate existing measures of entropies and directed divergence.

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1. Introduction

In this paper we have defined some functions which generate various measures of entropies and directed divergences.

In 1966, Golomb [5] defined

$$f\left(t\right) = -\sum_{i=1}^{n} p_i^t, \tag{1.1}$$

as generating function with the property that

$$f'\left(1\right) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{1.2}$$

That is Shannon's (see [16]) measure of entropy.

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Later in 1985, Guiasu and Reisher (see [6]) defined the generating function $g(t)$ for relative information or cross-entropy or directed divergence of one probability distribution $P = (p_1, p_2, \ldots, p_n)$ from another probability distribution $Q = (q_1, q_2, \ldots, q_n)$ by

$$g\,(t) = \sum_{i=1}^{n} q_i (p_{i/} q_i)^t, \tag{1.3}$$

with the property

$$g'\,(1) = -\sum_{i=1}^{n} p_i \ln p_i / q_i, \tag{1.4}$$

$$g^r\,(1) = \sum_{i=1}^{n} p_i (\ln p_i / q_i)^r, \quad r = 1, 2, 3. \tag{1.5}$$

In 1997, Kapur [12] defined the generating function

$$f_\alpha\,(t) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} (p_i)^t - 1\right), \quad \alpha \neq 1, \tag{1.6}$$

with the property

$$f_\alpha\,(1) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^\alpha - 1\right), \quad \alpha \neq 1 \tag{1.7}$$

and

$$f'_\alpha\,(0) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^\alpha, \quad \alpha \neq 1. \tag{1.8}$$

Kapur (see [12]) also defined the generating function for relative information or cross-entropy or directed divergence of $P = (p_1, p_2, \ldots, p_n)$ from another probability distribution $Q = (q_1, q_2, \ldots, q_n)$ by

$$g_\alpha(t) = \frac{1}{\alpha - 1}\left[\left(\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}\right)^t - 1\right], \quad \alpha \neq 1, \tag{1.9}$$

with the property that

$$g_\alpha(1) = \frac{1}{\alpha - 1}\left[\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} - 1\right], \quad \alpha \neq 1 \tag{1.10}$$

and

$$g'_{\alpha}(1) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}, \quad \alpha \neq 1. \tag{1.11}$$

In this paper we investigate some generating functions for measures of entropies in Subsection 2.1 and for measures of directed divergences in Subsection 2.2.

2. Main Results

2.1. Generating Functions for Measures of Information Based on Probability Distribution

Let

$$f_{\alpha,\beta}(t) = \frac{1}{\beta - \alpha} \left[\left(\sum_{i=1}^{n} p_i^{\alpha/\beta} \right)^{t} - \beta \right], \quad \alpha \neq \beta,\ \alpha > 1,\ \beta > 0, \tag{2.1.1}$$

then

$$f_{\alpha,1}(1) = \frac{1}{1 - \alpha} (\sum_{i=1}^{n} p_i^{\alpha} - 1), \quad \alpha \neq 1. \tag{2.1.2}$$

Therefore $f_{\alpha,1}(1)$ gives Havrda-Charvat's (see [7]) measure of entropy. Also

$$f'_{\alpha,1}(t) = \frac{1}{1 - \alpha} \left[\left(\sum_{i=1}^{n} p_i^{\alpha} \right)^{t} \ln \sum_{i=1}^{n} p_i^{\alpha} \right], \quad \alpha \neq 1, \tag{2.1.3}$$

then

$$f'_{\alpha,1}(0) = \frac{1}{1 - \alpha} \ln \sum_{i=1}^{n} p_i^{\alpha}, \quad \alpha \neq 1. \tag{2.1.4}$$

Therefore $f'_{\alpha,1}(0)$ gives Renyi's (see [15]) measure of entropy. Now let

$$f_{1,1}(1) = f_{\alpha,1}(1) = \frac{1}{1 - \alpha} \left(\sum_{i=1}^{n} p_i^{\alpha} - 1 \right), \tag{2.1.5}$$

$$f_{1,1}(1) = \left(- \sum_{i=1}^{n} p_i^{\alpha} \ln p_i \right),$$

thus

$$f_{1,1}(1) = -\sum_{i=1}^{n} p_i \ln p_i, \tag{2.1.6}$$

which is a trivial generating function for Shannon's (see [16]) measure of entropy.

Again, let

$$f_{2,1}(t) = 1 - \left(\sum_{i=1}^{n} p_i^2\right)^t. \tag{2.1.7}$$

Then

$$f_{2,1}(1) = 1 - \sum_{i=1}^{n} p_i^2. \tag{2.1.8}$$

Hence $f_{2,1}(1)$ gives Vajda's (see [18]) measures of entropy. Since

$$f'_{\alpha,1}(t) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^t \ln \sum_{i=1}^{n} p_i^{\alpha}\right], \quad \alpha \neq 1, \tag{2.1.9}$$

then

$$f'_{\alpha,1}(0) = -\frac{1}{\alpha-1} \ln \sum_{i=1}^{n} p_i^{\alpha}, \ \alpha \neq 1, \tag{2.1.10}$$

thus

$$f'_{\alpha,1}(0) = -\ln(1-\alpha)\frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{\alpha-1}}, \quad \alpha \neq 1 \tag{2.1.11}$$

which is antilogarithm of $(1-\alpha)$ times Behara-Chawla (see [4]) measure of entropy.

Now, let us define

$$f_{\alpha,\beta,\gamma}(t) = \frac{1}{\beta-\alpha}\left[\sum_{i=1}^{n}\left(p_i^{\alpha/\gamma}\right)^t - \beta\right], \quad \alpha \neq \beta, \beta, \gamma > 0. \tag{2.1.12}$$

Therefore

$$f_{2,3,2}(t) = \sum_{i=1}^{n} p_i^t - 3, \tag{2.1.13}$$

then

$$f'_{2,3,2}(t) = \sum_{i=1}^{n} p_i^t \ln p_i \tag{2.1.14}$$

and

$$f'_{2,3,2}(0) = \sum_{i=1}^{n} \ln p_i, \tag{2.1.15}$$

which is Burg's (see [2]) measure of entropy.

Because of

$$f'_{2,1,2}(t) = -\sum_{i=1}^{n} p_i^t \ln p_i, \tag{2.1.16}$$

we have

$$f'_{2,1,2}(1) = -\sum_{i=1}^{n} p_i \ln p_i, \tag{2.1.17}$$

so that $f'_{2,1,2}(1)$ gives Shannon's (see [16]) measure of entropy.

Since

$$f_{2,1,2}(t) = 1 - \sum_{i=1}^{n} p_i^t \tag{2.1.18}$$

then

$$f_{2,1,2}(2) = 1 - \sum_{i=1}^{n} p_i^2 \tag{2.1.19}$$

which is Vajda's (see [18]) measure of entropy.

Now let

$$f_{\alpha,1,1}(t) = \frac{1}{1-\alpha}\left[\sum_{i=1}^{n} (p_i^{\alpha})^t - 1\right], \quad \alpha \neq 1. \tag{2.1.20}$$

So,

$$f_{\alpha,1,1}(1) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right), \quad \alpha \neq 1, \tag{2.1.21}$$

which is Havrda-Charvat (see [7]) measure of entropy.

We define

$$f_{\alpha,\beta,\gamma}(t) = \frac{1}{\beta-\alpha}\left[\sum_{i=1}^{n} \left(p_i^{\alpha-\beta}\right)^t - \gamma\right], \quad \alpha \neq \beta \text{ and } \beta,\gamma > 0. \tag{2.1.22}$$

So,

$$f_{\alpha,\beta,1}(1) = \frac{1}{\beta-\alpha}\left[\frac{\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i^{\beta}}\right], \quad \alpha \neq \beta, \tag{2.1.23}$$

which is $\left(\sum\limits_{i=1}^{n} p_i^{\beta}\right)^{-1}$ times of Sharma and Taneja's (see [17]) measure of entropy. Now

$$f'_{\alpha,\beta,1}(0) = \frac{1}{\beta-\alpha} \ln \sum_{i=1}^{n} p_i^{\alpha-\beta}, \quad \alpha \neq \beta, \tag{2.1.24}$$

so that,

$$f'_{\alpha,\beta,1}(0) = \frac{1}{\beta-\alpha} \ln \frac{\sum\limits_{i=1}^{n} p_i^{\alpha}}{\sum\limits_{i=1}^{n} p_i^{\beta}}, \quad \alpha \neq \beta,\ \alpha > 1,\ \beta < 1, \tag{2.1.25}$$

or $\alpha > 1$, $\beta < 1$, which is the measure of entropy due to Aczel and Daroczy (see [1]) and Kapur. So, $f_{\alpha,\beta,\gamma}(t)$ is the generating function for Aczel-Daroczy-Kapur's measure of entropy. Now

$$f_{3,1,1}(1) = \frac{1}{2}(1 - \sum_{i=1}^{n} p_i^2), \tag{2.1.26}$$

which is half-times of Vajda's (see [18]) measure of entropy. Let

$$f_{\alpha,1,1}(1) = \frac{1}{1-\alpha}(\sum_{i=1}^{n} p_i^{\alpha} - 1), \quad \alpha \neq 1. \tag{2.1.27}$$

Therefore $f_{\alpha,1,1}(1)$ gives Havrda-Charvat (see [7]) measure of entropy. Also, let us nite that

$$f'_{\alpha,1,1}(t) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i^{\alpha-1}\right)^{t} \ln \sum_{i=1}^{n} p_i^{\alpha-1}\right], \quad \alpha \neq 1 \tag{2.1.28}$$

and

$$f'_{\alpha,1,1}(0) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^{\alpha}, \quad \alpha \neq 0,\ \alpha > 0. \tag{2.1.29}$$

Hence, it is Renyi's (see [15]) measure of entropy. Moreover

$$f_{1,1,1}(1) = f_{\alpha,1,1}(1) = \left[-(\sum_{i=1}^{n} p_i^{\alpha} \ln p_i)\right] = -\sum_{i=1}^{n} p_i \ln p_i$$

is Shannon's (see [16]) measure of entropy. Again, letting

$$f'_{\alpha,1,1}(0) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^{\alpha}, \quad \alpha \neq 0,\ \alpha > 0$$

so

$$f'_{\alpha,1,1}\left(0\right) = -\ln\left(1-\alpha\right).\frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/\alpha - 1}, \quad \alpha \neq 1, \tag{2.1.30}$$

is antilogarithm of (1-α) times Behara-Chawla (see [4]) measure of entropy.

Now, if we define

$$f_{\alpha,\beta}\left(t\right) = \frac{1}{\beta-\alpha}\left[\sum_{i=1}^{n}\left(p_i^{\alpha\beta}\right)^t - \beta\right], \quad \alpha \neq \beta,\ \alpha > 1,\ \beta > 0, \tag{2.1.31}$$

then

$$f_{\alpha,1}\left(1\right) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right), \quad \alpha \neq 1. \tag{2.1.32}$$

Thus $f_{\alpha,1}\left(1\right)$ gives Havrda-Charvat (see [7]) measure of entropy.

From

$$f'_{\alpha,1}\left(t\right) = \frac{1}{1-\alpha}\left[\sum_{i=1}^{n}\left(p_i^{\alpha}\right)^t \ln p_i^{\alpha}\right], \quad \alpha \neq 1, \tag{2.1.33}$$

it follows:

$$f'_{\alpha,1}\left(0\right) = \frac{1}{1-\alpha}\sum_{i=1}^{n} \ln p_i^{\alpha}, \quad \alpha \neq 1 \qquad 2.1.34)$$

$$f'_{\alpha,1}\left(0\right) = \frac{\alpha}{1-\alpha}\sum_{i=1}^{n} \ln p_i, \quad \alpha \neq 1, \tag{2.1.35}$$

which is $\frac{\alpha}{1-\alpha}$ times of Burg's (see [2]) measure of entropy. Again

$$f_{2,1}\left(t\right) = 1 - \left(\sum_{i=1}^{n} p_i^2\right)^t, \tag{2.1.36}$$

therefore

$$f_{2,1}\left(1\right) = 1 - \sum_{i=1}^{n} p_i^2 \tag{2.1.37}$$

and

$$f'_{2,1}\left(t\right) = -\sum_{i=1}^{n}\left(p_i^2\right)^t \ln p_i^2, \tag{2.1.38}$$

$$f'_{2,1}\left(1/2\right) = -2\sum_{i=1}^{n} p_i \ln p_i. \tag{2.1.39}$$

Thus $f_{2,1}(1)$ and $f'_{2,1}(1/2)$ gives Vajda's (see [18]) and two times of Shannon's (see [16]) measure of entropy, respectively.

Now we define

$$\bar{f}_a(t) = -\sum_{i=1}^{n} p_i^t + \frac{1}{a}\sum_{i=1}^{n}(1+ap_i)^t - \frac{1}{a}\sum_{i=1}^{n}(1+a)^t p_i. \tag{2.1.40}$$

Then

$$\bar{f}'_a(1) = -\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}\sum_{i=1}^{n}(1+ap_i)\ln(1+ap_i) - \frac{1}{a}\sum_{i=1}^{n}(1+a)\ln(1+a)p_i \tag{2.1.41}$$

is Kapur's (see [9]) measure of entropy. It is easy to check that

$$\bar{f}'_{-1}(1) = -\sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n}(1-p_i)\ln(1-p_i) \tag{2.1.42}$$

which is Fermi-Dirac (see [11]) measure of entropy. Also,

$$\bar{f}'_1(1) = -\sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n}(1+p_i)\ln(1+p_i) - 2\ln 2 \tag{2.1.43}$$

is Bose-Einstein[11] measure of entropy.

2.2. Generating Functions for Measures of Information Based on Two Probability Distribution

Measure of Directed Divergence

Let

$$g_{\alpha,\beta}(t) = \frac{1}{\alpha-\beta}\left[\left(\sum_{i=1}^{n} p_i^{\alpha\beta} q_i^{1-\alpha\beta}\right)^t - \beta\right], \quad \alpha \neq \beta,\ \beta > 0 \tag{2.2.1}$$

Therefore

$$g_{\alpha,1}(1) = \frac{1}{\alpha-1}\left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1\right], \quad \alpha \neq 1 \tag{2.2.2}$$

is Havrda-Charvat (see [7]) measure of directed divergence. Also,

$$g'_{\alpha,1}(t) = \frac{1}{\alpha - 1}\left[(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha})^t \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}\right], \quad \alpha \neq 1, \tag{2.2.3}$$

so,

$$g'_{\alpha,1}(0) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}, \quad \alpha \neq 1 \tag{2.2.4}$$

is Renyi's (see [15]) measure of directed divergence. Moreover

$$g_{1,1}(1) = g_{\alpha,1}(1) = \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} (\ln p_i - \ln q_i). \tag{2.2.5}$$

Hence

$$g_{1,1}(1) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \tag{2.2.6}$$

is Kullback-Leibler's (see [14]) measure of directed divergence.

Again, we define

$$g_{\alpha,\beta,\gamma}(t) = \frac{1}{\alpha - \beta}\left[\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{2-\overline{\alpha+\beta}}\right)^t - \gamma\right], \quad \alpha \neq \beta, \ \beta\gamma > 0. \tag{2.2.7}$$

So

$$g_{\alpha,1,1}(1) = \frac{1}{\alpha - 1}(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1), \quad \alpha \neq 1 \tag{2.2.8}$$

is Havrda-Charvat (see [7]) measure of directed divergence.

Now

$$g'_{\alpha,1,1}(0) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}, \ \alpha \neq 1 \tag{2.2.9}$$

which is Renyi's[15] measure of directed divergence, and

$$\begin{aligned} g_{1,1,1}(1) &= g'_{\alpha,1,1}(1) \\ &= \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} \frac{1}{\sum\limits_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}} \sum_{i=1}^{n} (p_i^{\alpha} \ln p_i q_i^{1-\alpha} - p_i^{\alpha} q_i^{1-\alpha} \ln q_i) \right. \\ &\qquad \left. + \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} \sum_{i=1}^{n} (p_i^{\alpha} \ln p_i q_i^{1-\alpha} - p_i^{\alpha} q_i^{1-\alpha} \ln q_i)\right]. \end{aligned} \tag{2.2.10}$$

Therefore

$$g'_{1,1,1}(1) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \tag{2.2.11}$$

is Kullback-Leibler's (see [14]) measure of directed divergence.

Now define

$$g_{\alpha,\beta}(t) = \frac{1}{\alpha-\beta}\left[\left(\sum_{i=1}^{n} p_i^{\alpha/\beta} q_i^{1-\alpha/\beta}\right)^t - \beta\right], \quad \alpha \neq \beta,\ \beta > 0 \tag{2.1.12}$$

then

$$g_{\alpha,1}(1) = \frac{1}{\alpha-1}\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1\right), \quad \alpha \neq 1 \tag{2.2.13}$$

is Havrda-Charvat (see [7]) measure of directed divergence. Also

$$g'_{\alpha,1}(0) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}, \quad \alpha \neq 1 \tag{2.2.14}$$

is Renyi's (see [15]) measure of directed divergence. Hence

$$g_{1,1}(1) = g_{\alpha,1}(1) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \tag{2.2.15}$$

is Kullback-Leibler's[14] measure of directed divergence.

Let

$$g_{\alpha}(t) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i^{\alpha} - q_i^{1-\alpha}\right)^t\right], \quad \alpha < 1. \tag{2.2.16}$$

Therefore

$$g_{1/2}(2) = 2\sum\nolimits_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 \tag{2.2.17}$$

is two times of Bhattacharya's (see [3]) measure of directed divergence.

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