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ON NEW INEQUALITIES OF NEWTON'S TYPE FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE CONVEX

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Abstract: In this paper, we obtain some new inequalities of Newton's type based on convexity. Some applications for special cases of real functions are also given.

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1. Introduction

The following inequality is well known in the literature as Newton's inequality.

Theorem 1.1. *Let f: [a, b]⟶ $\Re$ be a four times continuously differentiable mapping on the interval (a, b) and for the fourth derivative to be bounded on (a, b), that is*

$$\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty.$$

Then the following inequality holds:

$$\left|\frac{1}{8}\big[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{6480}\|f^{(4)}\|_\infty (b-a)^4. \quad (1.1)$$

For recent results, generalizations and new inequalities related to the trapez-

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ium's (Hermite-Hadamard type) and Simpson's types and applications for special cases (see [2,3,5,6] and therein).

In [4], G. Toader defined the concept of $m-convexity$ as the following:

Definition 1.1. The function f: [0,b]→ $\Re$ is said to be $m-convex$, where $m \in (0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

The purpose of the present paper is to establish Newton's type inequalities for the class of functions whose derivatives in absolute value at certain powers are convex functions.

2. Main Results

In order to prove our main theorems, we need the following Lemma.

Lemma 2.1. *Let $f : I^0 \subset \Re \to \Re$ be twice differentiable mapping on I^0 such that $f''(x) \in L_1[a,b]$, where $a, b \in I^0$ with $a < b$, then the following equality holds:*

$$\frac{1}{8}\left[f(a) + 3f(\frac{2a+b}{3}) + 3f(\frac{a+2b}{3}) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= (b-a)^2 \int_0^1 k(t) f''(tb + (1-t)a)dt, \quad (2.1)$$

where

$$k(t) = \begin{cases} -\frac{t^2}{2} + \frac{t}{8}, & t \in [0, \frac{1}{3}] \\ -\frac{t^2}{2} + \frac{t}{2} - \frac{1}{8}, & t \in [\frac{1}{3}, \frac{1}{3}] \\ -\frac{t^2}{2} + \frac{7t}{8} - \frac{3}{8}, & t \in [\frac{2}{3}, 1]. \end{cases}$$

Proof. By definition of $k(t)$, we have

$$I = \int_0^1 k(t) f''(tb + (1-t)a)dt$$
$$= \int_0^{\frac{1}{3}} (-\frac{t^2}{2} + \frac{t}{8}) f''(tb + (1-t)a)dt$$
$$+ \int_{\frac{1}{3}}^{\frac{2}{3}} (-\frac{t^2}{2} + \frac{t}{2} - \frac{1}{8}) f''(tb + (1-t)a)dt$$

$$
\begin{aligned}
&+\int_{\frac{2}{3}}^{1}(-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8})f''(tb+(1-t)a)dt\\
=&I_1+I_2+I_3. \qquad (2.2)
\end{aligned}
$$

Integrating by parts twice, we have:

$$
\begin{aligned}
I_1=&\int_0^{\frac{1}{3}}(-\frac{t^2}{2}+\frac{t}{8})f''(tb+(1-t)a)dt\\
=&-\frac{1}{72(b-a)}f'(\frac{2a+b}{3})+\frac{1}{b-a}\int_0^{\frac{1}{3}}f'(tb+(1-t)a)(t-\frac{1}{8})dt\\
=&-\frac{1}{72(b-a)}f'(\frac{2a+b}{3})+\frac{5}{24(b-a)^2}f(\frac{2a+b}{3})+\frac{1}{8(b-a)^2}f(a)\\
&-\frac{1}{(b-a)^2}\int_0^{\frac{1}{3}}f(tb+(1-t)a)dt. \qquad (2.3)
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
I_2=&\int_{\frac{1}{3}}^{\frac{2}{3}}(-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{8})f''(tb+(1-t)a)dt\\
=&\frac{1}{72(b-a)}f'(\frac{2a+b}{3})-\frac{1}{72(b-a)}f'(\frac{a+2b}{3})\\
&+\frac{1}{b-a}\int_{\frac{1}{3}}^{\frac{2}{3}}f'(tb+(1-t)a)(t-\frac{1}{2})dt\\
=&\frac{1}{72(b-a)}f'(\frac{2a+b}{3})-\frac{1}{72(b-a)}f'(\frac{a+2b}{3})\\
&+\frac{1}{6(b-a)^2}\left[f(\frac{2a+b}{3})+f(\frac{a+2b}{3})\right]\\
&-\frac{1}{(b-a)^2}\int_{\frac{1}{3}}^{\frac{2}{3}}f(tb+(1-t)a)dt. \qquad (2.4)
\end{aligned}
$$

$$
I_3=\int_{\frac{2}{3}}^{1}(-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8})f''(tb+(1-t)a)dt
$$

$$
=\frac{1}{72(b-a)}f'(\frac{a+2b}{3})+\frac{1}{b-a}\int_{\frac{2}{3}}^{1}f'(tb+(1-t)a)(t-\frac{7}{8})dt
$$

$$=\frac{1}{72(b-a)}f'(\frac{a+2b}{3})+\frac{5}{24(b-a)^2}f(\frac{a+2b}{3})+\frac{1}{8(b-a)^2}f(b)$$
$$-\frac{1}{(b-a)^2}\int_{\frac{2}{3}}^{1} f(tb+(1-t)a)dt.$$

Adding (2.3), (2.4) and (2.5),

$$I=I_1+I_2+I_3=\frac{1}{8(b-a)^2}\left[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\right]$$
$$-\frac{1}{(b-a)^2}\int_0^1 f(tb+(1-t)a)dt.$$

Using the change of the variable $x=tb+(1-t)a$ for $t\in[0,1]$ and multiplying the both sides by $(b-a)^2$, we obtain (2.1) which completes the proof. □

The next theorems give a Newton inequality for twice differentiable functions:

Theorem 2.1. *Let $f: I^0\subset\Re\to\Re$ be twice differentiable mapping on I^0 such that $f''(x)\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''(x)|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{(b-a)^2}{384}\left[|f''(a)|+|f''(b)|\right]. \quad (2.6)$$

Proof. From Lemma 2.1 and by used convexity of $|f''(x)|$, we get

$$\left|\frac{1}{8}\left[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq(b-a)^2\int_0^1 |k(t)||f''(tb+(1-t)a)dt|$$
$$\leq(b-a)^2\Big\{\int_0^{\frac{1}{3}}\left|-\frac{t^2}{2}+\frac{t}{8}\right|\left[\,t|f''(b)|+(1-t)|f''(a)|\,\right]dt$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{8}\right|\left[\,t|f''(b)|+(1-t)|f''(a)|\,\right]dt$$
$$+\int_{\frac{2}{3}}^{1}\left|-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8}\right|\left[\,t|f''(b)|+(1-t)|f''(a)|\,\right]dt\Big\}$$

$$= (b-a)^2(J_1 + J_2 + J_3),$$

where

$$J_1 = \int_0^{\frac{1}{3}} \left| -\frac{t^2}{2} + \frac{t}{8} \right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt,$$

$$J_2 = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| -\frac{t^2}{2} + \frac{t}{2} - \frac{1}{8} \right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt$$

and

$$J_3 = \int_{\frac{2}{3}}^{1} \left| -\frac{t^2}{2} + \frac{7t}{8} - \frac{3}{8} \right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt.$$

After simple computation, we obtain:

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{3}} \left| -\frac{t^2}{2} + \frac{t}{8} \right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \\ &= \int_0^{\frac{1}{4}} \frac{t}{2}\left(\frac{1}{4} - t\right) \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \\ &\quad + \int_{\frac{1}{4}}^{\frac{1}{3}} \frac{t}{2}\left(t - \frac{1}{4}\right) \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \\ &= \frac{125}{256 \times 324}|f''(a)| + \frac{1}{256 \times 12}|f''(b)|, \end{aligned}$$

$$\begin{aligned} J_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left| -\frac{t^2}{2} + \frac{t}{2} - \frac{1}{8} \right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \\ &= \frac{1}{8}\int_{\frac{1}{3}}^{\frac{2}{3}} (2t-1)^2 \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \\ &= \frac{1}{16 \times 81}[|f''(a)| + |f''(b)|], \end{aligned}$$

and

$$\begin{aligned} J_3 &= \int_{\frac{2}{3}}^{1} \left| -\frac{t^2}{2} + \frac{7t}{8} - \frac{3}{8} \right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \\ &= \int_{\frac{2}{3}}^{\frac{3}{4}} \frac{1}{8}(1-t)(3-4t) \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| \right] dt \end{aligned}$$

$$+\int_{\frac{3}{4}}^{1}\frac{1}{8}(1-t)(4t-3)\big[\ t\big|f''(b)\big|+(1-t)\big|f''(a)\big|\ \big]dt$$
$$=\frac{1}{12\times 256}|f''(a)|+\frac{125}{256\times 324}|f''(b)|,$$

which completes the proof. □

Theorem 2.2. *Let $f: I^0\subset\Re\to\Re$ be twice differentiable mapping on I^0 such that $f''(x)\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''(x)|^q$ is a convex on $[a,b]$ and $q\geq 1$, then the following inequality holds:*

$$|\frac{1}{8}\big[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\big]-\frac{1}{b-a}\int_a^b f(x)dx|$$
$$\leq(b-a)^2\{(\frac{19}{128\times 81})^{1-\frac{1}{q}}\big[(\frac{1}{12\times 256}|f''(b)|^q+\frac{125}{256\times 324}|f''(a)|^q)^{\frac{1}{q}}$$
$$+(\frac{1}{12\times 256}|f''(a)|^q+\frac{125}{256\times 324}|f''(b)|^q)^{\frac{1}{q}}\big]+\frac{1}{648}(\frac{|f''(a)|^q+|f''(b)|^q}{2})^{\frac{1}{q}}\}. \tag{2.7}$$

Proof. From Lemma 2.1 and by used convexity of $|f''(x)|^q$ and Cauchy-Schwartz, we get

$$|\frac{1}{8}\big[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\big]-\frac{1}{b-a}\int_a^b f(x)dx|$$
$$\leq(b-a)^2\int_0^1|k(t)||f''(tb+(1-t)a)dt|$$
$$=(b-a)^2\{\int_0^{\frac{1}{3}}\big|-\frac{t^2}{2}+\frac{t}{8}\big|\big|f''(tb+(1-t)a)\big|dt$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\big|-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{8}\big|\big|f''(tb+(1-t)a)\big|dt$$
$$+\int_{\frac{2}{3}}^{1}\big|-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8}\big|\big|f''(tb+(1-t)a)\big|dt\}$$
$$\leq(b-a)^2\{(\int_0^{\frac{1}{3}}\big|-\frac{t^2}{2}+\frac{t}{8}\big|dt)^{1-\frac{1}{q}}(\int_0^{\frac{1}{3}}\big|-\frac{t^2}{2}+\frac{t}{8}\big|\big|f''(tb+(1-t)a)\big|^q dt)^{\frac{1}{q}}$$
$$+(\int_{\frac{1}{3}}^{\frac{2}{3}}\big|-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{8}\big|dt)^{1-\frac{1}{q}}(\int_{\frac{1}{3}}^{\frac{2}{3}}\big|-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{8}\big|\big|f''(tb+(1-t)a)\big|^q)dt)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8}\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\Big\}$$

$$\leq (b-a)^2\Big\{\left(\frac{19}{128\times 81}\right)^{1-\frac{1}{q}}\Big[\left(\frac{125}{256\times 324}|f''(a)|^q+\frac{1}{256\times 12}|f''(b)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{12\times 256}|f''(a)|^q+\frac{125}{256\times 324}|f''(b)|^q\right)^{\frac{1}{q}}\Big]+\frac{1}{648}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\Big\}.$$

□

Remark 2.1. In Theorem 2.2, if $q=1$, then we have the inequality of (2.6).

Corollary 2.1. *In Theorem 2.2, if $|f''(x)|\leq K$, then we have*

$$\left|\frac{1}{8}\big[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\big]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2K}{192}$$

Theorem 2.3. *Let $f: I^0\subset\Re\to\Re$ be twice differentiable mapping on I^0 such that $f''(x)\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''(x)|$ is a m-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{8}\big[f(a)+3f(\frac{2a+b}{3})+3f(\frac{a+2b}{3})+f(b)\big]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{(b-a)^2}{384}\big[m|f''(\frac{a}{m})|+|f''(b)|\big].$$

Proof. With the m-convexity of $|f''(x)|$, we can complete the proof as similarly as Theorem 2.1. □

3. Applications to Special Cases

We shall consider the following special cases:

(a) The arithmetic mean:

$$A=A(a;b):=\frac{a+b}{2},\quad a,b>0.$$

(b) The harmonic mean:

$$H=H(a;b):=\frac{2ab}{a+b},\quad a,b>0.$$

(c) The weighted arithmetic mean:

$$W = W(\omega_1, \omega_2, a, b) := \frac{\omega_1 a + \omega_2 b}{\omega_1 + \omega_2}, \quad \omega_1, \omega_2, a, b, > 0.$$

(d) The logarithmic mean:

$$L = L(a; b) := \begin{cases} a, & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b, \end{cases} \qquad a, b > 0.$$

(e) The identric mean:

$$I = I(a, b) := 1/e(b^b/a^a)^{1/(b-a)}, \quad a, b > 0.$$

(f) The p-logarithmic mean:

$$L_p = L_p(a; b) := \begin{cases} a, & \text{if } a = b, \\ (\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)})^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases} \qquad p \in \Re - \{-1, 0\}, \ a, b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \Re$ with $L_{-1} := L$ and $L_0 := I$, In particular, we have the following inequalities

$$\min\{a, b\} < H(a, b) < L(a, b) < I(a, b) < A(a, b) < \max\{a, b\}.$$

Now, using the results of Section 2, some new inequalities is derived for the above cases.

Proposition 3.1. *Let $a, b \in \Re, 0 < a < b$ and $n \in \mathbb{N}, n > 2$. Then we have*

$$|\frac{1}{4}A(a^n, b^n) + \frac{3}{8}[W^n(2, 1, a, b) + W^n(1, 2, a, b)] - L_n^n(a, b)| \\ \leq \frac{(b-a)^2}{384} n(n-1)(a^{n-2} + b^{n-2}).$$

Proof. The assertion follows from Theorem 2.1 applied to convex mapping $f(x) = x^n, x \in [a, b]$ and $n \in \mathbb{N}$. □

Proposition 3.2. *Let $a, b \in \Re, 0 < a < b$, then we have*

$$|\frac{1}{4}H^{-1}(a, b) + \frac{3}{8}[W^{-1}(2, 1, a, b) + W^{-1}(1, 2, a, b)] - L^{-1}(a, b)| \\ \leq \frac{(b-a)^2}{96} H^{-1}(a^3, b^3).$$

Proof. The assertion follows from Theorem 2.1 applied to convex mapping $f(x) = \frac{1}{x}, x \in [a, b]$. □

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