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THE DISCRIMINANT OF A TRINOMIAL

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Abstract: In this paper we calculate the discriminant of a trinomial using two different methos and as an application we compute the discriminant of a cyclotomic number field.

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1. Introduction

One of the main problems in algebraic number theory is to compute the discriminant δ_K of a given number field K. However, this is not an easy task. In approaching this problem, it is useful to compute the discriminant of the polynomial defining such a field, since this is a square multiple of δ_K. In general, it is difficult to compute the discriminant of a polynomial, however, for special cases this can be done effectively in terms of the coefficients of the polynomial under consideration, that is the case for a trinomial. In this paper we compute the discriminant of a trinomial using two different methods.

2. First Method

For a given field K, let us denote by $\overline{K}$ a fixed algebraic closure of K and let $f(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n(x - \alpha_1)\cdots(x - \alpha_n)$ and $g(x) = b_m x^m + \cdots + b_1 x + b_0 = b_m(x - \beta_1)\cdots(x - \beta_m)$ be polynomials in $K[x]$, with α_i and β_j elements of $\overline{K}$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We recall one of the equivalent definition of the resultant of f and g which is a useful concept to identify common roots of polynomials. Let us denote the resultant of f and g by $R(f, g)$. Then:

$$R(f, g) := a_n^m \prod_{i=1}^{n} g(\alpha_i).$$

A straightforward calculation shows that

$$R(g, f) = (-1)^{mn} R(f, g). \tag{1}$$

If the polynomial f is as above, then the discriminant of f is defined by $D_f := a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2$. From the very definition of D_f, it is clear that f has multiple roots $\iff$ $D_f = 0$. Also, it is well known that the derivative of f provides a criterion to decide if a polynomial has multiple roots, actually the relationship between the discriminant of a polynomial and the resultant of f and its derivative is obtained in the following lines.

One has

$$f'(x) = a_n \sum_{j=1}^{n} \prod_{j \neq i} (x - \alpha_i),$$

hence

$$f'(\alpha_j) = a_n \prod_{i \neq j} (\alpha_j - \alpha_i).$$

Using this last equation and the definition of the resultant we have:

$$\begin{aligned} R(f, f') &= a_n^{n-1} \prod_{j=1}^{n} f'(\alpha_j) \\ &= a_n^{n-1} \prod_{j=1}^{n} \left(a_n \prod_{i \neq j} (\alpha_j - \alpha_i) \right) \\ &= (-1)^{n(n-1)/2} a_n D_f. \end{aligned}$$

Therefore, from this and Equation 1 we have

$$D_f = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n} R(f', f) \tag{2}$$

Our first method to compute the discriminant of a trinomial is based on Equation 2.

Theorem 1. *Let K be a field, $f(x) = x^n + Ax^k + B \in K[x]$ a trinomial. Assume that the characteristic of K does not divide $n(n-k)$ and let d be the gcd of n and $n-k$. Then the discriminant of f is given by:*

$$D_f = n^n (-1)^{\frac{n(n-1)}{2}} B^{k-1} \left[B^{\frac{n-k}{d}} - (-1)^{\frac{n}{d}} A^{\frac{n}{d}} \left(1 - \frac{k}{n}\right)^{\frac{n-k}{d}} \left(\frac{k}{n}\right)^{\frac{k}{d}} \right]^d. \tag{3}$$

Proof. (First Proof) In order to apply Equation 2, we need to compute the roots of $f'(x) = nx^{n-1} + kAx^{k-1} = 0$, and this can be attained by a straightforward computation, that is, the roots of f' are: $\beta_j = \zeta_{n-k}^j \sqrt[n-k]{-\frac{kA}{n}}$, $j = 0, \ldots, n-k-1$, and $\beta_j = 0$ for $j = n-k, \ldots, n-2$, where ζ_{n-k} denotes a primitive $(n-k)-$th root of unity. From this we have for $j = 0, \ldots, n-k-1$:

$$\begin{aligned} f(\beta_j) &= \left(\zeta_{n-k}^{j+\frac{1}{2}} \left(\frac{kA}{n}\right)^{\frac{1}{n-k}} \right)^n + A \left(\zeta_{n-k}^{j+\frac{1}{2}} \left(\frac{kA}{n}\right)^{\frac{1}{n-k}} \right)^k + B \\ &= -\zeta_{n-k}^{kj} \zeta_{n-k}^{\frac{k}{2}} \left(\frac{kA}{n}\right)^{\frac{k}{n-k}} \left(\frac{kA}{n}\right) + \zeta_{n-k}^{kj} \zeta_{n-k}^{\frac{k}{2}} \left(\frac{kA}{n}\right)^{\frac{k}{n-k}} A + B \\ &= \zeta_{n-k}^{kj} \zeta_{n-k}^{\frac{k}{2}} \left(\frac{kA}{n}\right)^{\frac{k}{n-k}} \left(-\frac{k}{n} + 1\right) A + B \\ &= \zeta_{n-k}^{kj} \zeta_{n-k}^{\frac{k}{2}} \left(\frac{kA}{n}\right)^{\frac{k}{n-k}} \left(\frac{n-k}{n}\right) A + B. \end{aligned}$$

Set $A' = \zeta_{n-k}^{\frac{k}{2}} \left(\frac{kA}{n}\right)^{\frac{k}{n-k}} \left(\frac{n-k}{n}\right) A$ and $B' = \dfrac{B}{A'}$, then we have: $f(\beta_j) = A'(\zeta_{n-k}^{kj} + B')$, for $j = 0, \ldots, n-k-1$ and $f(\beta_j) = B$, for $j = n-k, \ldots, n-2$.

From above and the definition of the resultant we have:

$$\begin{aligned} R(f', f) &= \prod_{j=0}^{n-2} f(\beta_j) \\ &= \prod_{j=0}^{n-k-1} (A'(\zeta_{n-k}^{kj} + B')) \prod_{j=n-k}^{n-2} B \\ &= (A')^{n-k} B^{k-1} \prod_{j=0}^{n-k-1} (\zeta_{n-k}^{kj} + B'). \end{aligned} \tag{4}$$

The product $\prod_{j=0}^{n-k-1} (\zeta_{n-k}^{kj} + B')$ can be obtained as follows: let $d = \gcd(k, n-k)$, then by the division algorithm applied to j and $\dfrac{n-k}{d}$ we have $j = q(\frac{n-k}{d}) + r$, with $r \in \{0, 1, \ldots, \frac{n-k}{d} - 1\}$. Notice that if $j \in \{0, 1, \ldots, n-k-1\}$ then q varies from 0 to $d-1$ in the previous representation. If r is fixed and q is an integer between 0 and $d-1$, then $\zeta_{n-k}^{kj} = \zeta_{n-k}^{k(\frac{n-k}{d}q + r)} = \zeta_{n-k}^{kr}$, hence we can compute the desired product by considering the polynomial $g(x) := (x - B')^{\frac{n-k}{d}} - 1$, whose roots are $\zeta_{n-k}^{kj} + B' = \zeta_{n-k}^{kr} + B'$, $r = 0, \ldots, \frac{n-k}{d} - 1$.

From this we obtain

$$\begin{aligned} \prod_{j=0}^{\frac{n-k}{d}-1} (\zeta_{n-k}^{jk} + B') &= \prod_{r=0}^{\frac{n-k}{d}-1} (\zeta_{n-k}^{rk} + B') \\ &= (-1)^{\frac{n-k}{d}} g(0) \\ &= (-1)^{\frac{n-k}{d}} \left((-1)^{\frac{n-k}{d}} (B')^{\frac{n-k}{d}} - 1\right) \\ &= (B')^{\frac{n-k}{d}} - (-1)^{\frac{n-k}{d}}. \end{aligned}$$

Consequently,

$$\prod_{j=0}^{n-k-1} (\zeta_{n-k}^{kj} + B') = [(B')^{\frac{n-k}{d}} - (-1)^{\frac{n-k}{d}}]^d.$$

From Equation 4 and the previous, one has

$$R(f, f') = B^{k-1} (A')^{n-k} [(B')^{\frac{n-k}{d}} - (-1)^{\frac{n-k}{d}}]^d$$

$$\begin{aligned}
&= B^{k-1}[(A'B')^{\frac{n-k}{d}} - (-1)^{\frac{n-k}{d}}(A')^{\frac{n-k}{d}}]^d \\
&= B^{k-1}\left[B^{\frac{n-k}{d}} - (-1)^{\frac{n-k}{d}} A^{\frac{n-k}{d}} \left(\frac{n-k}{n}\right)^{\frac{n-k}{d}} \left(\frac{kA}{n}\right)^{\frac{k}{d}} \zeta_{n-k}^{\frac{k}{2}\frac{n-k}{d}}\right]^d \\
&= B^{k-1}\left[B^{\frac{n-k}{d}} - (-1)^{\frac{n-k}{d}} A^{\frac{n}{d}} \left(1-\frac{k}{n}\right)^{\frac{n-k}{d}} \left(\frac{k}{n}\right)^{\frac{k}{d}} (-1)^{\frac{k}{d}}\right]^d \\
&= B^{k-1}\left[B^{\frac{n-k}{d}} - (-1)^{\frac{n}{d}} A^{\frac{n}{d}} \left(1-\frac{k}{n}\right)^{\frac{n-k}{d}} \left(\frac{k}{n}\right)^{\frac{k}{d}}\right]^d .
\end{aligned}$$

Finally, from Equation 2 we have

$$D_f = n^n(-1)^{\frac{n(n-1)}{2}} B^{k-1}\left[B^{\frac{n-k}{d}} - (-1)^{\frac{n}{d}} A^{\frac{n}{d}} \left(1-\frac{k}{n}\right)^{\frac{n-k}{d}} \left(\frac{k}{n}\right)^{\frac{k}{d}}\right]^d ,$$

as claimed. □

3. Second Method

Here we present a second method to compute the discriminant of an irreducible trinomial based on a result for evaluating determinants of special matrices, proposed by Greenfield and Drucker [1].

Assume that L/K is a finite separable extension of degree n. Then $L = K(\alpha)$, for some $\alpha \in L$. Let $f(x)$ be the irreducible polynomial of α over K.

Claim. If $N_{L/K}$ denotes the norm map from $L \to K$ then

$$D_f = (-1)^{\frac{n(n-1)}{2}} N_{L/K}(f'(\alpha)). \tag{5}$$

In fact, write $f(x) = \prod_{j=1}^{n}(x-\alpha_j)$, where $\alpha = \alpha_1$. The n different embeddings of L into an algebraic closure of K can be chosen so that $\sigma_i(\alpha_1) = \alpha_i$. It is straightforward to see that $f'(\alpha_j) = \prod_{i\neq j}(\alpha_j - \alpha_i)$ and $\sigma_k(f'(\alpha)) = f'(\alpha_k)$. From this remarks we have:

$$\sigma_1(f'(\alpha)) \quad = \quad (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n)$$

$$\begin{aligned}\sigma_2(f'(\alpha)) &= (\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\cdots(\alpha_2-\alpha_n)\\ \sigma_3(f'(\alpha)) &= (\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\cdots(\alpha_3-\alpha_n)\\ &\vdots\\ \sigma_n(f'(\alpha)) &= (\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\cdots(\alpha_n-\alpha_{n-1})\end{aligned}$$

Notice that each factor of $f'(\alpha_1)$ appears exactly once in $f'(\alpha_i)$ with opposite sign, $i>1$. All but the first factor of $f'(\alpha_2)$ appear in $f'(\alpha_i)$ exactly once with opposite sign, $i>2$, so when multiplying out $\prod_{j=1}^{n}\sigma_j(f'(\alpha))$, one obtains, $N(f'(\alpha))=\prod_{j=1}^{n}\sigma_j(f'(\alpha))=(-1)^l\prod_{i\neq j}(\alpha_i-\alpha_j)^2$, where l is the number of factors that have changed sign and can be counted as follows: in $\sigma_1(f'(\alpha))$ there were $n-1$; in $\sigma_2(f'(\alpha))$, there were $n-2$, and so, $l=1+2+\cdots+(n-1)=\frac{n(n-1)}{2}$, proving the claim.

3.1. Evaluating Norms of Elements

With the notation and assumptions as above, if $\beta\in L$, then β induces a K-lineal transformation $T_\beta : L\to L$ given by $T_\beta(\theta)=\beta\theta$. We shall denote the **characteristic polynomial** of T_β by $f_\beta(x)=\det(T_\beta-xI)$. Set $M=K(\beta)$ and let $\tau_1,\ldots,\tau_m$ be the different K-embeddings of M in $\overline{K}$, then each τ_i has exactly $k=[K(\alpha):K(\beta)]$ extensions to $K(\alpha)$ into $\overline{K}$. More precisely $\{\tau_{ij}\}$, $i=1,\ldots,m$, $j=1,\ldots,k$ are the K-embeddings of $K(\alpha)$ in $\overline{K}$ and $\tau_{ij}=\tau_i$ when restricted to M; the equality holds for every $j=1,\ldots,k$.

if $N_{L/K}$ denotes the norm map from $L\to K$, then from above we have $N_{L/K}(\beta)=[N_{M/K}(\beta)]^k$.

We also know, from basic linear algebra, that the characteristic and the minimum polynomial of a linear transformation have the same irreducible factors. Hence $f_\beta(x)=[\mathrm{irr}(\beta,K)]^k$.

For $a\in K$ we have

$$\begin{aligned}N_{L/K}(\beta+a) &= \prod\tau_{ij}(\beta+a)\\ &= [(\tau_1(\beta)+a)]^k\cdots[\tau_m(\beta)+a]^k\\ &= (-1)^n g(-a)^k\\ &= (-1)^n f_\beta(-a),\end{aligned}\tag{6}$$

where $g(x)=\mathrm{irr}(\beta,K)$ is the irreducible polynomial of β over K.

Let $f(x) = x^n + Ax^k + B$ be an irreducible trinomial in $K[x]$ and let α be a root of f. From Equation 5, one obtains:

$$D_f = (-1)^{\frac{n(n-1)}{2}}(-1)^{n(k-1)}n^n(B)^{k-1}N_{L/K}\left(\alpha^{n-k} + \frac{k}{n}A\right) \tag{7}$$

If $f_{\alpha^{n-k}}(x)$ denotes the characteristic polynomial of $T_{\alpha^{n-k}}$, then, from Equations 6 y 7, we have:

$$D_f = (-1)^{\frac{n(n-1)}{2}}(-1)^{n(k-1)}n^n B^{k-1} f_{\alpha^{n-k}}\left(-\frac{k}{n}A\right) \tag{8}$$

3.2. Calculating D_f

Let $\mathcal{B} = \left\{1, \alpha, ..., \alpha^{n-1}\right\}$ be the power basis of L over K generated by α and consider the transformation $T_{\alpha^{n-k}} : L \to L$, $T_{\alpha^{n-k}}(x) = \alpha^{n-k}x$. In order to calculate the discriminant of $f(x) = x^n + Ax^k + B$ we shall find the matrix associated to the transformation $T_{\alpha^{n-k}}$ respect to the base $\mathcal{B}$, $[T_{\alpha^{n-k}}]_{\mathcal{B}}$. We have:

$$[T_{\alpha^{n-k}}]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \dots & 0 & -B & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -B & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -A & 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & -A & 0 & \dots & -B \\ 0 & 1 & \dots & 0 & 0 & 0 & -A & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & -A \end{pmatrix}.$$

This is an $n \times n$ tridiagonal matrix with coefficients in K whose supradiagonal and subdiagonal, in general, are not symmetric respect to the main diagonal.

The value $f_{\alpha^{n-k}}\left(-\frac{k}{n}A\right)$ is given by the following determinant

$$\Delta = \begin{vmatrix} \frac{k}{n}A & 0 & \dots & 0 & -B & 0 & 0 & \dots & 0 \\ 0 & \frac{k}{n}A & \dots & 0 & 0 & -B & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{k}{n}A & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \dots & 0 & \frac{k}{n}A - A & 0 & 0 & \dots & -B \\ 1 & 0 & \dots & 0 & 0 & \frac{k}{n}A - A & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \frac{k}{n}A - A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & \frac{k}{n}A - A \end{vmatrix}.$$

Set $[T_{\alpha^{n-k}}]_{\mathcal{B}} + \dfrac{k}{n}AI_n = (b_{ij})$, then applying Greenfield and Drucker method to evaluate determinants we have:

$$\Delta = \prod_{s=1}^{d}\left(\prod_{t=0}^{\frac{n}{d}-1} b_{dt+s,dt+s} + (-1)^{\frac{n}{d}-1}\prod_{t=0}^{\frac{n-k}{d}-1} b_{td+s,td+s+k}\prod_{t=\frac{n-k}{d}}^{\frac{n}{d}-1} b_{td+s,td+s-(n-k)}\right), \quad (9)$$

where $d = \gcd(n-k,k) = \gcd(n,k)$. The following remarks are useful to evaluate the above product.

1. The determinant has exactly $n-k$ elements on the supradiagonal, all equal to $-B$.

2. The subdiagonal has exactly k elements, all equal to 1.

3. By (2) above, we have $\displaystyle\prod_{t=\frac{n-k}{d}}^{\frac{n}{d}-1} b_{td+s,td+s-(n-k)} = 1$.

4. On the main diagonal, the first k elements are equal to $\frac{k}{n}A$ and the remaining $n-k$ elements are equal to:
$$\frac{k}{n}A - A = (-1)\left(\frac{(n-k)A}{n}\right).$$

From the remarks above, the second term inside parenthesis in Equation 9 is equal to $(-1)^{\frac{n-k}{d}}B^{\frac{n-k}{d}}$; the first is obtained from (4.) and by noting that it is a product of $\frac{n}{d}$ elements of the main diagonal. In the second term just obtained, the product must have exactly $\frac{n-k}{d}$ factors of type $(-1)\frac{(n-k)}{d}A$, consequently it has $\frac{n}{d} - \frac{n-k}{d} = \frac{k}{d}$ factors equal to $\frac{k}{n}A$. Then, the first term inside parenthesis is:

$$\frac{(-1)^{\frac{n-k}{d}}(n-k)^{\frac{n-k}{d}}A^{\frac{n-k}{d}}}{n^{\frac{n-k}{d}}}\left(\frac{k}{n}A\right)^{\frac{k}{d}} = (-1)^{\frac{n-k}{d}}\left(1-\frac{k}{n}\right)^{\frac{n-k}{d}}\left(\frac{k}{n}\right)^{\frac{k}{d}}A^{\frac{n}{d}}.$$

Using all these remarks we have:

$$\begin{aligned}\Delta &= \left((-1)^{\frac{n-k}{d}}\left(1-\frac{k}{n}\right)^{\frac{n-k}{d}}\left(\frac{k}{n}\right)^{\frac{k}{d}}A^{\frac{n}{d}} + (-1)^{\frac{n}{d}-1}(-1)^{\frac{n-k}{d}}B^{\frac{n-k}{d}}\right)^d \\ &= \left((-1)^{\frac{n-k}{d}}\left(1-\frac{k}{n}\right)^{\frac{n-k}{d}}\left(\frac{k}{n}\right)^{\frac{k}{d}}A^{\frac{n}{d}} + (-1)^{\frac{k+d}{d}}B^{\frac{n-k}{d}}\right)^d \\ &= (-1)^{k+d}\left(B^{\frac{n-k}{d}} + (-1)^{\frac{n}{d}}\left(1-\frac{k}{n}\right)^{\frac{n-k}{d}}\left(\frac{k}{n}\right)^{\frac{k}{d}}A^{\frac{n}{d}}\right)^d.\end{aligned}$$

Hence, from Equation 8 and using the fact that $nk+k+d-n$ is even, we have:

$$D_f = n^n(-1)^{\frac{n(n-1)}{2}}B^{k-1}\left[B^{\frac{n-k}{d}} - (-1)^{\frac{n}{d}}A^{\frac{n}{d}}\left(1-\frac{k}{n}\right)^{\frac{n-k}{d}}\left(\frac{k}{n}\right)\right]^d,$$

as claimed.

4. An Application

The importance of computing the discriminant of the cyclotomic field $\mathbb{Q}(\zeta_m)$ is well known; it is also known that its ring of integers is $\mathbb{Z}[\zeta_m]$, hence the discriminant of $\mathbb{Q}(\zeta_m)$ is the same as the discriminant of the m-th cyclotomic polynomial $\Phi_m(x)$. We also know that $\mathbb{Q}(\zeta_m)$ is the composita of the p^n-th cyclotomic fields, where p^n is the exact power of p dividing m, p a rational prime. From Theorem 7Q [4], to compute the discriminant of $\mathbb{Q}(\zeta_m)$, it is enough to compute the discriminant of $\mathbb{Q}(\zeta_{p^n})$, for p a rational prime.

4.1. The Discriminant of $\mathbb{Q}(\zeta_{p^n})$

For a given field K, let $f(x), f_1(x), f_2(x), \ldots, f_k(x) \in K[x]$ be monic polynomials such that

$$f(x) = \prod_{i=1}^{k} f_i(x),$$

then the roots of each $f_i(x)$ can be arranged in order to compute the discriminant of $f(x)$ and obtain the next relationship between D_f and the discriminants D_{f_i} for each $f_i(x)$

$$D_f = \prod_{i=1}^{k} D_{f_i} \prod_{1\leq i,j\leq k} R(f_i, f_j)^2. \tag{10}$$

A binomial is a particular case of a trinomial, hence applying Equation 3 to compute the discriminant of $f_m(x) = x^m - 1$ one obtains

$$D_{f_m} = (-1)^{\frac{m(m-1)}{2}}(-1)^{m-1}m^m.$$

Set $m = p^n$, and considering that $p^n - 1$ is even we have

$$D_{f_{p^n}} = (-1)^{\frac{p^n(p^n-1)}{2}} p^{np^n}.$$

Using the fact that $f_m(x) = x^m - 1 = \prod_{d|m} \Phi_d(x)$ and applying it to $f_{p^n}(x)$ one obtains

$$\begin{aligned} f_{p^n}(x) &= \prod_{d|p^n} \Phi_d(x) \\ &= \Phi_{p^n}(x) \prod_{d|p^{n-1}} \Phi_d(x) \\ &= \Phi_{p^n}(x)(x^{p^{n-1}} - 1), \end{aligned}$$

hence, $f_{p^n}(x) = \Phi_{p^n}(x) f_{p^{n-1}}(x)$, and from this

$$\Phi_{p^n}(x) = \frac{(x^{p^{n-1}})^p - 1}{x^{p^{n-1}} - 1} = \Phi_p(x^{p^n-1}) .$$

In order to find the resultant of $\Phi_{p^n}(x)$ and $f_{p^{n-1}}(x)$, let $\alpha_1, \alpha_2, \ldots, \alpha_{p^{n-1}}$ be the roots of $f_{p^{n-1}}(x)$, then $\alpha_i^{p^{n-1}} = 1$. The resultant can be obtained as follows

$$\begin{aligned} R(f_{p^{n-1}}(x), \Phi_{p^n}(x)) &= \prod_{i=1}^{p^{n-1}} \Phi_{p^n}(\alpha_i) = \prod_{i=1}^{p^{n-1}} \Phi_p(\alpha_i^{p^{n-1}}) \\ &= \prod_{i=1}^{p^{n-1}} \Phi_p(1) = p^{p^{n-1}} . \end{aligned}$$

Using Equation (10) for $f_{p^n}(x) = \Phi_{p^n}(x) f_{p^{n-1}}(x)$ one has:

$$D_{f_{p^n}} = D_{\Phi_{p^n}} D_{f_{p^{n-1}}} (R(f_{p^{n-1}}(x), \Phi_{p^n}(x)))^2$$

then

$$(-1)^{\frac{p^n(p^n-1)}{2}} p^{np^n} = D_{\Phi_{p^n}} (-1)^{\frac{p^{n-1}(p^{n-1}-1)}{2}} p^{(n-1)p^{n-1}} p^{2p^{n-1}}$$

and finally

$$\begin{aligned} D_{\Phi_{p^n}} &= (-1)^{\frac{p^{n-1}(p-1)}{2}} p^{p^{n-1}(np-n-1)} \\ &= (-1)^{\frac{\varphi(p^n)}{2}} \frac{p^{n\varphi(p^n)}}{p^{\frac{\varphi(p^n)}{p-1}}} . \end{aligned}$$

From the result above and Theorem 7Q [4], we have for an integer m:

Theorem 2. *The discriminant of the cyclotomic field* $\mathbb{Q}(\zeta_m)$ *is given by:*

$$\delta_m = (-1)^{\frac{\varphi(m)}{2}} \frac{m^{\varphi(m)}}{\prod_{p|m} p^{\frac{\varphi(m)}{p-1}}}.$$

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