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ATTRACTORS, APPROXIMATIONS AND FIXED SETS OF EVOLUTION SYSTEMS

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Abstract: In this paper we study autonomous evolution inclusions in an evolution triple, and satisfying one sided Lipschitzian condition with some negative constant. It is known that the solution set is compact on every bounded interval. Using this fact we prove the existence of a unique strong forward attractor and a unique strong backward attractor when the one sided Lipschitz constant is positive. As a corollary some surjectivity and fixed point results are proved. An example of a parabolic system, satisfying our assumptions is discussed.

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1. Introduction and Preliminaries

In this paper we study parabolic systems in the form of differential inclusions in evolution triple.

Given a separable Hilbert space H. Let X be a separable and reflexive Banach space embedded continuously and densely into H. Moreover, we assume that the embedding operator $i : \mathsf{X} \to H$ is compact. The triple $\mathsf{X} \subset H \subset \mathsf{X}^*$ is called evolution triple.

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Let $I = [0,T]$. We denote by $C_0^\infty(0,T)$ the strong inductive limit of $C_K^\infty(0,T)$ (the space of all functions with compact support which have derivatives of arbitrary order). Furthermore, $\mathcal{D}^*(0,T,\mathsf{X})$ is the space of all linear continuous maps from $C_K^\infty(0,T)$ into X. Let $h \in \mathcal{D}^*(0,T,\mathsf{X})$, and for any $k \ge 1$, $D^k h(\varphi) = (-1)^k h(D^k \varphi) \in \mathcal{D}^*(0,T,\mathsf{X})$ (the k^{th} distributional derivative of $h(\cdot)$) for all $\varphi \in C_0^\infty(0,T)$. The elements of $\mathcal{D}^*(0,T,\mathsf{X})$ are called vector valued distributions.

For $1 < p,\ q < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ we let $W_{pq} = \{x \in L^p(I,\mathsf{X}) :\ \dot{x} \in L^q(I,\mathsf{X}^*)\}$, where the derivatives are understood in the sense of vector valued distributions. Notice that $W_{pq} \subset C(I,H)$ and the embedding is continuous. We denote by $(\cdot,\cdot)$ the scalar product in H and by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X^*. Then for any $x,y \in W_{pq}$ we have $\dfrac{d}{dt}(x(t),y(t)) = \langle \dot{x}(t), y(t)\rangle + \langle x(t), \dot{y}(t)\rangle$ and $\dfrac{d}{dt}|x(t)|^2 = 2\langle \dot{x}(t), x(t)\rangle$.

We refer to [12] ch. 1 and to [20] ch. II sect. 7 and 8, where all facts of differential equations (inclusions) on evolution triples not given here can be founded (e.g. [21]). The terminology is adopted from [5].

Let E be a Banach space with dual E^*. For the closed bounded sets A, B we define $Ex(A,B) = \sup\limits_{a\in A}\inf\limits_{b\in B}\{|a-b|\}$, and $D_H(A,B) = \max\{Ex(A,B), Ex(B,A))$ is the Hausdorff distance. The support function is $\sigma(l,A) = \sup\limits_{a\in A}\langle l,a\rangle$, where $A \subset E$ and $l \in E^*$. Also by $\mathbb{B}_E$ we denote the closed unit ball in E centered in the origin. We will skip the index, when it is clear in what space the unit ball is contained.

Given an evolution triple $\mathsf{X} \subset H \subset \mathsf{X}^*$ and a compact metric space U with metric ρ_U. We study the evolution inclusion

$$\dot{x}(t) + A(x) \in F(x,u),\ x(0) = x^0,\ u \in U,\ t \in I. \tag{1}$$

Recall that the operator $A : \mathsf{X} \to \mathsf{X}^*$ is said to be hemicontinuous if $A(x+\lambda y) \to A(x)$ as $\lambda \to 0+$ with respect to the weak topology of X^* for every $x,y \in \mathsf{X}$. Notice that $u \in U$ is a parameter, however, (1) is not considered as a control system.

Definition 1.1. A function $x \in W_{pq}(I)$ is said to be a solution of (1) if there exists a measurable $f(t) \in F(x(t),u)$, such that $x(0) = x^0$ and $\dot{x}(t) + A(x(t)) = f(t)$.

We say that $x(\cdot)$ is a solution of (1) on $[0,+\infty)$, when it satisfies (1) on $[0,T]$ for every $T > 0$.

In the paper we show that under one sided Lipschitz assumption the system (1) admits nonempty solution set as well as a minimal invariant attractor. We extend the main results of [7, 9, 10]. The existence of attractors of dynamical systems is important and has been studied extensively. We notice the books [2, 4] and also the papers [1, 3, 16, 15, 18].

Suppose the following hypotheses **(SH)** hold:

H(A): $A : \mathsf{X} \to \mathsf{X}^*$ is hemicontinuous, monotone and moreover there exist positive constants C, c, β and γ such that:

(i) $\|A(x)\|_* \leq C(1 + \|x\|^{p-1})$, where $\|\cdot\|_*$ is the norm in X^*, $2 \leq p < \infty$.

(ii) $\langle A(x), x\rangle \geq c[x]^p$ for any $x \in X$, where $[\cdot]$ is the semi-norm of X such that $[x] + \beta|x| \geq \gamma\|x\|$ ($|\cdot|$ is the norm in H).

H(F) $F : H \times U \rightrightarrows H$ has nonempty closed convex values and it is bounded on the bounded sets.

(i) $F(\cdot, u)$ is Upper Hemicontinuous (UHC) (its support function is upper semi-continuous as a real valued function, or equivalently $F(\cdot, u)$ has a closed graph in $H \times H_w$), and $F(x, \cdot)$ is continuous. Here H_w is actually H equipped with the weak topology.

(ii) (One sided Lipschitz (OSL) condition) There exists a real constant $L > 0$ such that

$$\sigma(x - y, F(x, u)) - \sigma(x - y, F(y, u)) \leq -L|x - y|^2, \tag{2}$$

where σ is the support function that was defined above. The OSL condition is introduced in [6].

The reachable set of (1) is defined as follows:

$$\mathrm{Reach}_{x^0}(t) = \{y \in H : \ \exists \text{ solution } x(\cdot) \text{ of (1) with } x(t) = y\}.$$

Sometimes we will skip x^0 for notation convenience.

Definition 1.2. The closed set $V \subset H$ is said to be a strong attractor of (1) if and only if

1) $\lim\limits_{t\to\infty} \mathrm{Ex}(\mathrm{Reach}_{x^0}(t), V) = 0$ for every $x^0 \in H$,

2) $\mathrm{Reach}_V(t) = V$ for every $t > 0$, i.e. V is invariant.

If V satisfies only 1) and it is minimal, then V will be called minimal attractor.

V is said to be strong backward attractor when in 1) $t \to \infty$ is replaced by $t \to -\infty$ and in 2) $t > 0$ by $t < 0$. It is easy to see that backward attractor is a forward attractor of (1) if t is replaced by $-t$.

The following lemma is in fact a different form of Theorem 1.2.2 in [12].

Lemma 1.3. *Let A and F satisfy **(SH)** and let (2) be replaced by:*

$$\sigma(x-y, F(x)) - \sigma(x-y, F(y)) \leq L|x-y|^2.$$

Then for any $T > 0$ the set of solutions to (1) is nonempty and $C([0,T], H)$ compact.

Proof. Since $F(\cdot)$ is bounded on bounded sets, one has that there exists $\delta > 0$ such that the solution set of (1) is nonempty and $C([0,\delta], H)$ is a compact thanks to Theorem 1.2.2 in [12].

If $x(\cdot)$ is a solution of (1), then replacing y by 0 in (2) we get

$$\sigma(x, F(x)) - \sigma(x, F(0)) \leq L|x|^2.$$

Furthermore, $\langle x-y, Ay - Ax\rangle \leq 0$ and hence $\langle x, A0 - Ax\rangle \leq 0$. Consequently $\sigma(x, F(x) - Ax) - \sigma(x, F(0) - A0) \leq L|x|^2$.

Thus $\langle x(t), \dot{x}(t)\rangle \leq \sigma(x, F(x) - Ax) \leq L|x|^2 + \sigma(x, F(0) - A\,0)$, which implies that $\langle x, \dot{x}\rangle \leq L|x|^2 + |x|(|F(0)| + |A0|)$. The latter shows that $\dfrac{1}{2}\dfrac{d}{dt}|x(t)|^2 \leq L|x|^2 + |x|(|F(0)| + |A0|)$.

The set $\mathtt{P} = \{t \in I : \ |x(t)| \neq 0\}$ is an open set and hence it is a countable union of open intervals. Taking $\mathtt{P} = \displaystyle\bigcup_{i=1}^{\infty}(p_i, q_i)$ on every such an interval one has that $\dfrac{1}{2}\dfrac{d}{dt}|x(t)|^2 = |x(t)|\dfrac{d}{dt}|x(t)|$. Hence $\dfrac{d}{dt}|x(t)| \leq L|x(t)| + |F(0)| + |A0|$.

Let $v(\cdot)$ be a solution of $\dot{v}(t) = Lv(t) + |F(0)| + |A0|$, $v(0) = |x^0|$.
We know that $v(\cdot)$ is strongly increasing and hence $v(t) > 0$ for all $t > 0$, therefore either $|x(t)| = 0$ or $\dfrac{d}{dt}|x(t)| \leq \dot{v}(t)$. Hence $|x(t)| \leq v(t)$. Thus we have $|x(t)| \leq e^{Lt}(|F(0)| + |A\,0| + |x^0|)$, which shows that $x(\cdot)$ exists on $[0,\infty)$ as well as the solution set of (1) is $C([0,T], H)$ compact for any $T < \infty$. $\square$

2. Main Results

In this section we prove the existence of attractor to the system (1). We provide also an approximation scheme for the attractor which is more flexible than the scheme used in [10].

Theorem 2.1. *Under **(SH)** there exists a unique strong attractor of (1).*

Proof. Since u is fixed, we will skip it and write $F(x)$ instead of $F(x,u)$ for convenience.

From Lemma 1.3 we know that the solution set of (1) is nonempty $C([0,T],H)$ compact for every $T>0$. Furthermore if $x(\cdot)$ is a solution of (1), then similarly to the proof of Lemma 1.3 we have

$$\langle \dot{x}(t), x(t)\rangle \le \langle x(t), Ax(t) - A\ 0\rangle + \sigma(x(t), F(x(t))) - \sigma(x(t), F(0)),$$
$$\text{i.e. } \langle \dot{x}(t), x(t)\rangle \le -L|x(t)|^2 + |A\ 0|\cdot|x(t)| + |x(t)|\cdot|F(0)|.$$

Thus $\frac{d}{dt}|x(t)|^2 \le -2L|x(t)|^2 + 2|x(t)|(|A\ 0| + |F(0)|)$. We want to derive some a priori bound of $|x(t)|$ for any $t \in [0,\infty)$.

If $|x(t)| \ne 0$ then $\frac{d}{dt}|x| \le -L|x| + |A0| + |F(0)|$.

Since $\frac{d}{dt}|x(t)|^2 = 2|x(t)|\frac{d}{dt}|x(t)|$, one has that $\frac{d}{dt}|x(t)| \le -L|x(t)| + C$, where $C = |A0| + |F(0)|$.

Consequently for $|x(t)| \ne 0$ obtain that $|x(t)| \le \frac{C}{L}$ or $|x(\cdot)|$ decreases. It is not difficult to see that

$$|x(t)| \le \max\left\{|x^0|, \frac{C}{L}\right\} = M. \tag{3}$$

So one can assume without loss of generality that there exists a constant $K = K(x^0)$ such that $|F(x)| \le K$. Thus the solution set is not empty and it is $C([0,T],H)$ compact for every $T>0$. Consider the reachable set $\mathrm{Reach}(t)$. It follows from (3) that $|\mathrm{Reach}(t)| \le M$.

Next we fix both $\bar{t}$ and $\bar{s}$.

Let $\dot{x}(t) + Ax(t) = f_x(t)$, where $f_x(t) \in F(x(t))$ and let $x(0) = \bar{x}^0$. Define the multifunction:

$$G(t,y) := \{z \in F(y) :\ \langle x(t) - y, f_x(t) - z\rangle \le -L|x(t) - y|^2\}.$$

It is easy to see that $G(\cdot,\cdot)$ is almost UHC, i.e. for every interval $[0,T]$ and every $\varepsilon > 0$ there exists a compact $I_\varepsilon \subset [0,T]$ such that G is USC on $I_\varepsilon \times H$. Given y^0 there exists a solution $y(\cdot)$ of

$$\dot{y}(t) + Ay(t) \in G(t, y(t)),\ y(0) = \bar{y}^0.$$

Since $\langle x - y, Ax - Ay\rangle \ge 0$, one has that $\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle \le -L|x(t) - y(t)|^2$. Consequently, $\frac{d}{dt}|x(t) - y(t)|^2 \le -2L|x(t) - y(t)|^2$ and hence $|x(t) - y(t)| \le e^{-Lt}|x^0 - y^0|$. Thus $D_H(\mathrm{Reach}_{x^0}(t), \mathrm{Reach}_{y^0}(t)) \le e^{-Lt}|x^0 - y^0|$.

If $\bar{x}^0 \in \text{Reach}_{x^0}(\bar{t})$ and $\bar{y}^0 \in \text{Reach}_{x^0}(\bar{s})$ then $|x(t)-y(t)| \le 2Me^{-Lt} \to 0$ as $t \to \infty$. Therefore $D_H(\text{Reach}(\bar{t}+t, \text{Reach}(\bar{s}+t)) \le 2Me^{-Lt}$. It is easy to see that the net $\{\text{Reach}_{x^0}(t)\}_{t>0}$ is a Cauchy net. Thus there exists an attractor $V(u) = \Lambda = \lim\limits_{t\to\infty} \text{Reach}_{x^0}(t)$ such that it does not depend on x^0 and moreover, $D_H(\text{Reach}_{x^0}(t), \Lambda)$ is decreasing and $\dot{x} + Ax \in F(x)$; $x(0) \in \Lambda$. Thus the compact set Λ is the unique strong attractor of (1). □

We need the following variant of Filippov - Pliss lemma on infinite interval.

Lemma 2.2. ***(Lemma of Filippov - Pliss)*** *Under* ***(SH)*** *there exists a constant C such that for small $\varepsilon > 0$ if $x(\cdot)$ is a solution of $\dot{x} + Ax \in F(x + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B}$, $x(0) = x^0$ on $[0, \infty)$ then there exists a solution $y(\cdot)$ of $\dot{y} + Ay \in F(y)$, $y(0) = y^0$ on $[0, \infty)$ such that $|x(t) - y(t)| \le C\sqrt{\varepsilon}$ for all $t \in \mathbb{R}^+$.*

Proof. Denote by $G(y) = F(y) + \varepsilon\mathbb{B}$. If $\dot{x}(t) + Ax \in F(x + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B}$, then $\dot{x}(t) + Ax \in G(x(t) + g_x(t)\mathbb{B})$, where $|g_x(t)| \le \varepsilon$. Also $\dot{x} + Ax = f_x(t)$, where $f_x(t) \in G(x(t) + g_x(t)\mathbb{B})$. Define $R(t, y) := \{v \in G(y) : \langle x + f_x(t) - y, \dot{x}(t) + Ax - v\rangle \le -L|x + f_x(t) - y|^2\}$. It is easy to see that $R(\cdot,\cdot)$ is almost UHC with nonempty convex weakly compact values. Let $\dot{y}(t) + Ay \in R(t, y)$. Evidently $\sigma(l, G(x)) - \sigma(l, G(y)) = \sigma(l, F(x)) - \sigma(l, F(y))$, for any $l \in H$. Hence $\langle x(t) - y(t), f_x(t) - f_y(t)\rangle \le -L|x + f(t) - y|^2$. This implies that

$$\begin{aligned}
\langle x - y, f_x - f_y\rangle &\le -L|x - y|^2 + |L|(|x + f_x(t) - y|^2 - |x - y|^2) \\
&\le -L|x - y|^2 + |L|(|f_x(t)||2x + f_x(t) + 2y|) \\
&\le -L|x - y|^2 + |L|\varepsilon(2|x| + 2|y| + \varepsilon) \\
&\le -L|x - y|^2 + (4M + \varepsilon)\varepsilon|L|.
\end{aligned}$$

Here $M = \max\left\{|x^0|, \dfrac{C}{L}\right\}$ is similar to one in the proof ot Theorem 2.1. Consequently

$$\langle x - y, \dot{x}(t) - \dot{y}(t)\rangle \le -L|x - y|^2 + |L|(4M + \varepsilon)\varepsilon.$$

This implies that $|x - y|^2 \le v$, where $\dot{v} \le -2Lv + 2(4M + \varepsilon)L + \varepsilon$. Hence $v \le (4M + \varepsilon)\varepsilon$. It follows that $|x(t) - y(t)| \le \sqrt{(4M + \varepsilon)\varepsilon} = C\sqrt{\varepsilon}$.

If $x(\cdot)$ is a solution of $\dot{x} + Ax \in F(x + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B}$, then there exists a solution $y(\cdot)$ of $\dot{y} + Ay \in F(y) + \varepsilon\mathbb{B}$ such that $|x(t) - y(t)| \le C\sqrt{\varepsilon}$. Obviously $\dot{y} + Ay = f_1(y) + f_2(y)$, where $f_1(y) \in F(y)$ and $|f_2(y)| \le \varepsilon$. Further we define $S(t, u) := \{v \in F(u) : \langle y(t) - u, f_1(y) + f_2(y) - v\rangle \le -L|y(t) - u|^2 + |f_2(y)||y(t) - v|\}$.

Let $\dot{z} + Az \in S(t, z)$, $z(0) = x^0 = y^0$. Obviously, we have

$$\langle y(t) - z(t), \dot{y}(t) - \dot{z}(t)\rangle \quad \le \quad \langle y - z, Az - Ay\rangle - L|y - z|^2 + \varepsilon|y - z|$$

$$\leq \ -L|y-z|^2 + \varepsilon|y-z|.$$

As in the proof of Theorem 2.1 we derive $\frac{d}{dt}|y(t)-z(t)| \leq -L|y(t)-z(t)| + \varepsilon$.

Hence $|y(t)-z(t)| \leq \frac{\varepsilon}{L}$, and by the triangle inequality we obtain $|x(t) - z(t)| \leq C(\sqrt{\varepsilon}+\varepsilon)$. □

Notice that although many versions of the Filippov - Pliss lemma were proved in the case of finite intervals (cf. [8]) to our knowledge there are no such kind of results in the literature in the case of $\mathbb{R}^+$.

Now we prove some corollaries of Theorem 2.1.

Corollary 2.3. *Skip again u. Under **(SH)** the operator $O(z) = F(z) - Az$ is surjective. If the OSL condition in **(SH)** is relaxed to*

$$\sigma(x-y, F(x)) - \sigma(x-y, F(y)) \leq l|x-y|^2, \tag{4}$$

where $l < 1$, then $O(\cdot)$ has a fixed point.

Notice that the fixed point is in X.

Proof. Fix $h > 0$, and from the proof of Theorem 2.1 we have that

$$D_H(\mathrm{Reach}_{x^0}(h), \mathrm{Reach}_{y^0}(h)) \leq \eta(h)|x^0 - y^0| \text{ with } \eta(h) = e^{-Lh} \in (0,1).$$

If $x^0 \in \Lambda$ then $\mathrm{Reach}_{x^0}(h) \subset \Lambda$. Thus the multimap $x^0 \rightrightarrows \mathrm{Reach}_{x^0}(h)$ is a set valued contraction mapping Λ into itself.

Therefore there exists a fixed point $z \in \Lambda$ such that $z \in \mathrm{Reach}_z(h)$ and hence there exists a periodic solution $z_h(\cdot)$ of (1) with period h. Let $h \to 0^+$ then the net of the corresponding h–periodic solutions $\{z_h(\cdot)\}_{h>0}$ satisfies the conditions of Arzela-Ascoli's theorem. Hence $\lim\limits_{h\to 0^+} z_h(t) = z(t)$, which shows that $z(\cdot)$ is periodic with period 0. Consequently $z(t)$ is constant and hence $\dot{z}(t) = 0$, thus $0 \in F(z) - Az$. To prove surjectivity we replace $F(x)$ by $F(x) - p$. As it was shown there exists q such that $0 \in F(q) - Aq - p$, which implies that $p \in F(q) - Aq$.

In the case (4) we have $\sigma(x-y, F(x)-x) - \sigma(x-y, F(y)-y) \leq (\eta - 1)|x-y|^2$. If $G(x) = F(x) - x$, then there exists z such that $0 \in G(z) - Az$, i.e. $z \in F(z) - Az$. □

The following corollary would be of interest, because here $G(\cdot)$ does not satisfy (2) in general, although due to Theorem 2.1 it follows easily from the main results of [13, 14, 18]. Here we prefer to provide a different proof.

Corollary 2.4. *Let $G(x,u) \subset F(x,u)$ be UHC with convex, weakly compact values. Under (**SH**) for every u the evolution inclusion*

$$\dot{x}(t) + A(x) \in G(x,u), \ x(0) = x^0, \tag{5}$$

admits an unique minimal attractor $V_G(u) \subset V(u)$. Furthermore, $V_G(\cdot)$ is USC.

Proof. We skip u again. Since $G(x) \subset F(x)$, one has that the solution set of (5) is contained in the solution set of (1). Denote by $\text{Reach}(\hat{t}, G, x^0)$ the reachable set of (5) at $t = \hat{t}$. Let $V(u)$ be the unique strong attractor of (1), then

$$\lim_{t\to\infty} Ex(\text{Reach}(t,G,x^0), V(u)) = 0.$$

Hence $V(u)$ is also an attractor of (5), and $t \to Ex(\text{Reach}(t,G,x^0), V(u))$ is strictly decreasing. Fix $\varepsilon > 0$ and denote $\mathbb{A}(t,\varepsilon) = \overline{\bigcup_{s\geq t} \text{Reach}(t,G,V(u+\varepsilon\mathbb{B}))}$.

However, $\lim\limits_{t\to\infty} Ex(\mathbb{A}(t,\varepsilon), V_G) = 0$ and $\mathbb{A}(t,\varepsilon) \subset \mathbb{A}(s,\varepsilon)$ whenever $t > s$. Denote $V_G(\varepsilon) = \bigcap\limits_{t>0} \mathbb{A}(t,\varepsilon)$ therefore $V_G(\varepsilon) \neq \emptyset$, $V_G(\varepsilon) \subset V(u)$ and hence $V_G(\varepsilon)$ is nonempty compact set. For every x^0, $\varepsilon > 0$ there exists $T = T(x^0, \varepsilon)$ such that $Ex(\text{Reach}(t,G,x^0), V(u)) < \varepsilon$ for all $t > T$.

Hence for $t > T$ one has that $\text{Reach}(t,G,x_0) \subset V(u) + \varepsilon\mathbb{B}$.

Thus $\lim\limits_{t\to\infty} Ex(\text{Reach}(t,G,x^0), V_G(\varepsilon)) = 0$ and it holds for any $x^0 \in H$. Since $V_G(\varepsilon) \subset V_G(\delta)$ for $\varepsilon > \delta$, one has that $V_G = \bigcap\limits_{\varepsilon>0} V_G(\varepsilon) \neq \emptyset$ is a compact set and moreover, $\lim\limits_{\varepsilon\to 0} V_G(\varepsilon) = V_G$. ☐

In the same way one can prove the following corollary:

Corollary 2.5. *Let $G(x,u) \subset F(x,u)$ be LSC with closed values. Under (**SH**) for every u_0 the evolution inclusion (5) admits a unique minimal attractor $V_G(u_0) \subset V(u_0)$.*

As an immediate corollary of Lemma 2.2, we have the following theorem:

Theorem 2.6. *Under (**SH**) if*

$$\lim_{i\to\infty} D_H(Graph(F(\cdot,u_i)), Graph(F(\cdot,u)))_{M\mathbb{B}} = 0$$

for any bounded set $M\mathbb{B}$ and $u_i \to u$, then $\Lambda(u_i) \to \Lambda(u)$, where $\Lambda(u_i)$ is the unique invariant attractor of (1).

Proof. Let $D_H(GraphF, GraphG) < \varepsilon$ on $2M\mathbb{B}$. It follows from Lemma 2.2 that the distance between the solution sets of $\dot{x}+Ax \in F(x)$ and $\dot{y}+Ay \in G(y)$ is less than $C\sqrt{\varepsilon}$. Consequently $D_H(\mathrm{Reach}(F(t)), \mathrm{Reach}(G(t))) \le C\sqrt{\varepsilon}$ and hence $D_H(\Lambda(u_i), \Lambda(u)) \le C\sqrt{\varepsilon}$. □

The following two theorems extend Theorem 3.2 and Theorem 3.4 of [10].

Theorem 2.7. *Under **(SH)** there exists a closed and bounded set B^∞ such that $B^\infty = \lim_{t\to\infty} \mathrm{Reach}(t, x^0)$. Moreover,*

(i) B^∞ attracts the reachable sets, i.e. $D_H(\mathrm{Reach}_C(T), B^\infty) < D_H(C, B^\infty)$ for any bounded $C \neq B^\infty$ and $T > 0$;

(ii) $\mathrm{Reach}_{B^\infty}(T) = B^\infty$ for every $T > 0$;

(iii) B^∞ does not depend on x^0;

(iv) B^∞ is a strongly invariant set for (1), i.e., every solution $x(\cdot)$ of (1) with $x(0) \in B^\infty$ satisfies $x(t) \in B^\infty$ for every $t > 0$;

(v) B^∞ depends continuously on u.

Consider the sequence $\{t_k\}_{k=1}^\infty$ with $t_i < t_{i+1}$ in $[0, +\infty)$ and $\lim_{k\to\infty} t_k = +\infty$. Fix $u \in U$. We study the discrete problem

$$\dot{x}(t) \in -Ax_i + F(x_i, u),\ x(0) = x^0,\ x_i = x(t_i),\ t \in (t_i, t_{i+1}], \tag{6}$$

at fixed $u \in U$. Further, we will show that the reachable set of the evolution inclusion (6) tends to the attractor of (1) for appropriately chosen sequence of subdivisions, i.e. we approximate the attractor of (1).

Theorem 2.8. *Under (**SH**) there exist approximation steps $t_{i+1} - t_i$ such that the system (6) has the same strong attractor as (1).*

Proof. Let $\dot{x}(\cdot)$ be a solution of (1). Then we construct an approximate solution as follows.

Assume that the approximate solution $y(\cdot)$ is already defined on $[0, t_k]$, where k is non-negative integer. We take $f_k(t) \in F(y_k, u)$ such that

$$\langle x(t) - y_k, f_x(t) - f_k(t)\rangle \le -L|x(t) - y_k|^2, \tag{7}$$

and $\dot{y}(t) + Ay = f_k(t)$ on $[t_k, t_{k+1}]$. Thus

$$\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle \le \langle x(t) - y(t), Ay(t) - Ax(t)\rangle - L|x(t) - y_k|^2.$$

Consequently on $[0, t_k]$, we have

$$\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t)\rangle \le -L|x(t) - y(t)|^2$$

$$+|L|\,\big||x(t)-y(t)|^2-|x(t)-y_k|^2\big| \qquad (8)$$
$$\leq -L|x(t)-y(t)|^2+|L||y(t)-y_k|(2|x(t)|+|y(t)|+|y_k|).$$

Having in mind the proof of Theorem 2.1 under (**SH**) we conclude that there exists a constant $M>0$ such that $|x(t)|\leq M$ for every solution $x(\cdot)$ of (1) and every $t\in[0,\infty)$. It follows from Lemma 1.3 that the solution set of

$$\dot{x}(t)+Ax\in M\mathbb{B},\ x(0)=x^0$$

is $C([0,T],H)$ compact for any $T>0$. Since the system (1) is autonomous, one has that there exists modulus of continuity

$$\Omega(\delta)=\max_{s,t\in[0,\infty)}\{|x(t)-x(s)|:\ |t-s|\leq\delta,\ \forall \text{ solution } \ x(\cdot) \text{ of } (1)\}.$$

It follows from (8) that $\langle x(t)-y(t),\dot{x}(t)-\dot{y}(t)\rangle\leq -L|x(t)-y(t)|^2+4M\Omega(h)$, where $h=t_{i+1}-t_i$. Thus $2\frac{d}{dt}|x(t)-y(t)|^2\leq -2L|x(t)-y(t)|^2+8M\Omega(h)$. Hence $|x(t)-y(t)|\leq e^{-Lt}|x^0-y^0|+\sqrt{8M\Omega(h)}$. Let fix $\varepsilon>0$, and h_i be so small that $\sqrt{8M\Omega(h)}<\frac{\varepsilon}{2}$. Let T be such that $e^{-LT}|x^0-y^0|\leq\frac{\varepsilon}{2}$. Then for $t>T$ we get $|x(t)-y(t)|\leq\varepsilon$. Consider now the sequences $\{\varepsilon_k\}_{k=1}^{\infty}$, $\{T_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ such that $\sqrt{8Mh}<\frac{\varepsilon}{2^{k+1}}$, $e^{-L(T_{k+1}-T_k)}\frac{\varepsilon}{2^k}\leq\frac{\varepsilon}{2^{k+1}}$ and $\lim_{k\to\infty}T_k=\infty$. Using the approximation scheme (6) on $[T_k,T_{k+1}]$ with step h_k, then we get that $\lim_{t\to\infty}|x(t)-y(t)|=0$. Obviously neither h_k nor T_k depend on the concrete solution $x(\cdot)$. Consequently one can use such a scheme to approximate the whole solution set of (1). Then we infer that the reachable set of the approximate solutions set, that is, of (6) will have the same attractor as (1). □

Finally we want to study backward behavior of the system (1). Consider the system

$$\dot{x}(s)\in Ax+G(x),\ x(0)=x^0,\ s\in[0,-\infty). \qquad (9)$$

where A and $-G(\cdot)$ satisfy (**SH**) without the OSL condition. The following theorem hold:

Theorem 2.9. *If $\sigma(x-y,G(x))-\sigma(x-y,G(y))\geq L|x-y|^2$ with $L>0$, then (1) admits a unique strong backward attractor, furthermore the operator $D(x)=Ax+G(x)$ is surjective. Moreover, if $L>1$, then $D(\cdot)$ admits a fixed point.*

Proof. If one changes $t=-s$ then the system becomes

$$\dot{x}(t)+Ax\in -G(x),\ x(0)=x^0,\ t\in[0,\infty).$$

It is easy to see that $\sigma(x-y,-G(x))-\sigma(x-y,-G(y)) \leq -L|x-y|^2$. One can apply Theorem 2.1 and Corollary 2.3. $\square$

3. Example

In this section we present an example of parabolic systems satisfying **(SH)**.

Let $\Upsilon \subset \mathbb{R}^m$ be a bounded domain with a smooth boundary $\partial\Upsilon \equiv \Gamma$, $Q_T = (0,T) \times \Upsilon$, $T \in (0,\infty)$. Also $2 \leq p < \infty$ and $p+q = pq$. Let $\beta = (\beta_1, \beta_2, \cdots, \beta_m)$, where β_j is nonnegative integer for all $j = 1, 2, \cdots, m$ and $\|\beta\| = \sum_{j=1}^{m} \beta_j$. For $n \in \mathbb{N}$, $W^{n,p}(\Upsilon)$ denotes the standard Sobolev space with the norm defined by

$$\|\psi\|_{W^{n,p}} = (\sum_{|\beta| \leq n} \|D^\beta \psi\|^p_{L^p(\Upsilon)})^{\frac{1}{p}}.$$

Taking $W_0^{n,p}(\Upsilon) = \{\psi \in W^{n,p} |\ D^\alpha \psi|_\Gamma = 0,\ |\alpha| < n-1\}$, then one has $C_0^\infty \hookrightarrow W_0^{n,p}(\Upsilon) \hookrightarrow L^2(\Upsilon) \hookrightarrow W_0^{-n,p}(\Upsilon)$. Furthermore, it turns out that the embedding $W_0^{n,p}(\Upsilon) \hookrightarrow L^2(\Upsilon)$ is compact. Denote $\mathsf{X} \equiv W_0^{n,p}(\Upsilon)$, $H \equiv L^2(\Upsilon)$, then $\mathsf{X}^* \equiv W_0^{-n,p}(\Upsilon)$.

Consider the following two dimensional system:

$$\frac{\partial}{\partial t} z(t,x) + \sum_{|\beta| \leq n} (-1)^{|\beta|} D^\beta A_\beta(x, z(t,x)) \in g(x, z(t,x)), \text{ on } Q_T, \tag{10}$$

$$z(0,x) = z_0(x), \text{ and } D^\alpha z(t,x) = 0 \text{ on } [0,\infty) \times \Gamma \text{ for all } |\alpha| \leq n-1.$$

Assume that $z_0(0) = 0$.

The solution $z(t,x) \equiv \bar{z}(x)$ is called steady state solution. Notice that $\bar{z}(x) \neq z_0(x)$, however $\bar{z}(0) = 0$.

The steady state solution is said to be weakly stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|z_0(x) - \bar{z}(x)| < \delta$ then there exists a solution $z(\cdot,\cdot)$ of (10) with $|z(t,x) - \bar{z}(x)| < \varepsilon$.

If moreover $\lim_{t\to\infty} |z(t,x) - \bar{z}(x)| \to 0$, then $\bar{z}(\cdot)$ is called weakly asymptotically stable.

For $z_1, z_2 \in W_0^{n,p}(\Upsilon)$ we define $a(z_1, z_2) = \int_\Upsilon \sum_{|\beta| \leq n} A_\beta(x, z_1(t,x)) D^\beta z_2 dx$

and the function $A_\beta : \Upsilon \times \mathbb{R}^N \to \mathbb{R}^2$ $\left(N = \dfrac{(m+n)!}{m!n!} \right)$ that satisfies the following properties:

a1. $x \to A_\beta(x,z)$ is measurable on Υ for $z \in \mathbb{R}^N$, $z \to A_\beta(x,z)$ is continuous on $\mathbb{R}^N$ for any $x \in \Upsilon$.

a2. For $z, \bar{z} \in \mathbb{R}^N$ there exist positive constants b_1, b_2, b_3 and b_4 such that

$$\sum_{|\beta| \le n} \langle A_\beta(x,z) - A_\beta(x,\bar{z}) \rangle (z - \bar{z}) \ge 0,$$

$$\sum_{|\beta| \le n} A_\beta(x,z) z \ge b_1 \sum_{|\delta| \le n} |z_\delta|^p - b_2,$$

$$|A_\beta(x,z)| \le b_4 + b_3 \sum_{|\delta| \le n} |z_\delta|^{p-1}.$$

It is easy to verify that $\bar{z} \to a(z,\bar{z})$ is a continuous linear form on X for any $z_1 \in \mathsf{X}$.

Hence there exists an operator $A : \mathsf{X} \to \mathsf{X}^*$ such that

$$\langle A(z), \bar{z} \rangle_{\mathsf{X},\mathsf{X}^*} = a(z,\bar{z}).$$

Let the function $g : \Upsilon \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ satisfies the following properties:

(1) $g(\cdot,\cdot)$ is almost UHC with nonempty closed convex bounded values.

(2) $g(\cdot,\cdot)$ is bounded on the bounded sets and there exists a constant $L > 0$, such that

$$\sigma((z_1 - z_2), g(x,z_1)) - \sigma((z_1 - z_2), g(x,z_2)) \le -L|z_1 - z_2|^2$$

for almost all $x \in \Upsilon$.

For $h \in H$ and $t \in I$, we consider

$$b(h,\eta) = \int_\Upsilon \sigma(\eta, g(x,h)) dx.$$

Then for $\eta \to b(h)$ there exists an operator $B : H \rightrightarrows H$ such that

$$b(h,\eta) = \sigma(\eta, B(h)).$$

Using for our purpose the operators A and B, then the problem under consideration can be written in the form

$$\dot{z}(t) + A(z(t)) \in B(z(t)), \ t \in [0,\infty). \tag{11}$$

The last evolution inclusion is in the form (1) although without having no parameters.

Let us drop the initial conditions at $t = 0$. From the proof of Corollary 2.3 we know that for every $T > 0$ the system (11) admits a T-periodic solution and hence the system

$$\begin{aligned} \frac{\partial}{\partial t} z(t,x) + \sum_{|\beta|\le n} (-1)^{|\beta|} D^\beta A_\beta(x, z(t,x)) \in g(x, z(t,x)),\ \text{on } Q_T, \qquad (12) \\ D^\alpha z(t,x) = 0 \text{ on } [0,\infty) \times \Gamma \text{ for all } |\alpha| \le n-1 \end{aligned}$$

also admits a T-periodic solution.

The surjectivity of the operator $O(z)$ implies that the system

$$\begin{aligned} \sum_{|\beta|\le n} (-1)^{|\beta|} D^\beta A_\beta(x, z(x)) &\in g(x, z(x)),\ \text{on } Q_T, \\ D^\alpha z(x) &= 0 \text{ on } \Gamma \text{ for all } |\alpha| \le n-1 \end{aligned}$$

admits a steady state solution $\hat{z}(\cdot) \equiv z(x)$, i.e. $O(\hat{z}) = 0$.

As a final conclusion we may assert that every such a solution is weakly asymptotically stable and weakly attracting.

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