ON THE HERMITE-HADAMARD-LIKE TYPE INEQUALITIES FOR CO-ORDINATED (s, r)-CONVEX MAPPINGS IN THE FIRST SENSE

Jaekeun Park
Department of Mathematics
Hanseo University
Haemi-Myun, Seosan-Si, Chungnam-Do
356-706, REPUBLIC OF KOREA

Abstract: In this article, by combining the notions of s-convexity in the first sense and r-convexity the author give the definition of the class of co-ordinated (s, r)-convex mappings in the first sense, and establish some Hadamard-type inequalities for mappings in this class.

AMS Subject Classification: s-convexity, r-convexity, Hadamard inequality, co-ordinated convexity
Key Words: 26A24, 26A51, 26B25

1. Introduction

The following inequality is well-known in the literature as Hadamard’s inequality: Let $f : \mathbb{I} \subset [0, b^*] \rightarrow \mathbb{R}$ be a convex mapping defined on an interval $\mathbb{I}$ in $\mathbb{R}$, where $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$. Then the following inequality holds:

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}.$$ 

In [11], C.E.M. Pearce, J. Pecaric and V. Simic generalized this Hadamard’s inequality to r-convex mapping $f$ which is defined on an interval $[a, b]$ as follows:
Definition 1. A mapping \( f : \mathbb{I} \subset [0, b^*] \to \mathbb{R} \) is called \( r \)-convex on \([a, b]\), where \( a, b \in \mathbb{I} \) with \( a < b \) and \( b^* > 0 \), if the following inequalities
\[
f(tx + (1 - t)y) \leq \begin{cases} \{ tf^{r}(x) + (1 - t)f^{r}(y) \}^{\frac{1}{r}} & \text{if } r \neq 0 \\ f(t)x + f(1-t)y & \text{if } r = 0. \end{cases}
\]
hold, for all \( x, y \in [a, b] \) and \( t \in [0, 1] \).

In [9], N.P.G. Ngoc, N.V. Vinh and P.T.T. Hien proved the following theorems for \( r \)-convex mappings:

**Theorem 1.1.** Let \( f : \mathbb{I} \subset [0, b^*] \to \mathbb{R} \) be a convex mapping defined on an interval \( \mathbb{I} \) where \( a, b \in \mathbb{I} \) with \( a < b \) and \( b^* > 0 \). For \( 0 < r \leq 1 \), the following inequality holds:
\[
\frac{1}{b-a} \int_{a}^{b} f(x)dx \leq (\frac{r}{r+1})^{\frac{1}{r}} \{ f^{r}(a) + f^{r}(b) \}^{\frac{1}{r}}.
\]

**Theorem 1.2.** Let \( f, g : [a, b] \to \mathbb{R} \) be \( r_{1} \)-convex and \( r_{2} \)-convex mappings, respectively, on \([a, b]\) with \( a < b \). For \( r_{1} > 1 \) with \( \frac{1}{r_{1}} + \frac{1}{r_{2}} = 1 \) the following inequality holds:
\[
\frac{1}{b-a} \int_{a}^{b} f(x)g(x)dx \leq \frac{1}{2} \{ f^{r_{1}}(a) + f^{r_{1}}(b) \}^{\frac{1}{r_{1}}} \{ g^{r_{2}}(a) + g^{r_{2}}(b) \}^{\frac{1}{r_{2}}}.
\] (1)

Similar results can be found for several kind of convexity, in [2, 3, 6, 7].

In [10], M.E. Özdemir and A.O. Akdemir gave the definition of the co-ordinated convex mappings as follows:

**Definition 2.** A mapping \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is called co-ordinated convex on \( \Delta \) if the partial mappings \( f_{y} : [a, b] \to \mathbb{R}, f_{y}(u) = f(u, y) \) and \( f_{x} : [c, d] \to \mathbb{R}, f_{x}(v) = f(x, v) \) are convex for all \( y \in [c, d] \) and \( x \in [a, b] \).

**Theorem 1.3.** For a co-ordinated convex mapping \( f : \Delta \to \mathbb{R} \) on \( \Delta = [a, b] \times [c, d] \), the following inequalities hold:
\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c} \int_{c}^{d} f\left(\frac{a+b}{2}, y\right)dy \right]
\]
\[
\leq \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y)dxdy
\]
\[ \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b \{ f(x, c) + f(x, d) \} dx + \frac{1}{d-c} \int_c^d \{ f(a, y) + f(b, y) \} dy \right] \]
\[ \leq \frac{1}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right].\]

The above inequalities are sharp.

In [1], M. Alomari and M. Darus proved some inequalities of the Hadamard and Jensen types for co-ordinated log-convex function. In [4, 5, 8], M.K. Bakula and J. Pecaric improved several inequalities of Jensen’s type for convex and \( s \)-convex functions on the co-ordinates. In [10], M.E. Özdemir, Erhan Set, M.Z. Sarikaya, S. Hussain, M.I. Bhatti and M. Iqbal eastablished Hadamard’s type inequalities for co-ordinated \( m \)-convex, \((\alpha, m)\)-convex and \( s \)-convex mappings, respectively.

In [10], M.E. Özdemir and A.O. Akdemir defined \( r \)-convex mappings on the co-ordinates as follows:

**Definition 3.** A mapping \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+ \) is called \( r \)-convex on \( \Delta \) if the following inequalities hold:

\[ f(tx + (1-t)y, \lambda u + (1-\lambda)v) \leq \begin{cases} 
   \{ tf^r(x, u) + t(1-\lambda) f^r(x, v) 
   \} \quad \text{if } r \neq 0 \\
   + (1-t)\lambda f^r(y, u) + (1-t)(1-\lambda) f^r(y, v) \} \quad \text{if } r = 0 
\end{cases} \]

for all \( t, \lambda \in [0, 1] \) and \((x, y), (u, v) \in \Delta \).

**Definition 4.** A mapping \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+ \) is called co-ordinated \( r \)-convex on \( \Delta \) if the partial mappings \( f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v) \) are \( r \)-convex for all \( y \in [c, d] \) and \( x \in [a, b] \).

Note that every \( r \)-convex mapping \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+ \) is \( r \)-convex on the co-ordinates [10].

In this article the main purpose is newly to define the co-ordinated \( (s, r) \)-convex mapping in the first sense and prove some Hadamard-type inequalities for co-ordinated \( (s, r) \)-convex mappings in the first sense.

**2. Main Results**

We can define \( (s, r) \)-convex mapping on \([a, b] \) and \((s, r)\)-convex mapping in the first sense on \( \Delta = [a, b] \times [c, d] \) as follows:
Definition 5. A mapping $f : [a, b] \to \mathbb{R}^+$ is called $(s, r)$-convex in the first sense on $[a, b]$, for $s \in (0, 1]$, if the following inequalities hold:

$$f(tx + (1-t)y) \leq \begin{cases} t^s f^r(x) + (1-t^s) f^r(y) & \text{if } r \neq 0 \\ t^s f(x) f^{(1-t^s)}(y) & \text{if } r = 0. \end{cases}$$

for all $t \in [0, 1]$ and $x, y \in [a, b]$.

Definition 6. A mapping $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+$ is called $(s, r)$-convex in the first sense on $\Delta$, for $s \in (0, 1]$, if the following inequalities hold:

$$f(tx + (1-t)y, \lambda u + (1-\lambda)v)$$

$$\leq \begin{cases} [t^s \lambda^s f^r(x, u) + t^s(1-\lambda^s) f^r(x, v) & \text{if } r \neq 0 \\ t^s \lambda^s (x, u) f^{(1-t^s)}(x, v) f^{(1-t^s)}(y, u) f^{(1-t^s)}(1-\lambda^s)(y, v) & \text{if } r = 0. \end{cases}$$

for any $t, \lambda \in [0, 1]$ and $(x, y), (u, v) \in \Delta$.

Definition 7. A mapping $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+$ is called co-ordinated $(s, r)$-convex in the first sense on $\Delta$, for $s \in (0, 1]$, if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are $(s, r)$-convex in the first sense, for all $y \in [c, d]$ and $x \in [a, b]$.

To begin with, let us consider the following lemma:

Lemma 1. Every $(s, r)$-convex mapping $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+$ in the first sense is a co-ordinated $(s, r)$-convex mapping in the first sense, for $s \in (0, 1]$.

Proof. Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}^+$ is $(s, r)$-convex in the first sense on $\Delta$. Consider the partial mapping $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$, for all $y \in [c, d]$ and $x \in [a, b]$.

Case 1: For $r = 0$ and $u_1, u_2 \in [a, b]$, we have:

$$f_y(tu_1 + (1-t)u_2)$$

$$= f(tu_1 + (1-t)u_2, y)$$

$$= f(tu_1 + (1-t)u_2, \lambda y + (1-\lambda)y)$$

$$\leq f_t^s \lambda^s (u_1, y) f^{(1-t^s)}(u_1, y) f^{(1-t^s)}(u_2, y) f^{(1-t^s)}(1-\lambda^s)(u_2, y)$$

$$= f_t^s \lambda^s + t^s(1-\lambda^s)(u_1, y) f^{(1-t^s)}(1-\lambda^s)(u_2, y)$$

$$= f_y^s (u_1) f_y^s (1-\lambda^s)(u_2)$$
Case 2: For $r \neq 0$ and $u_1, u_2 \in [a, b]$, we have:

\[
\begin{align*}
  f_y(tu_1 + (1-t)u_2) \\
  = f(tu_1 + (1-t)u_2, \lambda y + (1-\lambda)y) \\
  \leq \left[ t^s \lambda^s f'_y(u_1) + t^s(1-\lambda^s)f'_y(u_1) \\
  + (1-t^s)\lambda^s f''_y(u_2) + (1-t^s)(1-\lambda^s)f''_y(u_2) \right]^\frac{1}{s} \\
  = \left[ t^s f'_y(u_1) + (1-t^s)f'_y(u_2) \right]^\frac{1}{s}.
\end{align*}
\]

Therefore $f_y(u) = f(u, y)$ is $(s, r)$-convex in the first sense on $[a, b]$. By a similar argument one can see that $f_x(v) = f(x, v)$ is $(s, r)$-convex in the first sense on $[c, d]$.

**Theorem 2.1.** If $f, g : \Delta \to \mathbb{R}_+$ are two co-ordinated $(s, r)$-convex mappings in the first sense on $\Delta$, then the product $(fg)$ is also a co-ordinated $(s, r)$-convex mapping in the first sense on $\Delta$.

**Proof.** (i) Case 1: $r = 0$:

\[
\begin{align*}
  f(tx + (1-t)y, \lambda u + (1-\lambda)v)g(tx + (1-t)y, \lambda u + (1-\lambda)v) \\
  = \left[ f^{\ell s}(x, u)f^{(1-\ell s)}(y, v)f^{(1-\ell s)(1-\lambda^s)}(y, v) \right] \\
  \times \left[ g^{\ell s}(x, u)g^{(1-\ell s)}(y, v)g^{(1-\ell s)(1-\lambda^s)}(y, v) \right] \\
  = \left\{ f(x, u)g(x, u) \right\}^{\ell s} \left\{ f(x, v)f(x, v) \right\}^{(1-\ell s)} \\
  \times \left\{ f(y, u)g(y, u) \right\}^{(1-\ell s)} \left\{ f(y, v)g(y, v) \right\}^{(1-\ell s)(1-\lambda^s)} \\
  = (fg)^{(1-\ell s)}(1-\lambda^s)(x, v)(fg)^{(1-\ell s)(1-\lambda^s)}(y, u)(fg)^{(1-\ell s)(1-\lambda^s)}(y, v)
\end{align*}
\]

(ii) Case 2: $r \neq 0$:

\[
\begin{align*}
  f(tx + (1-t)y, \lambda u + (1-\lambda)v)g(tx + (1-t)y, \lambda u + (1-\lambda)v) \\
  \leq \left[ t^s \lambda^s f^r(x, u) + t^s(1-\lambda^s)f^r(x, v) + (1-t^s)\lambda^s f^r(y, u) \\
  + (1-t^s)(1-\lambda^s)f^r(y, v) \right]^\frac{1}{s} \left[ t^s \lambda^s g^r(x, u) + t^s(1-\lambda^s)g^r(x, v) \\
  + (1-t^s)\lambda^s g^r(y, u) + (1-t^s)(1-\lambda^s)g^r(y, v) \right]^\frac{1}{s} \\
  = \left[ t^s \lambda^s f^r(x, u)g^r(x, u) + t^s(1-\lambda^s)f^r(x, v)g^r(x, v) \\
  + (1-t^s)\lambda^s f^r(y, u)g^r(y, u) + (1-t^s)(1-\lambda^s)f^r(x, u)g^r(x, u) \right]^\frac{1}{s}.
\end{align*}
\]
Theorem 2.2. Let \( f : [a, b] \to \mathbb{R}_+ \) be an \((s, r)\)-convex mapping in the first sense on \([a, b]\) with \(a < b\). Then for \(r, s > 0\), we have
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \left[ \left\{ \frac{r}{s + r} \right\}^s f^r(a) + \left\{ \frac{\Gamma(1 + \frac{r}{s}) \Gamma(1 + \frac{1}{s})}{\Gamma(1 + \frac{r}{s}) + \frac{1}{s}} \right\}^s f^r(b) \right]^{\frac{1}{s}}.
\]

Proof. Since \( f \) is an \((s, r)\)-convex mapping in the first sense on \([a, b]\) with \(a < b\) for \(r, s > 0\), we have
\[
f(ta + (1-t)b) \leq \left\{ ts f^r(a) + (1-t)s f^r(b) \right\}^{\frac{1}{s}}
\]
for all \(t \in [0,1]\), which implies that
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \left[ \int_0^1 \left\{ t^s f^r(a) + (1-t)^s f^r(b) \right\}^{\frac{1}{s}} \, dt \right]^{\frac{1}{s}}
\]
\[
\leq \left[ \left\{ \int_0^1 t^r f(a) \, dt \right\}^s + \left\{ \int_0^1 (1-t)^r f(b) \, dt \right\}^s \right]^{\frac{1}{s}}
\]
\[
= \left[ \left\{ \frac{r}{s + r} \right\}^s f^r(a) + \left\{ \frac{\Gamma(1 + \frac{r}{s}) \Gamma(1 + \frac{1}{s})}{\Gamma(1 + \frac{r}{s}) + \frac{1}{s}} \right\}^s f^r(b) \right]^{\frac{1}{s}}.
\]

Remark 1. Theorem 2.2 is a generalization of Theorem 1.1.

Theorem 2.3. Let \( f, g : [a, b] \to \mathbb{R}_+ \) be, respectively, \((s_1, r_1)\)-convex and \((s_2, r_2)\)-convex mappings in the first sense on \([a, b]\) with \(a < b\). Then for \(0 < r_1, r_2 \leq 2\) the following inequality holds:
\[
\frac{1}{b-a} \int_a^b f(x) g(x) \, dx
\]
\[
\leq \frac{1}{2} \left[ \left\{ \left( \frac{r_1}{r_1 + 2s_1} \right)^{\frac{r_1}{2}} f^{r_1}(a) + \left( \frac{\Gamma(1 + \frac{r_1}{s_1}) \Gamma(1 + \frac{1}{s_1})}{\Gamma(1 + \frac{r_1}{s_1}) + \frac{1}{s_1}} \right)^{\frac{r_1}{2}} f^{r_1}(b) \right\}^{\frac{2}{r_1}}
\]
\[
+ \left\{ \left( \frac{r_2}{r_2 + 2s_2} \right)^{\frac{r_2}{2}} g^{r_2}(a) + \left( \frac{\Gamma(1 + \frac{r_2}{s_2}) \Gamma(1 + \frac{1}{s_2})}{\Gamma(1 + \frac{r_2}{s_2}) + \frac{1}{s_2}} \right)^{\frac{r_2}{2}} g^{r_2}(b) \right\}^{\frac{2}{r_2}} \right].
\]

Proof. Since \( f \) and \( g \) are, respectively, \((s_1, r_1)\)-convex and \((s_2, r_2)\)-convex in the first sense on \([a, b]\) with \(a < b\) for \(r, s > 0\), we have
\[
f(ta + (1-t)b) \leq \left\{ ts f^{r_1}(a) + (1-t) s f^{r_1}(b) \right\}^{\frac{1}{r_1}}
\]
ON THE HERMITE-HADAMARD-LIKE TYPE INEQUALITIES...

\[ g(ta + (1-t)b) \leq \left\{ ts_2 g^{r_2}(a) + (1-ts_2)g^{r_2}(b) \right\}^{\frac{1}{r_2}}, \quad (3) \]

for all \( t \in [0, 1] \), which implies that, by using the property \( ab \leq \frac{1}{2} \{ a^2 + b^2 \} \) for any \( a, b \in \mathbb{R}_+ \),

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \\
\leq \int_0^1 \left\{ ts_1 f^{r_1}(a) + (1-ts_1)f^{r_1}(b) \right\}^{\frac{1}{r_1}} dt \\
\times \left\{ ts_2 g^{r_2}(a) + (1-ts_2)g^{r_2}(b) \right\}^{\frac{1}{r_2}} dt \\
\leq \frac{1}{2} \left[ \int_0^1 \left\{ ts_1 f^{r_1}(a) + (1-ts_1)f^{r_1}(b) \right\}^{\frac{2}{r_1}} dt \\
+ \int_0^1 \left\{ ts_2 g^{r_2}(a) + (1-ts_2)g^{r_2}(b) \right\}^{\frac{2}{r_2}} dt \right]. \quad (4)
\]

By using Minkowski’s inequality, we have

\[
\int_0^1 \left\{ ts_1 f^{r_1}(a) + (1-ts_1)f^{r_1}(b) \right\}^{\frac{2}{r_1}} dt \\
\leq \left[ \left\{ f^2(a) \int_0^1 t^{\frac{s_1}{r_1}} dt \right\}^{\frac{r_1}{2}} + \left\{ f^2(b) \int_0^1 (1-t)^{\frac{s_1}{r_1}} dt \right\}^{\frac{r_1}{2}} \right]^{\frac{2}{r_1}} \\
= \left\{ \left( \frac{r_1}{r_1 + 2s_1} \right)^{\frac{r_1}{2}} f^{r_1}(a) + \left( \frac{\Gamma(1 + \frac{2}{r_1})\Gamma(1 + \frac{1}{s_1})}{\Gamma(1 + \frac{2}{r_1} + \frac{1}{s_1})} \right)^{\frac{r_1}{2}} f^{r_1}(b) \right\}^{\frac{2}{r_1}}. \quad (5)
\]

Similarly we have

\[
\int_0^1 \left\{ ts_2 [g(a)]^{r_2} + (1-ts_2)g^{r_2}(b) \right\}^{\frac{2}{r_2}} dt \\
\leq \left\{ \left( \frac{r_2}{r_2 + 2s_2} \right)^{\frac{r_2}{2}} g^{r_2}(a) + \left( \frac{\Gamma(1 + \frac{2}{r_2})\Gamma(1 + \frac{1}{s_2})}{\Gamma(1 + \frac{2}{r_2} + \frac{1}{s_2})} \right)^{\frac{r_2}{2}} g^{r_2}(b) \right\}^{\frac{2}{r_2}}. \quad (6)
\]

By (4)-(6), the assertion in this theorem holds.

**Theorem 2.4.** Let \( f, g : [a, b] \to \mathbb{R}_+ \) be, respectively, \((s_1, r_1)\)-convex and \((s_2, r_2)\)-convex mappings in the first sense on \([a, b]\) with \( a < b \). Then for \( r_1 > 1 \) with \( \frac{1}{r_1} + \frac{1}{r_2} = 1 \) the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \\
\leq \left( \frac{1}{s_1 + 1} \right)^{\frac{r_1}{2}} \left( \frac{1}{s_2 + 1} \right)^{\frac{r_2}{2}} \left\{ f^{r_1}(a) + s_1 f^{r_1}(b) \right\}^{\frac{1}{r_1}} \left\{ g^{r_2}(a) + s_2 g^{r_2}(b) \right\}^{\frac{1}{r_2}}.
\]
Proof. Since \( f \) and \( g \) are, respectively, \((s_1, r_1)\)-convex and \((s_2, r_2)\)-convex mappings in the first sense on \([a, b]\) with \(a < b\), for \(r_1, r_2 > 0\) and \(s_1, s_2 > 0\) we have the inequalities (4) for all \(t \in [0, 1]\), which implies that

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \\
\leq \int_0^1 \left\{ t^{s_1}f^{r_1}(a) + (1-t^{s_1})f^{r_1}(b) \right\}^{\frac{1}{r_1}} \\
\times \left\{ t^{s_2}g^{r_2}(a) + (1-t^{s_2})g^{r_2}(b) \right\}^{\frac{1}{r_2}} dt
\]

\[
\leq \left\{ \int_0^1 \left( t^{s_1}f^{r_1}(a) + (1-t^{s_1})f^{r_1}(b) \right) dt \right\}^{\frac{1}{r_1}} \\
\times \left\{ \int_0^1 \left( t^{s_2}g^{r_2}(a) + (1-t^{s_2})g^{r_2}(b) \right) dt \right\}^{\frac{1}{r_2}}
\]

\[
\leq \left\{ \frac{f^{r_1}(a) + s_1f^{r_1}(b)}{s_1 + 1} \right\}^{\frac{1}{r_1}} \left\{ \frac{g^{r_2}(a) + s_2g^{r_2}(b)}{s_2 + 1} \right\}^{\frac{1}{r_2}},
\]

which completes the proof.

Corollary 1. In Theorem 2.4, if we choose \(r_1 = r_2 = 2\) and \(s_1 = s_2 = 1\), then we get

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{2} \left\{ f^{r_1}(a) + f^{r_1}(b) \right\}^{\frac{1}{r_1}} \left\{ g^{r_2}(a) + g^{r_2}(b) \right\}^{\frac{1}{r_2}},
\]

which implies Theorem 1.2.

Theorem 2.5. Suppose that \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}_+ \) is an \((s, r)\)-convex mapping in the first sense on \(\Delta\). Then the following inequality holds:

\[
\frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy
\]

\[
\leq \int_a^b \left\{ \left( \frac{r}{s+r} \right)^r f_x^r(c) + \frac{\Gamma(1 + \frac{r}{s})\Gamma(1 + \frac{r}{r})}{\Gamma(1 + \frac{r}{s} + \frac{r}{r})} f_x^r(d) \right\}^{\frac{1}{r}} dx
\]

\[
+ \int_c^d \left\{ \left( \frac{r}{s+r} \right)^r f_y^r(a) + \frac{\Gamma(1 + \frac{r}{s})\Gamma(1 + \frac{r}{r})}{\Gamma(1 + \frac{r}{s} + \frac{r}{r})} f_y^r(b) \right\}^{\frac{1}{r}} dy.
\] (7)

Proof. Since \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}_+ \) is a co-ordinated \((s, r)\)-convex mapping in the first sense on \(\Delta\), the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}_+ \), \( f_y(u) = \)
ON THE HERMITE-HADAMARD-LIKE TYPE INEQUALITIES... 259

\( f(u, y) \) and \( f_x : [c, d] \to \mathbb{R}_+ \), \( f_x(v) = f(x, v) \) are \((s, r)\)-convex in the first sense for all \( y \in [c, d] \) and \( x \in [a, b] \), which implies that, by Theorem 2.2,

\[
\frac{1}{d-c} \int_c^d f_x(y)dy \\
\leq \left\{ \left( \frac{r}{s+r} \right)^{r} f_x^r(c) + \left( \frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})} \right)^{r} f_x^r(d) \right\}^{\frac{1}{r}}. \tag{8}
\]

Dividing both side of the inequality (8) by \((b-a)\) and integrating with respect to \( x \) on \([a, b] \), we get

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dxdy \\
\leq \int_a^b \left\{ \left( \frac{r}{s+r} \right)^{r} f_x^r(c) + \left( \frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})} \right)^{r} f_x^r(d) \right\}^{\frac{1}{r}} dx. \tag{9}
\]

By a similar argument, for the partial mapping \( f_y : [a, b] \to \mathbb{R} \), \( f_y(u) = f(u, y) \), we have

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dxdy \\
\leq \int_c^d \left\{ \left( \frac{r}{s+r} \right)^{r} f_y^r(a) + \left( \frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})} \right)^{r} f_y^r(b) \right\}^{\frac{1}{r}} dy. \tag{10}
\]

By (9) and (10), the inequality (7) is proved.

**Theorem 2.6.** Let \( f, g : [a, b] \to \mathbb{R}_+ \) be, respectively, co-ordinated \((s_1, r_1)\)-convex and co-ordinated \((s_2, r_2)\)-convex mappings in the first sense on \( \Delta \). Then for \( r_1 > 0 \) and \( r_2 \leq 2 \), the following inequality holds:

\[
\frac{4}{(b-a)(d-c)} \int_a^b \int_c^d f_x(y)g_x(y)dxdy \\
\leq \left( \frac{r_1}{r_1 + 2s_1} \right) \left( \frac{1}{b-a} \right) \int_a^b \left\{ f^{r_1}(x, c) + f^{r_1}(x, d) \right\}^{\frac{2}{r_1}} dx \\
+ \left( \frac{r_2}{r_2 + 2s_2} \right) \left( \frac{1}{b-a} \right) \int_a^b \left\{ g^{r_2}(x, c) + g^{r_2}(x, d) \right\}^{\frac{2}{r_2}} dx \\
+ \left( \frac{r_1}{r_1 + 2s_1} \right) \left( \frac{1}{d-c} \right) \int_c^d \left\{ f^{r_1}(a, y) + f^{r_1}(b, y) \right\}^{\frac{2}{r_1}} dy \\
+ \left( \frac{r_2}{r_2 + 2s_2} \right) \left( \frac{1}{d-c} \right) \int_c^d \left\{ g^{r_2}(a, y) + g^{r_2}(b, y) \right\}^{\frac{2}{r_2}} dy.
\]
Proof. Since \( f \) is a co-ordinated \((s_1, r_1)\)-convex mapping in the first sense and \( g \) is a co-ordinated \((s_2, r_2)\)-convex mapping in the first sense on \( \Delta \), for \( r_1, r_2 > 0 \) and \( s_1, s_2 > 0 \) the partial mappings \( f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \to \mathbb{R}_+, f_x(v) = f(x, v) \) are \((s_1, r_1)\)-convex in the first sense, and the partial mappings \( g_y : [a, b] \to \mathbb{R}_+, g_y(u) = g(u, y) \) and \( g_x : [c, d] \to \mathbb{R}_+, g_x(v) = g(x, v) \) are \((s_2, r_2)\)-convex in the first sense, for all \( y \in [c, d] \) and \( x \in [a, b] \).

By Theorem 2.3, we have

\[
\frac{2}{b-a} \int_a^b f_y(x)g_y(x)dx 
\leq \frac{r_1}{r_1 + 2s_1} \left\{ f_y^{r_1}(a) + f_y^{r_1}(b) \right\}^{\frac{2}{r_1}} + \frac{r_2}{r_2 + 2s_2} \left\{ g_y^{r_2}(a) + g_y^{r_2}(b) \right\}^{\frac{2}{r_2}},
\]

(11)

and

\[
\frac{2}{d-c} \int_c^d f_x(y)g_x(y)dy 
\leq \frac{r_1}{r_1 + 2s_1} \left\{ f_x^{r_1}(c) + f_x^{r_1}(d) \right\}^{\frac{2}{r_1}} + \frac{r_2}{r_2 + 2s_2} \left\{ g_x^{r_2}(c) + g_x^{r_2}(d) \right\}^{\frac{2}{r_2}}.
\]

(12)

By dividing both side of the inequalities (11) and (12) \((b - a)\) and \((d - c)\), and integrating with respect to \( x \) on \([a, b]\) and \( y \) on \([c, d]\), respectively, we get:

\[
\frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx 
\leq \left( \frac{r_1}{r_1 + 2s_1} \right) \left( \frac{1}{d-c} \right) \int_c^d \left\{ f_y^{r_1}(a) + f_y^{r_1}(b) \right\}^{\frac{2}{r_1}} dy 
+ \left( \frac{r_2}{r_2 + 2s_2} \right) \left( \frac{1}{d-c} \right) \int_c^d \left\{ g_y^{r_2}(a) + g_y^{r_2}(b) \right\}^{\frac{2}{r_2}} dy 
\]

(13)

and

\[
\frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx 
\leq \left( \frac{r_1}{r_1 + 2s_1} \right) \left( \frac{1}{b-a} \right) \int_a^b \left\{ f_x^{r_1}(c) + f_x^{r_1}(d) \right\}^{\frac{2}{r_1}} dx 
+ \left( \frac{r_2}{r_2 + 2s_2} \right) \left( \frac{1}{b-a} \right) \int_a^b \left\{ g_x^{r_2}(c) + g_x^{r_2}(d) \right\}^{\frac{2}{r_2}} dx.
\]

(14)
By adding the inequalities (13) and (14), we have
\[
\frac{4}{(b-a)(d-c)} \int_a^b \int_c^d f_x(y)g_x(y)dydx \\
\leq \left( \frac{r_1}{r_1+2s_1} \right) \left( \frac{1}{b-a} \right) \int_a^b \{ f_x^r(c) + f_x^r(d) \} \frac{1}{r_1} dx \\
+ \left( \frac{r_2}{r_2+2s_2} \right) \left( \frac{1}{b-a} \right) \int_a^b \{ g_x^r(c) + g_x^r(d) \} \frac{1}{r_2} dx \\
+ \left( \frac{r_1}{r_1+2s_1} \right) \left( \frac{1}{d-c} \right) \int_c^d \{ f_y^r(a) + f_y^r(b) \} \frac{1}{r_1} dy \\
+ \left( \frac{r_2}{r_2+2s_2} \right) \left( \frac{1}{d-c} \right) \int_c^d \{ g_y^r(a) + g_y^r(b) \} \frac{1}{r_2} dy,
\]
which completes the proof.

**Corollary 2.** In Theorem 2.7, if we choose \( r_1 = r_2 = 2 \) and \( s_1 = s_2 = 1 \), then we have
\[
\frac{8}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx \\
\leq \left( \frac{1}{b-a} \right) \int_a^b \{ f^2(x,c) + f^2(x,d) + g^2(x,c) + g^2(x,d) \} dx \\
+ \left( \frac{1}{d-c} \right) \int_c^d \{ f^2(a,y) + f^2(b,y) + g^2(a,y) + g^2(b,y) \} dy.
\]

**Theorem 2.7.** Let \( f, g : [a,b] \to \mathbb{R}_+ \) be, respectively, \((s_1, r_1)\)-convex and \((s_2, r_2)\)-convex mappings in the first sense on \([a,b]\) with \( a < b \). Then for \( r_1 > 1 \) with \( \frac{1}{r_1} + \frac{1}{r_2} = 1 \) the following inequality holds:
\[
\frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx \\
\leq \left( \frac{1}{s_1+1} \right) ^{\frac{1}{r_1}} \left( \frac{1}{s_2+1} \right) ^{\frac{1}{r_2}} \left( \frac{1}{b-a} \right) ^2 \\
\times \int_a^b \{ f^{r_1}(x,c) + f^{r_1}(x,d) \} \frac{1}{r_1} dx \int_a^b \{ g^{r_2}(x,c) + g^{r_2}(x,d) \} \frac{1}{r_2} dx \\
+ \left( \frac{1}{s_1+1} \right) ^{\frac{1}{r_1}} \left( \frac{1}{s_2+1} \right) ^{\frac{1}{r_2}} \left( \frac{1}{d-c} \right) ^2 \\
\times \int_c^d \{ f^{r_1}(a,y) + f^{r_1}(b,y) \} \frac{1}{r_1} dy \int_c^d \{ g^{r_2}(a,y) + g^{r_2}(b,y) \} \frac{1}{r_2} dy.
Proof. Since $f$ is a co-ordinated $(s_1, r_1)$-convex mapping in the first sense and $g$ is a co-ordinated $(s_2, r_2)$-convex mapping in the first sense on $\Delta$ for $r_1 > 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$ and $s_1, s_2 > 0$, the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}_+$, $f_x(v) = f(x, v)$ are $(s_1, r_1)$-convex in the first sense for all $y \in [c, d]$ and $x \in [a, b]$, and the partial mappings $g_y : [a, b] \to \mathbb{R}_+$, $g_y(u) = g(u, y)$ and $g_x : [c, d] \to \mathbb{R}_+$, $g_x(v) = g(x, v)$ are $(s_2, r_2)$-convex in the first sense for all $y \in [c, d]$ and $x \in [a, b]$.

By Theorem 2.5, we have

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx$$

$$\leq \left( \frac{1}{s_1 + 1} \right)^{\frac{1}{r_1}} \left( \frac{1}{b-a} \right)^{\frac{1}{r_1}} \int_a^b \left\{ f^{r_1}(x, c) + f^{r_1}(x, d) \right\}^{\frac{1}{r_1}} dx$$

$$\times \left( \frac{1}{s_2 + 1} \right)^{\frac{1}{r_2}} \left( \frac{1}{b-a} \right)^{\frac{1}{r_2}} \int_a^b \left\{ g^{r_2}(x, c) + g^{r_2}(x, d) \right\}^{\frac{1}{r_2}} dx$$

(15)

and

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx$$

$$\leq \left( \frac{1}{s_1 + 1} \right)^{\frac{1}{r_1}} \left( \frac{1}{d-c} \right)^{\frac{1}{r_1}} \int_c^d \left\{ f^{r_1}(a, y) + f^{r_1}(b, y) \right\}^{\frac{1}{r_1}} dy$$

$$\times \left( \frac{1}{s_2 + 1} \right)^{\frac{1}{r_2}} \left( \frac{1}{d-c} \right)^{\frac{1}{r_2}} \int_c^d \left\{ g^{r_2}(a, y) + g^{r_2}(b, y) \right\}^{\frac{1}{r_2}} dy.$$  (16)

By the inequalities (15) and (16), the assertion in this theorem holds.

References


264