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ON THE HERMITE-HADAMARD-LIKE TYPE INEQUALITIES FOR CO-ORDINATED (s, r)-CONVEX MAPPINGS IN THE FIRST SENSE

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Abstract: In this article, by combining the notions of s-convexity in the first sense and r-convexity the author give the definition of the class of co-ordinated (s, r)-convex mappings in the first sense, and establish some Hadamard-type inequalities for mappings in this class.

AMS Subject Classification: s-convexity, r-convexity, Hadamard inequality, co-ordinated convexity
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1. Introduction

The following inequality is well-known in the literature as Hadamard's inequality: Let $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ be a convex mapping defined on an interval $\mathbb{I}$ in $\mathbb{R}$, where $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$. Then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [11], C.E.M. Pearce, J. Pecaric and V. Simic generalized this Hadamard's inequality to r-convex mapping f which is defined on an interval $[a, b]$ as follows:

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Definition 1. A mapping $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ is called r-convex on $[a, b]$, where $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$. if the following inequalities

$$f(tx + (1-t)y) \leq \begin{cases} \{tf^r(x) + (1-t)f^r(y)\}^{\frac{1}{r}} & \text{if } r \neq 0 \\ f^t(x)f^{1-t}(y) & \text{if } r = 0. \end{cases}$$

hold, for all $x, y \in [a, b]$ and $t \in [0, 1]$.

In [9], N.P.G. Ngoc, N.V. Vinh and P.T.T. Hien proved the following theorems for r-convex mappings:

Theorem 1.1. *Let $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ be a convex mapping defined on an interval $\mathbb{I}$ where $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$. For $0 < r \leq 1$, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\{f^r(a) + f^r(b)\}^{\frac{1}{r}}.$$

Theorem 1.2. *Let $f, g : [a, b] \to \mathbb{R}$ be r_1-convex and r_2-convex mappings, respectively, on $[a, b]$ with $a < b$. For $r_1 > 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$ the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{2}\{f^{r_1}(a) + f^{r_1}(b)\}^{\frac{1}{r_1}}\{g^{r_2}(a) + g^{r_2}(b)\}^{\frac{1}{r_2}}. \quad (1)$$

Similar results can be found for several kind of convexity, in [2, 3, 6, 7]. In [10], M.E. Özdemir and A.O. Akdemir gave the definition of the co-ordinated convex mappings as follows:

Definition 2. A mapping $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is called co-ordinated convex on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $y \in [c, d]$ and $x \in [a, b]$.

Theorem 1.3. *For a co-ordinated convex mapping $f : \Delta \to \mathbb{R}$ on $\Delta = [a, b] \times [c, d]$, the following inequalities hold:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$
$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)dxdy$$

$$\leq \frac{1}{4}\Big[\frac{1}{b-a}\int_a^b \{f(x,c)+f(x,d)\}dx + \frac{1}{d-c}\int_c^d \{f(a,y)+f(b,y)\}dy\Big]$$
$$\leq \frac{1}{4}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big].$$

The above inequalities are sharp.

In [1], M. Alomari and M. Darus proved some inequalities of the Hadamard and Jensen types for co-ordinated log-convex function. In [4, 5, 8], M.K. Bakula and J. Pecaric improved several inequalities of Jensen's type for convex and s-convex functions on the co-ordinates. In [10], M.E. Özdemir, Erhan Set, M.Z. Sarikaya, S. Hussain, M.I. Bhatti and M. Iqbal eastablished Hadamard's type inequalities for co-ordinated m-convex, (α, m)-convex and s-convex mappings, respectively.

In [10], M.E. Özdemir and A.O. Akdemir defined r-convex mappings on the co-ordinates as follows:

Definition 3. A mapping $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ is called r-convex on Δ if the following inequalities hold:

$$f(tx+(1-t)y, \lambda u + (1-\lambda)v)$$
$$\leq \begin{cases} \big\{t\lambda f^r(x,u) + t(1-\lambda)f^r(x,v) \\ \qquad +(1-t)\lambda f^r(y,u) + (1-t)(1-\lambda)f^r(y,v)\big\}^{\frac{1}{r}} & \text{if } r \neq 0 \\ f^{t\lambda}(x,u)f^{t(1-\lambda)}(x,v)f^{(1-t)\lambda}(y,u)f^{(1-t)(1-\lambda)}(y,v) & \text{if } r = 0. \end{cases}$$

for all $t, \lambda \in [0,1]$ and $(x,y), (u,v) \in \Delta$.

Definition 4. A mapping $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ is called co-ordinated r-convex on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are r-convex for all $y \in [c,d]$ and $x \in [a,b]$.

Note that every r-convex mapping $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ is r-convex on the co-ordinates [10].

In this article the main purpose is newly to define the co-ordinated (s,r)-convex mapping in the first sense and prove some Hadamard-type inequalities for co-ordinated (s,r)-convex mappings in the first sense.

2. Main Results

We can define (s,r)-convex mapping on $[a,b]$ and (s,r)-convex mapping in the first sense on $\Delta = [a,b] \times [c,d]$ as follows:

Definition 5. A mapping $f : [a,b] \to \mathbb{R}_+$ is called (s,r)-convex in the first sense on $[a,b]$, for $s \in (0,1]$, if the following inequalities hold:

$$f(tx+(1-t)y) \leq \begin{cases} \left\{t^s f^r(x) + (1-t^s) f^r(y)\right\}^{\frac{1}{r}} & \text{if } r \neq 0 \\ f^{t^s}(x) f^{(1-t^s)}(y) & \text{if } r = 0. \end{cases}$$

for all $t \in [0,1]$ and $x, y \in [a,b]$.

Definition 6. A mapping $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ is called (s,r)-convex in the first sense on Δ, for $s \in (0,1]$, if the following inequalities hold:

$$f(tx+(1-t)y, \lambda u + (1-\lambda)v)$$

$$\leq \begin{cases} \big[t^s \lambda^s f^r(x,u) + t^s(1-\lambda^s) f^r(x,v) \\ \qquad + (1-t^s)\lambda^s f^r(y,u) + (1-t^s)(1-\lambda^s) f^r(y,v)\big]^{\frac{1}{r}} & \text{if } r \neq 0 \\ f^{t^s\lambda^s}(x,u) f^{t^s(1-\lambda^s)}(x,v) f^{(1-t^s)\lambda^s}(y,u) f^{(1-t^s)(1-\lambda^s)}(y,v) & \text{if } r = 0. \end{cases}$$

for any $t, \lambda \in [0,1]$ and $(x,y), (u,v) \in \Delta$.

Definition 7. A mapping $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ is called co-ordinated (s,r)-convex in the first sense on Δ, for $s \in (0,1]$, if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are (s,r)-convex in the first sense, for all $y \in [c,d]$ and $x \in [a,b]$.

To begin with, let us consider the following lemma:

Lemma 1. *Every (s,r)-convex mapping $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ in the first sense is a co-ordinated (s,r)-convex mapping in the first sense, for $s \in (0,1]$.*

Proof. Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}_+$ is (s,r)-convex in the first sense on Δ. Consider the partial mapping $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$, for all $y \in [c,d]$ and $x \in [a,b]$.

Case 1: For $r = 0$ and $u_1, u_2 \in [a,b]$, we have:

$$\begin{aligned} & f_y(tu_1 + (1-t)u_2) \\ & = f(tu_1 + (1-t)u_2, y) \\ & = f(tu_1 + (1-t)u_2, \lambda y + (1-\lambda)y) \\ & \leq f^{t^s\lambda^s}(u_1,y) f^{t^s(1-\lambda^s)}(u_1,y) f^{(1-t^s)\lambda^s}(u_2,y) f^{(1-t^s)(1-\lambda^s)}(u_2,y) \\ & = f^{t^s\lambda^s + t^s(1-\lambda^s)}(u_1,y) f^{(1-t^s)\lambda^s + (1-t^s)(1-\lambda^s)}(u_2,y) \\ & = f_y^{t^s}(u_1) f_y^{(1-t^s)}(u_2) \end{aligned}$$

Case 2: For $r \neq 0$ and $u_1, u_2 \in [a,b]$, we have:

$$\begin{aligned}
&f_y(tu_1 + (1-t)u_2) \\
&= f(tu_1 + (1-t)u_2, \lambda y + (1-\lambda)y) \\
&\leq \Big[t^s\lambda^s f_y^r(u_1) + t^s(1-\lambda^s) f_y^r(u_1) \\
&\qquad + (1-t^s)\lambda^s f_y^r(u_2) + (1-t^s)(1-\lambda^s) f_y^r(u_2)\Big]^{\frac{1}{r}} \\
&= \Big[t^s f_y^r(u_1) + (1-t^s) f_y^r(u_2)\Big]^{\frac{1}{r}}.
\end{aligned}$$

Therefore $f_y(u) = f(u,y)$ is (s,r)-convex in the first sense on $[a,b]$. By a simillar argument one can see that $f_x(v) = f(x,v)$ is (s,r)-convex in the first sense on $[c,d]$.

Theorem 2.1. *If $f, g : \Delta \to \mathbb{R}_+$ are two co-ordinated (s,r)-convex mappings in the first sense on Δ, then the product (fg) is also a co-ordinated (s,r)-convex mapping in the first sense on Δ.*

Proof. (i) Case 1: $r = 0$:

$$\begin{aligned}
&f(tx + (1-t)y, \lambda u + (1-\lambda)v) g(tx + (1-t)y, \lambda u + (1-\lambda)v) \\
&= \Big[f^{t^s\lambda^s}(x,u) f^{t^s(1-\lambda^s)}(x,v) f^{(1-t^s)\lambda^s}(y,u) f^{(1-t^s)(1-\lambda^s)}(y,v)\Big] \\
&\quad \times \Big[g^{t^s\lambda^s}(x,u) g^{t^s(1-\lambda^s)}(x,v) g^{(1-t^s)\lambda^s}(y,u) g^{(1-t^s)(1-\lambda^s)}(y,v)\Big] \\
&= \big\{f(x,u)g(x,u)\big\}^{t^s\lambda^s} \big\{f(x,v)f(x,v)\big\}^{t^s(1-\lambda^s)} \\
&\quad \times \big\{f(y,u)g(y,u)\big\}^{(1-t^s)\lambda^s} \big\{f(y,v)g(y,v)\big\}^{(1-t^s)(1-\lambda^s)} \\
&= \big(fg\big)^{t^s\lambda^s}(x,u) \big(fg\big)^{t^s(1-\lambda^s)}(x,v) \big(fg\big)^{(1-t^s)\lambda^s}(y,u) \big(fg\big)^{(1-t^s)(1-\lambda^s)}(y,v)
\end{aligned}$$

(ii) Case 2: $r \neq 0$:

$$\begin{aligned}
&f(tx + (1-t)y, \lambda u + (1-\lambda)v) g(tx + (1-t)y, \lambda u + (1-\lambda)v) \\
&\leq \Big[t^s\lambda^s f^r(x,u) + t^s(1-\lambda^s) f^r(x,v) + (1-t^s)\lambda^s f^r(y,u) \\
&\quad + (1-t^s)(1-\lambda^s) f^r(y,v)\Big]^{\frac{1}{r}} \Big[t^s\lambda^s g^r(x,u) + t^s(1-\lambda^s) g^r(x,v) \\
&\quad + (1-t^s)\lambda^s g^r(y,u) + (1-t^s)(1-\lambda^s) g^r(y,v)\Big]^{\frac{1}{r}} \\
&= \Big[t^s\lambda^s f^r(x,u) g^r(x,u) + t^s(1-\lambda^s) f^r(x,v) g^r(x,v) \\
&\quad + (1-t^s)\lambda^s f^r(y,u) g^r(y,u) + (1-t^s)(1-\lambda^s) f^r(x,u) g^r(x,u)\Big]^{\frac{1}{r}}.
\end{aligned}$$

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}_+$ be an (s, r)-convex mapping in the first sense on $[a, b]$ with $a < b$. Then for $r, s \in (0, 1]$ the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \Big[\Big\{\frac{r}{s+r}\Big\}^r f^r(a) + \Big\{\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})}\Big\}^r f^r(b)\Big]^{\frac{1}{r}}. \qquad (2)$$

Proof. Since f is an (s, r)-convex mapping in the first sense on $[a, b]$ with $a < b$ for $r, s > 0$, we have

$$f(ta + (1-t)b) \le \big\{t^s f^r(a) + (1-t^s) f^r(b)\big\}^{\frac{1}{r}}$$

for all $t \in [0, 1]$, which implies that

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &\le \Big[\int_0^1 \big\{t^s f^r(a) + (1-t^s) f^r(b)\big\}^{\frac{1}{r}} dt\Big]^{\frac{1}{r}}\\ &\le \Big[\Big\{\int_0^1 t^{\frac{s}{r}} f(a)dt\Big\}^r + \Big\{\int_0^1 (1-t^s)^{\frac{1}{r}} f(b)dt\Big\}^r\Big]^{\frac{1}{r}}\\ &= \Big[\Big\{\frac{r}{s+r}\Big\}^r f^r(a) + \Big\{\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})}\Big\}^r f^r(b)\Big]^{\frac{1}{r}}.\end{aligned}$$

Remark 1. Theorem 2.2 is a generalization of Theorem 1.1.

Theorem 2.3. *Let $f, g : [a, b] \to \mathbb{R}_+$ be, respectively, (s_1, r_1)-convex and (s_2, r_2)-convex mappings in the first sense on $[a, b]$ with $a < b$. Then for $0 < r_1, r_2 \le 2$ the following inequality holds:*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)dx\\ &\le \frac{1}{2}\Big[\Big\{\Big(\frac{r_1}{r_1+2s_1}\Big)^{\frac{r_1}{2}} f^{r_1}(a) + \Big(\frac{\Gamma(1+\frac{2}{r_1})\Gamma(1+\frac{1}{s_1})}{\Gamma(1+\frac{2}{r_1}+\frac{1}{s_1})}\Big)^{\frac{r_1}{2}} f^{r_1}(b)\Big\}^{\frac{2}{r_1}}\end{aligned}$$

$$+\Big\{\Big(\frac{r_2}{r_2+2s_2}\Big)^{\frac{r_2}{2}} g^{r_2}(a) + \Big(\frac{\Gamma(1+\frac{2}{r_2})\Gamma(1+\frac{1}{s_2})}{\Gamma(1+\frac{2}{r_2}+\frac{1}{s_2})}\Big)^{\frac{r_2}{2}} g^{r_2}(b)\Big\}^{\frac{2}{r_2}}\Big].$$

Proof. Since f and g are, respectively, (s_1, r_1)-convex and (s_2, r_2)-convex in the first sense on $[a, b]$ with $a < b$ for $r, s > 0$, we have

$$f(ta + (1-t)b) \le \big\{t^{s_1} f^{r_1}(a) + (1-t^{s_1}) f^{r_1}(b)\big\}^{\frac{1}{r_1}}$$

$$g(ta+(1-t)b) \le \left\{t^{s_2}g^{r_2}(a)+(1-t^{s_2})g^{r_2}(b)\right\}^{\frac{1}{r_2}}, \tag{3}$$

for all $t \in [0,1]$, which implies that, by using the property $ab \le \frac{1}{2}\{a^2+b^2\}$ for any $a,b \in \mathbb{R}_+$,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\le \int_0^1 \left\{t^{s_1}f^{r_1}(a)+(1-t^{s_1})f^{r_1}(b)\right\}^{\frac{1}{r_1}} \\
&\qquad \times \left\{t^{s_2}g^{r_2}(a)+(1-t^{s_2})g^{r_2}(b)\right\}^{\frac{1}{r_2}} dt \\
&\le \frac{1}{2}\Big[\int_0^1 \left\{t^{s_1}f^{r_1}(a)+(1-t^{s_1})f^{r_1}(b)\right\}^{\frac{2}{r_1}} dt \\
&\qquad + \int_0^1 \left\{t^{s_2}g^{r_2}(a)+\big(1-t^{s_2}\big)g^{r_2}(b)\right\}^{\frac{2}{r_2}} dt\Big].
\end{aligned} \tag{4}$$

By using Minkowski's inequality, we have

$$\begin{aligned}
&\int_0^1 \left\{t^{s_1}f^{r_1}(a)+(1-t^{s_1})f^{r_1}(b)\right\}^{\frac{2}{r_1}} dt \\
&\le \Big[\left\{f^2(a)\int_0^1 t^{\frac{2s_1}{r_1}} dt\right\}^{\frac{r_1}{2}} + \left\{f^2(b)\int_0^1 (1-t^{s_1})^{\frac{2}{r_1}} dt\right\}^{\frac{r_1}{2}}\Big]^{\frac{2}{r_1}} \\
&= \left\{\Big(\frac{r_1}{r_1+2s_1}\Big)^{\frac{r_1}{2}} f^{r_1}(a) + \Big(\frac{\Gamma(1+\frac{2}{r_1})\Gamma(1+\frac{1}{s_1})}{\Gamma(1+\frac{2}{r_1}+\frac{1}{s_1})}\Big)^{\frac{r_1}{2}} f^{r_1}(b)\right\}^{\frac{2}{r_1}}.
\end{aligned} \tag{5}$$

Similarily we have

$$\begin{aligned}
&\int_0^1 \left\{t^{s_2}[g(a)]^{r_2}+(1-t^{s_2})g^{r_2}(b)\right\}^{\frac{2}{r_2}} dt \\
&\le \left\{\Big(\frac{r_2}{r_2+2s_2}\Big)^{\frac{r_2}{2}} g^{r_2}(a) + \Big(\frac{\Gamma(1+\frac{2}{r_2})\Gamma(1+\frac{1}{s_2})}{\Gamma(1+\frac{2}{r_2}+\frac{1}{s_2})}\Big)^{\frac{r_2}{2}} g^{r_2}(b)\right\}^{\frac{2}{r_2}}.
\end{aligned} \tag{6}$$

By (4)-(6), the assertion in this theorem holds.

Theorem 2.4. *Let $f,g:[a,b]\to\mathbb{R}_+$ be, respectively, (s_1,r_1)-convex and (s_2,r_2)-convex mappings in the first sense on $[a,b]$ with $a<b$. Then for $r_1>1$ with $\frac{1}{r_1}+\frac{1}{r_2}=1$ the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\le \Big(\frac{1}{s_1+1}\Big)^{\frac{1}{r_1}}\Big(\frac{1}{s_2+1}\Big)^{\frac{1}{r_2}}\left\{f^{r_1}(a)+s_1f^{r_1}(b)\right\}^{\frac{1}{r_1}}\left\{g^{r_2}(a)+s_2g^{r_2}(b)\right\}^{\frac{1}{r_2}}.
\end{aligned}$$

Proof. Since f and g are, respectively, (s_1, r_1)-convex and (s_2, r_2)-convex mappings in the first sense on $[a, b]$ with $a < b$, for $r_1, r_2 > 0$ and $s_1, s_2 > 0$ we have the inequalities (4) for all $t \in [0, 1]$, which implies that

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\leq \int_0^1 \Big\{t^{s_1} f^{r_1}(a) + (1-t^{s_1}) f^{r_1}(b)\Big\}^{\frac{1}{r_1}} \\
&\qquad \times \Big\{t^{s_2} g^{r_2}(a) + (1-t^{s_2}) g^{r_2}(b)\Big\}^{\frac{1}{r_2}} dt \\
&\leq \Big\{\int_0^1 \big(t^{s_1} f^{r_1}(a) + (1-t^{s_1}) f^{r_1}(b)\big) dt\Big\}^{\frac{1}{r_1}} \\
&\qquad \times \Big\{\int_0^1 \big(t^{s_2} g^{r_2}(a) + (1-t^{s_2}) g^{r_2}(b)\big) dt\Big\}^{\frac{1}{r_2}} \\
&\leq \Big\{\frac{f^{r_1}(a) + s_1 f^{r_1}(b)}{s_1+1}\Big\}^{\frac{1}{r_1}} \Big\{\frac{g^{r_2}(a) + s_2 g^{r_2}(b)}{s_2+1}\Big\}^{\frac{1}{r_2}},
\end{aligned}$$

which completes the proof.

Corollary 1. *In Theorem 2.4, if we choose $r_1 = r_2 = 2$ and $s_1 = s_2 = 1$, then we get*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{2}\Big\{f^{r_1}(a) + f^{r_1}(b)\Big\}^{\frac{1}{r_1}} \Big\{g^{r_2}(a) + g^{r_2}(b)\Big\}^{\frac{1}{r_2}},$$

which implies Theorem 1.2.

Theorem 2.5. *Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is an (s, r)-convex mapping in the first sense on Δ. Then the following inequality holds:*

$$\begin{aligned}
&\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
&\leq \int_a^b \Big\{\big(\frac{r}{s+r}\big)^r f_x^r(c) + \big(\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})}\big)^r f_x^r(d)\Big\}^{\frac{1}{r}} dx \\
&\quad + \int_c^d \Big\{\big(\frac{r}{s+r}\big)^r f_y^r(a) + \big(\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})}\big)^r f_y^r(b)\Big\}^{\frac{1}{r}} dy. \qquad (7)
\end{aligned}$$

Proof. Since $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is a co-ordinated (s, r)-convex mapping in the first sense on Δ, the partial mappings $f_y : [a, b] \to \mathbb{R}_+$, $f_y(u) =$

$f(u,y)$ and $f_x : [c,d] \to \mathbb{R}_+$, $f_x(v) = f(x,v)$ are (s,r)-convex in the first sense for all $y \in [c,d]$ and $x \in [a,b]$, which implies that, by Theorem 2.2,

$$\frac{1}{d-c}\int_c^d f_x(y)dy \leq \Big\{(\frac{r}{s+r})^r f_x^r(c) + (\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})})^r f_x^r(d)\Big\}^{\frac{1}{r}}. \tag{8}$$

Dividing both side of the inequality (8) by $(b-a)$ and integrating with respect to x on $[a,b]$, we get

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \leq \int_a^b \Big\{(\frac{r}{s+r})^r f_x^r(c) + (\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})})^r f_x^r(d)\Big\}^{\frac{1}{r}} dx \tag{9}$$

By a similar argument, for the partial mapping $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \leq \int_c^d \Big\{(\frac{r}{s+r})^r f_y^r(a) + (\frac{\Gamma(1+\frac{1}{r})\Gamma(1+\frac{1}{s})}{\Gamma(1+\frac{1}{r}+\frac{1}{s})})^r f_y^r(b)\Big\}^{\frac{1}{r}} dy. \tag{10}$$

By (9) and (10), the inequality (7) is proved.

Theorem 2.6. *Let $f, g : [a,b] \to \mathbb{R}_+$ be, respectively, co-ordinated (s_1, r_1)-convex and co-ordinated (s_2, r_2)-convex mappings in the first sense on Δ. Then for $r_1 > 0$ and $r_2 \leq 2$, the following inequality holds:*

$$\begin{aligned}
&\frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f_x(y)g_x(y)dydx \\
&\leq \Big(\frac{r_1}{r_1+2s_1}\Big)\Big(\frac{1}{b-a}\Big)\int_a^b \{f^{r_1}(x,c)+f^{r_1}(x,d)\}^{\frac{2}{r_1}}dx \\
&\quad + \Big(\frac{r_2}{r_2+2s_2}\Big)\Big(\frac{1}{b-a}\Big)\int_a^b \{g^{r_2}(x,c)+g^{r_2}(x,d)\}^{\frac{2}{r_2}}dx \\
&\quad + \Big(\frac{r_1}{r_1+2s_1}\Big)\Big(\frac{1}{d-c}\Big)\int_c^d \{f^{r_1}(a,y)+f^{r_1}(b,y)\}^{\frac{2}{r_1}}dy \\
&\quad + \Big(\frac{r_2}{r_2+2s_2}\Big)\Big(\frac{1}{d-c}\Big)\int_c^d \{g^{r_2}(a,y)+g^{r_2}(b,y)\}^{\frac{2}{r_2}}dy.
\end{aligned}$$

Proof. Since f is a co-ordinated (s_1, r_1)-convex mapping in the first sense and g is a co-ordinated (s_2, r_2)-convex mapping in the first sense on Δ, for $r_1, r_2 > 0$ and $s_1, s_2 > 0$ the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}_+$, $f_x(v) = f(x, v)$ are (s_1, r_1)-convex in the first sense, and the partial mappings $g_y : [a, b] \to \mathbb{R}_+$, $g_y(u) = g(u, y)$ and $g_x : [c, d] \to \mathbb{R}_+$, $g_x(v) = g(x, v)$ are (s_2, r_2)-convex in the first sense, for all $y \in [c, d]$ and $x \in [a, b]$.

By Theorem 2.3, we have

$$\frac{2}{(b-a)} \int_a^b f_y(x) g_y(x) dx \leq \frac{r_1}{r_1 + 2s_1} \left\{ f_y^{r_1}(a) + f_y^{r_1}(b) \right\}^{\frac{2}{r_1}} + \frac{r_2}{r_2 + 2s_2} \left\{ g_y^{r_2}(a) + g_y^{r_2}(b) \right\}^{\frac{2}{r_2}}, \tag{11}$$

and

$$\frac{2}{(d-c)} \int_c^d f_x(y) g_x(y) dy \leq \frac{r_1}{r_1 + 2s_1} \left\{ f_x^{r_1}(c) + f_x^{r_1}(d) \right\}^{\frac{2}{r_1}} + \frac{r_2}{r_2 + 2s_2} \left\{ g_y^{r_2}(c) + g_y^{r_2}(d) \right\}^{\frac{2}{r_2}}. \tag{12}$$

By dividing both side of the inequalities (11) and (12) $(b - a)$ and $(d - c)$, and integrating with respect to x on $[a, b]$ and y on $[c, d]$, respectively, we get:

$$\begin{aligned} &\frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dy dx \\ &\leq \left(\frac{r_1}{r_1 + 2s_1} \right) \left(\frac{1}{d-c} \right) \int_c^d \left\{ f_y^{r_1}(a) + f_y^{r_1}(b) \right\}^{\frac{2}{r_1}} dy \\ &\quad + \left(\frac{r_2}{r_2 + 2s_2} \right) \left(\frac{1}{d-c} \right) \int_c^d \left\{ g_y^{r_2}(a) + g_y^{r_2}(b) \right\}^{\frac{2}{r_2}} dy \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dy dx \\ &\leq \left(\frac{r_1}{r_1 + 2s_1} \right) \left(\frac{1}{b-a} \right) \int_a^b \left\{ f_x^{r_1}(c) + f_x^{r_1}(d) \right\}^{\frac{2}{r_1}} dx \\ &\quad + \left(\frac{r_2}{r_2 + 2s_2} \right) \left(\frac{1}{b-a} \right) \int_a^b \left\{ g_x^{r_2}(c) + g_x^{r_2}(d) \right\}^{\frac{2}{r_2}} dx. \end{aligned} \tag{14}$$

By adding the inequalities (13) and (14), we have

$$\begin{aligned}
&\frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f_x(y)g_x(y)dydx\\
&\le\Big(\frac{r_1}{r_1+2s_1}\Big)\Big(\frac{1}{b-a}\Big)\int_a^b\Big\{f_x^{r_1}(c)+f_x^{r_1}(d)\Big\}^{\frac{2}{r_1}}dx\\
&\quad+\Big(\frac{r_2}{r_2+2s_2}\Big)\Big(\frac{1}{b-a}\Big)\int_a^b\Big\{g_x^{r_2}(c)+g_x^{r_2}(d)\Big\}^{\frac{2}{r_2}}dx\\
&\quad+\Big(\frac{r_1}{r_1+2s_1}\Big)\Big(\frac{1}{d-c}\Big)\int_c^d\Big\{f_y^{r_1}(a)+f_y^{r_1}(b)\Big\}^{\frac{2}{r_1}}dy\\
&\quad+\Big(\frac{r_2}{r_2+2s_2}\Big)\Big(\frac{1}{d-c}\Big)\int_c^d\Big\{g_y^{r_2}(a)+g_y^{r_2}(b)\Big\}^{\frac{2}{r_2}}dy,
\end{aligned}$$

which completes the proof.

Corollary 2. *In Theorem 2.7, if we choose $r_1=r_2=2$ and $s_1=s_2=1$, then we have*

$$\begin{aligned}
&\frac{8}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\
&\le\Big(\frac{1}{b-a}\Big)\int_a^b\Big\{f^2(x,c)+f^2(x,d)+g^2(x,c)+g^2(x,d)\Big\}dx\\
&\quad+\Big(\frac{1}{d-c}\Big)\int_c^d\Big\{f^2(a,y)+f^2(b,y)+g^2(a,y)+g^2(b,y)\Big\}dy.
\end{aligned}$$

Theorem 2.7. *Let $f,g:[a,b]\to\mathbb{R}_+$ be, respectively, (s_1,r_1)-convex and (s_2,r_2)-convex mappings in the first sense on $[a,b]$ with $a<b$. Then for $r_1>1$ with $\frac{1}{r_1}+\frac{1}{r_2}=1$ the following inequality holds:*

$$\begin{aligned}
&\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\
&\le\Big(\frac{1}{s_1+1}\Big)^{\frac{1}{r_1}}\Big(\frac{1}{s_2+1}\Big)^{\frac{1}{r_2}}\Big(\frac{1}{b-a}\Big)^2\\
&\quad\times\int_a^b\Big\{f^{r_1}(x,c)+f^{r_1}(x,d)\Big\}^{\frac{1}{r_1}}dx\int_a^b\Big\{g^{r_2}(x,c)+g^{r_2}(x,d)\Big\}^{\frac{1}{r_2}}dx\\
&+\Big(\frac{1}{s_1+1}\Big)^{\frac{1}{r_1}}\Big(\frac{1}{s_2+1}\Big)^{\frac{1}{r_2}}\Big(\frac{1}{d-c}\Big)^2\\
&\quad\times\int_c^d\Big\{f^{r_1}(a,y)+f^{r_1}(b,y)\Big\}^{\frac{1}{r_1}}dy\int_c^d\Big\{g^{r_2}(a,y)+g^{r_2}(b,y)\Big\}^{\frac{1}{r_2}}dy.
\end{aligned}$$

Proof. Since f is a co-ordinated (s_1,r_1)-convex mapping in the first sense and g is a co-ordinated (s_2,r_2)-convex mapping in the first sense on Δ for $r_1>1$ with $\frac{1}{r_1}+\frac{1}{r_2}=1$ and $s_1,s_2>0$, the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}_+$, $f_x(v)=f(x,v)$ are (s_1,r_1)-convex in the first sense for all $y\in[c,d]$ and $x\in[a,b]$, and the partial mappings $g_y:[a,b]\to\mathbb{R}_+$, $g_y(u)=g(u,y)$ and $g_x:[c,d]\to\mathbb{R}_+$, $g_x(v)=g(x,v)$ are (s_2,r_2)-convex in the first sense for all $y\in[c,d]$ and $x\in[a,b]$.

By Theorem 2.5, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\
&\le\left(\frac{1}{s_1+1}\right)^{\frac{1}{r_1}}\left(\frac{1}{b-a}\right)\int_a^b\left\{f^{r_1}(x,c)+f^{r_1}(x,d)\right\}^{\frac{1}{r_1}}dx\\
&\quad\times\left(\frac{1}{s_2+1}\right)^{\frac{1}{r_2}}\left(\frac{1}{b-a}\right)\int_a^b\left\{g^{r_2}(x,c)+g^{r_2}(x,d)\right\}^{\frac{1}{r_2}}dx
\end{aligned}\tag{15}$$

and

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\
&\le\left(\frac{1}{s_1+1}\right)^{\frac{1}{r_1}}\left(\frac{1}{d-c}\right)\int_c^d\left\{f^{r_1}(a,y)+f^{r_1}(b,y)\right\}^{\frac{1}{r_1}}dy\\
&\quad\times\left(\frac{1}{s_2+1}\right)^{\frac{1}{r_2}}\left(\frac{1}{d-c}\right)\int_c^d\left\{g^{r_2}(a,y)+g^{r_2}(b,y)\right\}^{\frac{1}{r_2}}dy.
\end{aligned}\tag{16}$$

By the inequalities (15) and (16), the assertion in this theorem holds.

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