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SPIN$^{\mathbb{C}}$ QUANTIZATION IN ODD DIMENSIONS

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Abstract: We define and discuss an extension of the Spin$^{\mathbb{C}}$ quantization concept to odd-dimensional manifolds. After that we describe its relation to (the usual) even-dimensional Spin$^{\mathbb{C}}$ quantization and how its famous properties like "Quantization commutes with reduction" can be regained in odd dimensions. At the end, we analyse the situation on 3-manifolds and give some examples.

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1. Odd-Dimensional Spin$^{\mathbb{C}}$ Quantization

Unlike geometric quantization efforts using symplectic or almost complex structures, Spin$^{\mathbb{C}}$ quantization, as it is discussed in [5] and [4], seems to be not completely dependent on the assumption $\dim M = 2m$. Of course, if you naively try to replace $2m$ by the odd number $2m-1$ in all definitions, you fail because the equivariant Spin$^{\mathbb{C}}$ Dirac operator will not split, so you cannot define a (non-zero) index.

Nevertheless, if you instead consider a family of operators parametrised by S^1 you get an equivariant spectral flow, i.e. an element of $K^1_{S^1}(S^1) \cong K^1(S^1) \otimes R(S^1)$. Through the isomorphism $K^1(S^1) \cong K^0(\{\mathrm{pt}\})$ this can be seen as quantization in the sense of [5]. We give a more precise definition in the next section.

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This index is closely related to the even-dimensional index, using maps of the form $M \mapsto M \times S^1$.

2. Definition of the Structure

Let M be a (closed, Riemannian, oriented) $\mathrm{Spin}^{\mathbb{C}}$ manifold of dimension $2m-1$ with differentiable S^1-action. We assume that an S^1-equivariant $\mathrm{Spin}^{\mathbb{C}}$ structure $\tilde{P}_M$ on M is chosen. In the following discussion, α always denotes an element of $H^1_{S^1}(M;\mathbb{Z})$, which will be interpreted either as the S^1-invariant first cohomology group or as S^1-equivariant harmonic one-forms on M. Our quantization will be a map, which associates to a pair $(\tilde{P}_M, \alpha)$ an element in $K^1_{S^1}(S^1)$, being a group homomorphism in the second component.

The $\mathrm{Spin}^{\mathbb{C}}$ structure $\tilde{P}_M$ has an associated vector bundle $\mathcal{S}_{\mathbb{C}}(M)$ that comes along with a family of Dirac operators parametrised by connection one forms on the determinant line bundle. All these constructions should be made S^1-equivariant (and in the following discussions all maps between spaces with S^1-action should be equivariant). Now fix one Dirac operator $\mathcal{D}_0$ and describe all other Dirac operators by $\mathcal{D}_\gamma := \mathcal{D}_0 + \mathtt{i}c_\gamma$, $\gamma \in \Omega^1(M)$, where c means Clifford multiplication.

Now we can associate to α a family $\mathcal{D}_{t\alpha}$, $t \in [0,1]$ of Dirac operators. Since $H^1_{S^1}(M;\mathbb{Z})$ is isomorphic to the space $[M, S^1]_{S^1}$ of S^1-invariant homotopy classes of maps to S^1, we can choose an S^1-invariant map u_α for our fixed element α. A direct calculation shows that $\mathcal{D}_\alpha = u_\alpha^{-1}\mathcal{D}_0 u_\alpha$, which particularly shows that the beginning and end of our family have the same spectrum. Furthermore, we can interpret u_α as invertible S^1-invariant linear map on the Hilbert space H_M defined by $\Gamma_{L^2}(\mathcal{S}_{\mathbb{C}}(M))$ (with inherited S^1-action). The space $\mathrm{Gl}_{S^1}(H_M)$ of S^1-equivariant invertible endomorphisms is path-connected (it is in general *not* contractible, as Kuiper's theorem proves for the non-equivariant case), so we can transform u_α into 1 by a family u^t_α. Here we have to be aware of the fact that u^t_α will in general be no bundle isomorphism but just a Hilbert space map. Nevertheless we know that conjugating with u^t_α will not change the S^1-equivariant spectrum of an operator on H_M.

So putting together the family $\mathcal{D}_{t\alpha}$ with $(u^t_\alpha)^{-1}\mathcal{D}_\alpha u^t_\alpha$ we get a cyclic family of operators, where the spectrum is constant during the second part. Therefore, the equivariant spectral flow of this family (which equals the equivariant spectral flow of the first half) defines an element of $K^1_{S^1}(S^1)$. Now $K^1_{S^1}(S^1) \cong K^0_{S^1}(\{\mathrm{pt}\})$; we choose an isomorphism in the following way: The embedding $\{\mathrm{pt}\} \subset S^1$ induces an isomorphism $H_0(\{\mathrm{pt}\}) \cong H_0(S^1)$, which by tensoring with $R(S^1)$, Poincare-duality and the Chern character defines the isomorphism map. The composition of the map to $K^1_{S^1}(S^1)$ with this last isomorphism will be our *quantization*. It is a virtual representation of S^1.

Lemma 1. *For fixed $\tilde{P}_M$, the quantization is a group homomorphism $Q(M)$ from $H^1_{S^1}(M;\mathbb{Z})$ to $K^1_{S^1}(S^1) \cong K^0_{S^1}(\{pt\})$.*

Proof. Let $\mathcal{D}_0$ be a fixed Dirac operator. Then the spectral flow of $\mathcal{D}_\beta + \mathtt{tic}_\alpha$ is independent of β; furthermore, it is independent of the path which connects $\mathcal{D}_\beta$ and $\mathcal{D}_{\beta+\alpha}$ by homotopy invariance of the spectral flow. Therefore, we can connect $\mathcal{D}_0$ and $\mathcal{D}_{\alpha+\beta}$ by touching $\mathcal{D}_\alpha$. This shows that the spectral flow of $\alpha + \beta$ is just the sum of the two spectral flows. □

3. Going Up the Stairs

We now want to relate even and odd dimensional quantizations. For this section, let M^{2m-1} be an odd-dimensional and N^{2n} be an even-dimensional manifold; X will serve as placeholder for both.

The easiest way to "go up one dimension" is to replace X by $X \times S^1$. This again should be an S^1-manifold, so that we have to combine the action on X with an action on S^1. Every non-trivial action on S^1 will force $X \times S^1$ to be a fixed-point-free space; this leads to a zero index (we already know this from the even-dimensional case, but we will later see that is also true in odd dimensions). So we take an S^1 with trivial action.

Taking the only possible Spin$^{\mathbb{C}}$ structure on S^1, we get a Spin$^{\mathbb{C}}$ structure $\tilde{P}_{X \times S^1}$ on $X \times S^1$.

Let data(X) be the data we need on X to define a quantization, i.e.

- for $\dim X$ even this is $\{\text{Spin}^{\mathbb{C}}$ structures of $X\}$ and
- for $\dim X$ odd we have $\{\text{Spin}^{\mathbb{C}}$ structures of $X\} \times H^1_{S^1}(X;\mathbb{Z})$.

Now we want to find a map $\nearrow$: data$(X) \to$ data$(X \times S^1)$ so that the following diagram commutes:

$$\begin{array}{ccc} \text{data}(X) & \xrightarrow{Q(X)} & K^0_{S^1}(S^1) \\ \nearrow \big\downarrow & \nearrow_{Q(X\times S^1)} & \\ \text{data}(X \times S^1) & & \end{array} \tag{1}$$

Definition 2. Let e_{S^1} be the positive generator of $H^1_{S^1}(S^1) = H^1(S^1)$. Then we define $\nearrow$ to be

- the map which sends $(\tilde{P}_M, \alpha)$ to $\tilde{P}_M + \alpha \cup e_{S^1}$, where the addition of $\alpha \cup e_{S^1} \in H^2_{S^1}(M;\mathbb{Z})$ means twisting with the respective line bundle (odd case).
- or the map which sends $\tilde{P}_N$ to $(\tilde{P}_N, e_{S^1})$ (even case).

Theorem 3. *The map $\nearrow$ just defined makes (1) commutative.*

Proof. We first argue for M:

An equivalence of indices of this kind was mentioned in [1]. In a similar fashion, we want to construct an argument out of [3] and [2]. Since the periodicity of $\mathcal{D}_t$ is produced by a twist, we first look at $M \times [0, 1]$ with an equivariant spectral flow from 0 to 1. Following [3], p.95, we can identify this spectral flow with the APS-index of $\frac{\partial}{\partial t} + \mathcal{D}_t$ on the manifold $M \times [0, 1]$. Now the boundary terms only depend on the equivariant spectra of $\mathcal{D}_0$ and $\mathcal{D}_1$, which are equal. So the boundary terms vanish and we get the index of $\frac{\partial}{\partial t} + \mathcal{D}_t$ over a twisted bundle over $M \times S^1$. This is the same as the index of the even-dimensional Dirac operator after identifying the two vector bundle components $\mathcal{S}^+$ and $\mathcal{S}^-$ with the help of $\frac{\partial}{\partial t}$.

Now for N: If we apply $\nearrow$ twice, we again get an even-dimensional manifold $N \times S^1 \times S^1$, where the Spin$^{\mathbb{C}}$-structure on $\tilde{P}_{N\times S^1\times S^1}$ is given by $\tilde{P}_N + e^1_{S^1} \cup e^2_{S^1}$. If we take $\tilde{P}_{Sq\times S^1} + e^1_{S^1} \cup e^2_{S^1}$ on $S^1 \times S^1$, we get a Dirac operator $\mathcal{D}_{S^1\times S^1}$ with index 1. The construction leads to the situation, that the index on $M \times S^1 \times S^1$ is just the product of the indices on both spaces, so we see that after applying $\nearrow$ twice, we again have the same quantization. As we already proved the odd-dimensional case, this is enough for the even-dimensional case. □

Corollary 4. *In every dimension, there are manifolds with non-trivial quantization.*

Proof. We know that $Q(S^2) \neq 0$ and can then proceed by induction. □

4. The Fixed Point Formula

The even-dimensional Spin$^{\mathbb{C}}$ quantization, described by its character χ, can be calculated by a fixed point formula stemming from the Atiyah-Segal-Singer-index theorem. It looks like this:

$$\chi\big(\exp_{S^1}(v)\big) = \sum_{F\subset N^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot \int_F \exp\left(\tfrac{1}{2}\tilde{c}_1(L|_F)\right)(v) \cdot \hat{A}(TF) \cdot \hat{A}_e(NF)(v).$$

Here, $\exp_{S^1}$ is the exponential map of the Lie-Group S^1, mapping an element $v \in \mathfrak{s}_1 \cong \mathbb{R}$ to S^1, so that the left hand side is an element of $\mathbb{C}$. Its value in any neighbourhood of $0 \in \mathfrak{s}_1$ determines χ and therefore $Q(M)$.

On the right side, we sum integrals over the (finitely many) fixed point components. Here, $m(F)$ is the complex codimension of F and $(-1)^F$ a sign depending on orientations (which is discussed in [5] and is of no great importance here). In the integral itself we use equivariant characteristic classes for the trivial S^1-space F. Here, $\tilde{c}_1$ describes the equivariant first Chern class, $\hat{A}$ the A-roof-class and $\hat{A}_e$ the quotient of the equivariant A-roof-class and Euler-class. The very last term only exists for a bundle which does not contain trivial representations and is only well-defined for small v. This formula was introduced in [4]; a thorough discussion and proof can be found in [6]. Notice that the terms in the integral are of the form

$$\sum (\text{cohomology-class}) \cdot (\text{representation of } S^1),$$

so that the integral transforms this terms into the character of a (virtual) representation (integrals of classes of the wrong dimension are defined to be zero).

Since $N = M \times S^1$ has the same quantization as M, we can replace N by $M \times S^1$ in this formula. Our aim is to derive a simplified version of this result, eliminating the detour over N that we took.

First of all, all fixed point sets in $M \times S^1$ are of the form $F \times S^1$ because the action on S^1 is trivial. We especially see that all fixed point sets in M are odd-dimensional; particularly, we have no isolated fixed points anymore. The integral $\int_{F\times S^1}$ will be thought of as iterated integral $\int_F \int_{S^1}$; the inner integral will be computed in the next paragraphs.

We now want to understand the three terms involved in the formula and start at the very end: The bundle $N(F\times S^1)$ in $T(M\times S^1)$ is (equivariantly) the same as the bundle $\pi_{S^1}^*(NF)$ where $\pi_{S^1} : F \times S^1 \to F$ denotes the projection (sometimes also the projection of $M \times S^1$ to M). Since $\pi_{S^1}^*$ is a group homomorphism commuting with characteristic classes, we can replace $A_e(N(F\times S^1))$ by $\pi_{S^1}^*(A_e(NF))$.

For $A(T(F \times S^1))$ notice that $T(F \times S^1) = \pi_{S^1}^*(TF) \oplus \pi_F^*(TS^1)$. The A-roof class is multiplicative under direct sums of vector bundles; furthermore it is just 1 on trivial bundles like TS^1. So we get $\pi_{S^1}^*(A(TF))$.

The first Chern class of the determinant line bundle splits into two parts: The bundle L over M gives us a class $\exp\left(\frac{1}{2}\pi_{S^1}^*(\tilde{c}_1(L))\right)$ while the twisting with the bundle of (equivariant) first Chern class $\alpha \cup e_{S^1}$ gives an extra term $\exp(\alpha \cup e_{S^1})$. So the situation looks like this:

$$\begin{aligned}
&\chi\big(\exp_{S^1}(v)\big) \\
&= \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot \int_F \int_{S^1} \exp(\alpha|_F \cup e_{S^1}) \\
&\qquad\qquad \cdot \pi_{S^1}^*\Big(\exp\left(\tfrac{1}{2}\tilde{c}_1(L|_F)\right)(v) \cdot \hat{A}(TF) \cdot \hat{A}_e(NF)(v)\Big) \\
&= \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot \int_F \Big(\exp\left(\tfrac{1}{2}\tilde{c}_1(L|_F)\right)(v) \cdot \hat{A}(TF) \\
&\qquad\qquad\qquad\qquad \cdot \hat{A}_e(NF)(v) \int_{S^1} \exp(\alpha|_F \cup e_{S^1})\Big) \\
&= \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \\
&\qquad\qquad \cdot \int_F \exp\left(\tfrac{1}{2}\tilde{c}_1(L|_F)\right)(v) \cdot \hat{A}(TF) \cdot \hat{A}_e(NF)(v) \cdot \alpha|_F.
\end{aligned}$$

Note that this again shows that the quantization is a group homomorphism in $H^1_{S^1}(M;\mathbb{Z})$ and furthermore, that it only depends on the value of α on the different fixed point components (if $\alpha|_F = 0$ the component F does not deliver anything for $Q(M)$).

5. Additivity and $[Q, R] = 0$

The important theorems for "Additivity under Cutting" and "Quantization commutes with Reduction" can now be easily transferred to the odd-dimensional world. For that we first have to make some definitions.

As in the even-dimensional case, let $Z \subset M$ be a splitting hypersurface with a free S^1-action. Then the reduced manifold M_{red} is given by Z/S^1. The cut-spaces $M_{\text{cut}}^{\pm}$ are constructed as in the even-dimensional case. To define the quantization of M_{red} and $M_{\text{cut}}^{\pm}$ we have to carry the invariant one-form α with us: We take $(\alpha|Z)/S^1$ on M_{red} and a similar restriction on $M_{\text{cut}}^{\pm}$. Then we have:

Theorem 5. *In the case of odd-dimensional manifolds, assuming the adjusted hypotheses above, "Additivity under Cutting", which means*

$$Q(M) = Q(M^+_{cut}) + Q(M^-_{cut}),$$

and "Quantization commutes with reduction", which is

$$[Q,\, R] = 0,$$

holds.

Proof. Take the manifold $M \times S^1$ with splitting hypersurface $Z \times S^1$. Since the addition of $\alpha \cup e_{S^1}$ and the constructions on the Spin$^{\mathbb{C}}$ structures commute, the theorems for $M \times S^1$ imply the same ones for M. □

6. The Situation in 3 Dimensions

The lowest dimension that seems worth investigating is three. The fixed point set of a non-trivial S^1-action on M consists of a finite union of circles. The fixed point formula becomes significantly easier.

6.1. The Fixed Point Formula Revisited

Since we are integrating over circles, and we already have a one-form α involved, we know that all integrals with further "form-parts" have to vanish. So it is enough to extract the "pure representation part" of each of the other three terms. For $\exp\left(\frac{1}{2}\tilde{c}_1(L|_F)\right)$, this is $z^{\frac{1}{2}\mu_F}$ (where μ_F is the degree of the representation of S^1 on $L|_F$). For $\hat{A}_e(NF)$ this is just the action of S^1 on the (real, two-dimensional) vector bundle NF (called $z^{\frac{1}{2}n_F}$). So we get

$$\chi(z) = \sum_F z^{\frac{1}{2}(\mu_F + n_F)} \cdot \int_F \alpha|_F.$$

6.2. Invariant Hypersurfaces

Let $Z \subset M$ be a 2-dimensional hypersurface, invariant under the action of S^1. The fixed point set of this action may consist of the whole of Z; otherwise it just consists of isolated points (or vanishes completely). Since the fixed point set on Z is part of the fixed point set on M, which consists of odd-dimensional manifolds, the first case implies that Z lies in a three-dimensional component F of the fixed point set; since F has to be open and closed, we have $F = M$, which is boring.

So we rule out cases in which the action on Z is trivial. This leaves us only with the cases $Z \cong S^2$ and $Z \cong T^2$, since surfaces of higher genus do not offer us non-trivial S^1-actions. We investigate the two cases:

6.2.1. The 2-Sphere

The 2-sphere lacks a fixed-point free S^1-action, so we cannot use it as a splitting hypersurface in the sense of quantization. Nevertheless, we can look at this surface in the context of connected sums. We equip S^2 with the rotational action (around the z-axis) of speed $l \in \mathbb{Z}$ (which is essentially the only action on S^2), calling it S^2_l. Now, if two manifolds M_1 and M_2 have open balls $B^3_{\pm l}$ bounding S^2_l and S^2_{-l} we can form an equivariant connected sum. $M_1 \# M_2$ inherits an equivariant $\text{Spin}^{\mathbb{C}}$ structure, since both $\text{Spin}^{\mathbb{C}}$ structures can be identified over $B^3_{\pm l}$.

What happens to the fixed point set in this construction? The S^1-components not touching $S^2_{\pm l}$ do not change, the two 1-spheres going through the poles of $S^2_{\pm l}$ will be connected to one big S^1. Before we discuss the fixed points further, we have a look at our invariant first cohomology and line bundles:

Since the cohomology in dimension 1 and 2 is additive, we write every element of $H^1_{S^1}(M_1 \# M_2)$ as $\alpha_1 + \alpha_2$, where α_i are representatives of $H^1_{S^1}(M_i)$ with $\alpha_i | B^3_{\pm l} \equiv 0$. Every (equivariant) line bundle L will be written as $L_1 \otimes L_2$, where L_i is a line bundle over M_i extended over M_{2-i} by trivialising it over $B^3_{\pm l}$ in a non-equivariant fashion (i.e. it becomes topologically trivial with maybe non-trivial S^1-action). Notice that if L_1 and L_2 are the determinant line bundles for the two $\text{Spin}^{\mathbb{C}}$ structures on M_1 and M_2, then the resulting $\text{Spin}^{\mathbb{C}}$ structure will have $L_1 \otimes \overline{L_2}$ as determinant line bundle (in the sense just explained).

By F_1 and F_2 we denote the two fixed point components which are connected by $\#$. We have to analyse the resulting component F of $M_1 \# M_2$ with the fixed point formula.

First of all, notice that the integral splits into $\int_{F_1} \alpha_1 + \int_{F_2} \alpha_2$, since we chose our one-forms to be supported only in their own "half" of the connected sum. The spinning number l of the sphere implies that $n_F = l$. Since our determinant line bundle is given by $L_1 \otimes L_2$, we get $\mu_F = \mu_{F_1} + \mu_{F_2}$. So we have

$$Q(M_1 \# M_2) = Q(M_1) + Q(M_2) + D(F),$$

where $D(F)$ is given as

$$\left(z^{\frac{1}{2}(l+\mu_{F_1}+\mu_{F_2})} - z^{\frac{1}{2}(l+\mu_{F_1})}\right)\int_{F_1}\alpha_1 + \left(z^{\frac{1}{2}(l+\mu_{F_1}+\mu_{F_2})} - z^{\frac{1}{2}(l+\mu_{F_2})}\right)\int_{F_2}\alpha_2.$$

6.2.2. The 2-Torus

Every non-trivial S^1-action on the two-torus is free. From the discussion above we know that toruses with free S^1-action are the only possible splitting hypersurfaces. The reduction then is an S^1 (without a group action). We have a short look at the quantization of such an S^1:

There is just one Spin$^{\mathbb{C}}$ structure, combined with a one-dimensional space $H^1_{S^1}(S^1) \cong H^1(S^1) \cong \mathbb{Z}$. The quantization map is just the Chern character $H^1(S^1) \cong K^1(S^1)$.

So if M_{red} consists of k components, we get a map of the form

$$H^1(S^1) \oplus H^1(S^1) \oplus \cdots \oplus H^1(S^1) \to K^1(S^1)$$

where we get one summand for every component of M_{red}. Please note, that in the $[Q,R]$-theorem, we just apply the quantization map to the subspace of $H^1(S^1) \oplus H^1(S^1) \oplus \cdots \oplus H^1(S^1)$ which consists of the image of $\iota^*\big(H^1(M)\big)$ where $\iota : M_{\text{red}} \to M$ is the embedding.

Let us now come to some examples.

6.3. Examples

For the connected sums, we get a nice playground by repeatedly connecting $S^2 \times S^1$ along balls around the poles; the action is given by a twist of speed l on the S^2 and the trivial action on S^1. Of course, for $S^2 \times S^1$, the index is just given by mapping the S^2-index with $\nearrow$ to $S^2 \times S^1$ (the S^2-index is calculated in detail in [5] and [6]).

If we look at S^3, we can do the following: Representing elements as (z_1, z_2) with $|z| = 1$, we find splitting hypersurfaces for fixed $0 < |z| < 1$. For every $(n_1, n_2) \in \mathbb{Z}^2$ we get an S^1-action on S^3. For it to be free on a surface Z we have to assume that $\gcd(n_1, n_2) = 1$. If both n_1 and n_2 are non-zero, there are no fixed points on S^3, so $Q(S^3)$ is the zero-map. In general, we know that $R \circ Q = Q \circ R = 0$, since $H^1_{S^1}(S^3) \to H^1(S^1)$ is the zero map which implies that the quantization on the reduced manifold is always zero.

What about $T^3 = \mathbb{R}^3/\mathbb{Z}^3$? We get a number of S^1-actions on T^3 of the form $(z^{n_1}, z^{n_2}, z^{n_3})$. Now we choose Z to consist of two 2-tori, on which the induced action is free. The splitting construction shows that $M^{\pm}_{\text{cut}} \cong S^3$ with two fixed point circles. The "Additivity under Cutting" now implies that their quantizations are the opposite of each other.

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