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CONVERGENCE AND STABILITY OF AN ITERATIVE SCHEME FOR GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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Abstract: In this paper, a new class of variational inequalities, which are called the generalized nonlinear variational-like inequalities are introduced and analyzed in Hilbert spaces. Making use of the auxiliary principle technique, some existence theorems of solutions for the generalized nonlinear variational-like inequalities are proved and an iterative algorithm to compute the approximate solutions is constructed. Some stability results of iterative sequences generated by the iterative algorithm are also discussed. These results presented here are the extensions and improvements of the earlier and recent results in this field.

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1. Introduction

It is well known that variational inequality theory plays an important and fundamental role in mechanics, optimization, operations research, management sciences, and other branches of mathematical and engineering sciences, and have become a rich source of inspiration for scientists and engineers. For details, we refer to [1]-[7], [9]-[11] and the references therein. A useful and important generalization of variational inequalities is variational-like inequalities. The variational-like inequalities have potential and significant applications in optimization theory, structural analysis and economics. On the other hand, one of the most interesting and important problems in the variational inequality theory is the development of efficient and implementable numerical methods. There are a substantial number of numerical methods including the projection method and its variant forms, linear approximation and Newton's methods for variational inequalities. However, there are very few methods for some special cases of variational-like inequalities. It is worth mentioning that the projection type technique cannot be used to suggest iterative algorithms for variational-like inequalities, since it is not possible to find the projection of the solution. Glowinski, Lions and Tremolieres [3] have suggested another technique which does not depend upon the projection. This technique is called the auxiliary principle technique.

Huang and Deng [6] extended the auxiliary principle technique to study a class of generalized set-valued strongly nonlinear mixed variational-like inequalities, proved the existence of a solution of the auxiliary problem by using Theorem 1 of Chang [1], constructed an iterative algorithm and show the existence of a solution of the generalized set-valued strongly nonlinear mixed variational-like inequality by using the auxiliary principle technique. Verma [11] studied some nonlinear variational inequalities.

Inspired and motivated by the recent research works [1], [2], [6], [7], [9], [11], in this paper, we introduce and study a new class of generalized nonlinear variational-like inequalities that include variational inequalities studied in [9], [11] as special cases. The auxiliary problem for this class of generalized nonlinear variational-like inequalities given here is different from that of Huang and Deng [6] and Luo [7]. Using Theorem 1 of Chang [1], we prove the existence of a solution of the auxiliary problem for the generalized nonlinear variational-like inequality, and obtain existence and uniqueness of solution for the generalized nonlinear variational-like inequality. We also suggest an iterative algorithm for solving the class of the generalized nonlinear variational-like inequalities and obtain convergence and stability results of iterative sequences generated by the iterative algorithm.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and I denote the identity mapping on H.

Let $b : H \times H \to (-\infty, +\infty)$ be nondifferentiable and satisfy the following conditions:

(c1) b is linear in the first argument;

(c2) b is convex in the second argument;

(c3) b is bounded, that is, there exists a constant $\gamma > 0$ satisfying

$$|b(u, v)| \leq \gamma \|u\| \|v\|, \quad \forall u, v \in H;$$

(c4) $b(u, v) - b(u, w) \leq b(u, v - w)$, $\forall u, v, w \in H$;

Let $f : H \times H \to (-\infty, +\infty)$ satisfy the following conditions:

(c5) f is ϵ-Lipschitz continuous, that is, there exists a constant $\epsilon > 0$ satisfying

$$|f(u, v)| \leq \epsilon \|u - v\|, \quad \forall u, v \in H;$$

(c6) f is lower semi-continuous in the first argument;

(c7) f is concave in the second argument;

(c8) $f(u, v) + f(v, u) \geq 0$, $\forall u, v \in H$.

Remark 2.1. Property (e) implies that $f(v, v) = 0$ for arbitrary $v \in H$. In fact,

$$0 \leq |f(v, v)| \leq \epsilon \|v - v\| = 0, \quad \forall v \in H.$$

Assume that A, B, C, $g : H \to H$, $\eta : H \times H \to H$ and $N : H \times H \times H \to H$ are mappings. We consider the following generalized nonlinear variational-like inequality: Find $u \in H$ such that

$$\langle N(Au, Bu, Cu), \eta(v, u) \rangle + b(g(u), v) - b(g(u), u) \geq f(u, v), \quad \forall v \in H. \quad (2.1)$$

Some special cases of the generalized nonlinear variational-like inequality (2.1) are as follows:

(A) If $f(u,v)=0$, $g=I$, $N(u,v,w)=u-v$, $\eta(v,u)=Gv-Gu$ and $b(u,v)=\phi(v)$ for all $u,v,w\in H$, where $G:H\to H$ and $\phi:H\to(-\infty,+\infty)$, then the generalized nonlinear variational-like inequality (2.1) collapses to finding $u\in H$ such that

$$\langle Au-Bu, Gv-Gu\rangle+\phi(v)-\phi(u)\geq 0,\quad \forall v\in H, \tag{2.2}$$

which is called the *generalized variational inequality* and studied by Yao [9].

(B) If $f(u,v)=0$, $A=B=I$, $N(u,v,w)=N(u,v)$, $\eta(v,u)=v-u$ and $b(u,v)=0$ for all $u,v,w\in H$, K is a nonempty closed convex subset of H, then the the generalized nonlinear variational-like inequality (2.1) is equivalent to the following problem: Find $u\in K$ such that

$$\langle N(u,u), v-u\rangle\geq 0,\quad \forall v\in K. \tag{2.3}$$

(C) If $f(u,v)=0$, $A=I$, $N(u,v,w)=u-v$, $\eta(v,u)=v-u$ and $b(u,v)=0$ for all $u,v,w\in H$, K is a nonempty closed convex subset of H, then the the generalized nonlinear variational-like inequality (2.1) is equivalent to finding $u\in K$ such that

$$\langle u-Bu, v-u\rangle\geq 0,\quad \forall v\in K, \tag{2.4}$$

which is called the *nonlinear variational inequality* and introduced by Verma [11].

We need the following definitions and lemmas.

Definition 2.1. Let $\eta:H\times H\to H$ and $T:H\to H$ be two mappings. T is said to be

(1) *η-α-strongly monotone* if there exists a constant $\alpha>0$ such that

$$\langle Tu-Tv, \eta(u,v)\rangle\geq\alpha\|u-v\|^2,\quad \forall u,v\in H;$$

(2) *β-Lipschitz continuous* if there exists a constant $\beta>0$ such that

$$\|Tu-Tv\|\leq\beta\|u-v\|,\quad \forall u,v\in H.$$

Definition 2.2. Let $T,S:H\to H$ and $N:H\times H\times H\to H$ be mappings.

(1) N is said to be *t-Lipschitz continuous* in the first argument if there exists a constant $t>0$ such that

$$\|N(x,u,v)-N(y,u,v)\|\leq t\|x-y\|,\quad \forall x,y,u,v\in H;$$

(2) T is said to be *S-ς-strongly monotone* with respect to the first argument of N if there exists a constant $\varsigma>0$ such that

$$\langle N(Tx,u,v)-N(Ty,u,v), Sx-Sy\rangle\geq\varsigma\|x-y\|^2,\quad \forall x,y,u,v\in H;$$

(3) T is said to be *S-τ-relaxed monotone* with respect to the first argument of N if there exists a constant $\tau > 0$ such that

$$\langle N(Tx, u, v) - N(Ty, u, v), Sx - Sy\rangle \geq -\tau\|x - y\|^2, \quad \forall x, y, u, v \in H.$$

In a similar way, we can define that N is Lipschitz continuous in the second or the third argument, T is S-strongly monotone and S-relaxed monotone with respect to the second and the third arguments of N, respectively.

Definition 2.3. A mapping $\eta : H \times H \to H$ is said to be

(1) *σ-strongly monotone* if there exists a constant $\sigma > 0$ such that

$$\langle u - v, \eta(u, v)\rangle \geq \sigma\|u - v\|^2, \quad \forall u, v \in H;$$

(2) *δ-Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|, \quad \forall u, v \in H.$$

Definition 2.4. Let $T : H \to H$ and $\eta : H \times H \to H$ be two mappings. T is said to be *η-hemicontinuous* if, for any fixed $x, y \in H$, the mapping $g : [0, 1] \to (-\infty, +\infty)$ defined by $g(t) = \langle T(x + t(y - x)), \eta(y, x)\rangle$ is continuous at 0^+.

Definition 2.5. Let T be a self-mapping of H, $x_0 \in H$ and $x_{n+1} = f(T, x_n)$ define an iterative procedure that yields a sequence of points $\{x_n\}_{n\geq 0}$ in H. Suppose that $\{x \in H : x = Tx\} \neq \emptyset$ and $\{x_n\}_{n\geq 0}$ converges to a fixed point p of T. Let $\{y_n\}_{n\geq 0}$ be an arbitrary sequence in H and $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$, then the iterative procedure defined by $x_{n+1} = f(T, x_n)$ is said to be *T-stable* or *stable* with respect to T.

Harder and Hicks [4] demonstrated the importance of investigating the stability of various iterative procedures for various classes of nonlinear mappings.

Lemma 2.1. (see [1]) *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E and ϕ, $\psi : X \times X \to (-\infty, +\infty)$ be mappings satisfying the following conditions:*

(a) $\psi(x, y) \leq \phi(x, y)$, $\forall x, y \in X$ *and* $\psi(x, x) \geq 0$, $\forall x \in X$;

(b) *for each* $x \in X$, $\phi(x, \cdot)$ *is upper semi-continuous on* X;

(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x, y) < 0\}$ *is convex;*

(d) *there exists a nonempty compact set* $K \subset X$ *and* $x_0 \in K$ *such that* $\psi(x_0, y) < 0$, $\forall y \in X \setminus K$.

Then there exists a $\bar{y} \in K$ *such that*

$$\phi(x, \bar{y}) \geq 0, \quad \forall x \in K.$$

Lemma 2.2. (see [8]) *Let* $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_n\}_{n\geq 0}$ *be nonnegative sequences satisfying*

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \delta_n\beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\delta_n\}_{n\geq 0} \subset [0,1]$, $\sum_{n\geq 0}\delta_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n\geq 0}\gamma_n < \infty$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

In order to obtain our results, we need the following assumption.

Assumption 2.1. Let $\eta : H \times H \to H$ be a mapping and

(1) $\eta(v,u) = -\eta(u,v)$, $\forall u, v \in H$;

(2) for given $x, u \in H$, mapping $v \mapsto \langle x,\ \eta(u,v)\rangle$ is concave and upper semi-continuous on H.

3. Main Results

In this section, we extend the auxiliary principle technique to study the generalized nonlinear variational-like inequality (2.1). First, we show that the auxiliary problem for the generalized mixed variational-like inequality (2.1) has a unique solution, then based on this existence conclusion we construct an iterative algorithm for solving the generalized mixed variational-like inequality (2.1). At last, we obtain the convergence and stability results of iterative sequences generated by the iterative algorithm.

Let's consider the following auxiliary problem $P(u)$: for any given $u \in H$, find $w \in H$ such that

$$\begin{aligned}&\langle Sw, \eta(v,w)\rangle + \rho b(g(w),v) - \rho b(g(w),w)\\ &\geq \langle Su, \eta(v,w)\rangle - \rho\langle N(Au,Bu,Cu), \eta(v,w)\rangle + \rho f(w,v), \quad \forall v \in H,\end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant and $S : H \to H$ is a mapping.

Remark 3.1. Our auxiliary variational inequality problem is different from that in [6].

Theorem 3.1. *Let* $\eta : H \times H \to H$ *be* ξ*-Lipschitz continuous,* $S : H \to H$ *be* η*-*δ*-strongly monotone and* η*-hemicontinuous,* $g : H \to H$ *be* α*-Lipschitz continuous and the functions* b, $f : H \times H \to (-\infty, +\infty)$ *satisfy the conditions* (c1)-(c8). *If Assumption 2.1 holds and there exists* $\rho \in (0, \delta(\gamma\alpha)^{-1})$, *then for any given* $u \in H$, *the auxiliary problem* $P(u)$ *has a unique solution.*

Proof. Let u be an arbitrary element in H. Define $\phi,\ \psi : H\times H \to (-\infty,+\infty)$ by

$$\begin{aligned}\phi(v,w) = \langle Sv,\eta(v,w)\rangle - \langle Su,\eta(v,w)\rangle + \rho\langle N(Au,Bu,Cu),\eta(v,w)\rangle \\ - \rho b(g(w),w) + \rho b(g(w),v) - \rho f(w,v), \quad \forall v,w\in H,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\psi(v,w) = \langle Sw,\eta(v,w)\rangle - \langle Su,\eta(v,w)\rangle + \rho\langle N(Au,Bu,Cu),\eta(v,w)\rangle \\ - \rho b(g(w),w) + \rho b(g(w),v) - \rho f(w,v), \quad \forall v,w\in H.\end{aligned} \tag{3.3}$$

We verify that the mappings ϕ, ψ satisfy all the conditions of Lemma 2.1. It follows from the strong monotonicity of S, the conditions (c1)-(c8) and Assumption 2.1 that ϕ, ψ satisfy the conditions (a)-(c) of Lemma 2.1. Let v^* be an arbitrary element in H. Put $r = (\delta-\rho\alpha\gamma)^{-1}(\xi\|Sv^*-Su\|+\rho\xi\|N(Au,Bu,Cu)\|+\rho\gamma\|gv^*\|+\rho\epsilon)$ and $K=\{w\in H : \|w-v^*\|\le r\}$.

Let w be in $H\setminus K$. From Assumption 2.1, the Lipschitz continuity of η, g and strong monotonicity of S, we have

$$\begin{aligned}\psi(v^*,w) &= \langle Sw,\eta(v^*,w)\rangle - \langle Su,\eta(v^*,w)\rangle + \rho\langle N(Au,Bu,Cu),\eta(v^*,w)\rangle \\ &\quad - \rho b(g(w),w) + \rho b(g(w),v^*) - \rho f(w,v^*) \\ &\le -\langle Sw-Sv^*,\eta(w,v^*)\rangle + \xi\|Sv^*-Su\|\|w-v^*\| \\ &\quad + \rho\xi\|N(Au,Bu,Cu)\|\|w-v^*\| + \rho b(g(w),v^*-w) \\ &\quad - \rho b(g(v^*),v^*-w) + \rho b(g(v^*),v^*-w) + \rho\epsilon\|w-v^*\| \\ &\le (\rho\alpha\gamma-\delta)\|w-v^*\|\{\|w-v^*\| - (\delta-\rho\alpha\gamma)^{-1}[\xi\|Sv^*-Su\| \\ &\quad + \rho\xi\|N(Au,Bu,Cu)\| + \rho\gamma\|g(v^*)\| + \rho\epsilon]\} \\ &< 0.\end{aligned}$$

Therefore, the condition (d) of Lemma 2.1 holds. By Lemma 2.1, there exists a $\bar{w}\in H$ such that $\phi(v,\bar{w})\ge 0$, $\forall v\in H$, that is,

$$\begin{aligned}\langle Sv,\eta(v,\bar{w})\rangle \ge \langle Su,\eta(v,\bar{w})\rangle - \rho\langle N(Au,Bu,Cu),\eta(v,\bar{w})\rangle \\ + \rho b(g(\bar{w}),\bar{w}) - \rho b(g(\bar{w}),v) + \rho f(\bar{w},v).\end{aligned} \tag{3.4}$$

For each $t\in(0,1]$ and $v\in H$, let $x_{\mathsf{t}} = tv+(1-t)\bar{w}$. Replacing v by x_{t} in (3.4), we get that

$$\begin{aligned}& t\langle S(x_{\mathsf{t}}) - Su,\eta(v,\bar{w})\rangle \\ &\ge \langle S(x_{\mathsf{t}}) - Su,\eta(tv+(1-t)\bar{w}),\bar{w}\rangle \\ &\ge -\rho\langle N(Au,Bu,Cu),\eta(tv+(1-t)\bar{w},\bar{w})\rangle + \rho b(g(\bar{w}),\bar{w}) \\ &\quad - \rho b(g(\bar{w}),tv+(1-t)\bar{w}) + \rho f(\bar{w},tv+(1-t)\bar{w}) \\ &\ge -\rho t\langle N(Au,Bu,Cu),\eta(v,\bar{w})\rangle - \rho t b(g(\bar{w}),v) \\ &\quad + \rho t b(g(\bar{w}),\bar{w}) + t\rho f(\bar{w},v),\end{aligned}$$

that is,

$$\begin{aligned}\langle S(x_{\mathrm{t}}), \eta(v, \bar{w})\rangle \geq \langle Su, \eta(v, \bar{w})\rangle - \rho\langle N(Au, Bu, Cu), \eta(v, \bar{w})\rangle \\ - \rho b(g(\bar{w}), v) + \rho b(g(\bar{w}), \bar{w}) + \rho f(\bar{w}, v).\end{aligned}$$

Letting $t \to 0^+$ in the above inequality, we have

$$\begin{aligned}\langle S(\bar{w}), \eta(v, \bar{w})\rangle \geq \langle Su, \eta(v, \bar{w})\rangle - \rho\langle N(Au, Bu, Cu), \eta(v, \bar{w})\rangle \\ - \rho b(g(\bar{w}), v) + \rho b(g(\bar{w}), \bar{w}) + \rho f(\bar{w}, v).\end{aligned}$$

Therefore $\bar{w}$ is a solution of the auxiliary problem $P(u)$.

Next we prove the uniqueness of the solution of the auxiliary problem $P(u)$. For any given $u \in H$, suppose that w_1 and w_2 are two different solutions of the auxiliary problem $P(u)$, then we have

$$\begin{aligned}&\langle S(w_1), \eta(v, w_1)\rangle \\ &\geq \langle Su, \eta(v, w_1)\rangle - \rho\langle N(Au, Bu, Cu), \eta(v, w_1)\rangle \\ &\quad - \rho b(g(w_1), v) + \rho b(g(w_1), w_1) + \rho f(w_1, v), \quad \forall v \in H,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\langle S(w_2), \eta(v, w_2)\rangle \\ &\geq \langle Su, \eta(v, w_2)\rangle - \rho\langle N(Au, Bu, Cu), \eta(v, w_2)\rangle \\ &\quad - \rho b(g(w_2), v) + \rho b(g(w_2), w_2) + \rho f(w_2, v), \quad \forall v \in H.\end{aligned} \tag{3.6}$$

Taking $v = w_2$ in (3.5) and $v = w_1$ in (3.6), respectively, and adding them, we obtain that

$$\begin{aligned}&\langle S(w_1) - S(w_2), \eta(w_1, w_2)\rangle \\ &\leq -\rho b(g(w_2) - g(w_1), w_2) + \rho b(g(w_2) - g(w_1), w_1),\end{aligned}$$

which implies that

$$\begin{aligned}\delta\|w_1 - w_2\|^2 &\leq \langle S(w_1) - S(w_2), \eta(w_1, w_2)\rangle \\ &\leq \rho b(g(w_2) - g(w_1), w_1) - \rho b(g(w_2) - g(w_1), w_2) \\ &\leq \rho\gamma\alpha\|w_2 - w_1\|^2,\end{aligned}$$

which is impossible since $\delta > \rho\gamma\alpha$. Hence the auxiliary problem $P(u)$ has a unique solution. This completes the proof. □

According to Theorem 3.1, we know that the auxiliary problem $P(u)$ yields a mapping $F : H \to H$ as $F(u) = w$ for each $u \in H$, where w satisfies (3.1). Now we construct an iterative algorithm for solving the generalized mixed variational-like inequality (2.1) based on Theorem 3.1.

Algorithm 3.1. Let $N : H\times H\times H \to H$, $\eta : H\times H \to H$ and $S : H \to H$ be mappings. For given $u_0 \in H$, compute sequences $\{u_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ by

$$\begin{aligned}&\langle S(z_n), \eta(v, z_n)\rangle\\ &\geq \langle S(u_n), \eta(v, z_n)\rangle - \rho\langle N(Au_n, Bu_n, Cu_n), \eta(v, z_n)\rangle\\ &\quad - \rho b(g(z_n), v) + \rho b(g(z_n), z_n) + \rho f(z_n, v),\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\langle S(u_{n+1}), \eta(v, u_{n+1})\rangle\\ &\geq \langle S(z_n), \eta(v, u_{n+1})\rangle - \rho\langle N(Az_n, Bz_n, Cz_n), \eta(v, u_{n+1})\rangle\\ &\quad - \rho b(g(u_{n+1}), v) + \rho b(g(u_{n+1}), u_{n+1}) + \rho f(u_{n+1}, v)\end{aligned} \tag{3.8}$$

for all $v \in H$ and $n \geq 0$ where $\rho > 0$ is a constant.

Next we study the convergence and stability of Algorithm 3.1.

Theorem 3.2. *Let $N : H \times H \times H \to H$ be r-Lipschitz continuous in the first argument, s-Lipschitz continuous in the second argument and t-Lipschitz continuous in the third argument. Let $g : H \to H$ be α-Lipschitz continuous, $\eta : H \times H \to H$ be ξ-Lipschitz continuous and $S : H \to H$ be η-δ-strongly monotone and σ-Lipschitz continuous. Let A, B, $C : H \to H$ be l-Lipschitz continuous, m-Lipschitz continuous and p-Lipschitz continuous, respectively, B be S-τ-strongly monotone with respect to the second argument of N and C be S-ς-relaxed strongly monotone with respect to the third argument of N. Suppose that $\{y_n\}_{n\geq 0}$ is an arbitrary sequence in H and $\{\varepsilon_n\}_{n\geq 0} \subset [0, +\infty)$ is defined by $\varepsilon_n = \|y_{n+1} - x_{n+1}\|$, where x_{n+1} is generated by*

$$\begin{aligned}&\langle S(v_n), \eta(v, v_n)\rangle\\ &\geq \langle S(y_n), \eta(v, v_n)\rangle - \rho\langle N(Ay_n, By_n, Cy_n), \eta(v, v_n)\rangle\\ &\quad - \rho b(g(v_n), v) + \rho b(g(v_n), v_n) + \rho f(v_n, v)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\langle S(x_{n+1}), \eta(v, x_{n+1})\rangle\\ &\geq \langle S(v_n), \eta(v, x_{n+1})\rangle - \rho\langle N(Av_n, Bv_n, Cv_n), \eta(v, x_{n+1})\rangle\\ &\quad - \rho b(g(x_{n+1}), v) + \rho b(g(x_{n+1}), x_{n+1}) + \rho f(x_{n+1}, v)\end{aligned} \tag{3.10}$$

for all $v \in H$ and $n \geq 0$. Suppose that $a = (sm + tp)^2\xi^2 - (\gamma\alpha + \xi rl)^2$, $b = (\tau - \varsigma)\xi^2 - \delta(\gamma\alpha + \xi rl)$ and $c = \xi^2\sigma^2 - \delta^2$. If Assumption 2.1 holds and there exists a constant $\rho > 0$ satisfying

$$\rho < \delta(\gamma\alpha + \xi rl)^{-1} \tag{3.11}$$

and one of the following conditions:

$$a > 0, \quad b^2 > ac, \quad |\rho - ba^{-1}| < a^{-1}\sqrt{b^2 - ac}; \tag{3.12}$$

$$a < 0, \quad |\rho - ba^{-1}| > -a^{-1}\sqrt{b^2 - ac}, \tag{3.13}$$

then (a) *the generalized nonlinear variational-like inequality* (2.1) *has a solution* $u \in H$ *and the sequence* $\{u_n\}_{n\geq 0}$ *defined by Algorithm 3.1 converges strongly to* u.

(b) $\lim_{n\to\infty} y_n = u$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. In order to show that the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in H$, it is enough to prove that the mapping $F : H \to H$ defined by (3.1) has a unique fixed point $u \in H$. Let x, y be arbitrary elements in H. It follows from Theorem 3.1 that

$$\begin{aligned} &\langle S(Fx), \eta(v, Fx)\rangle \\ &\geq \langle Sx, \eta(v, Fx)\rangle - \rho\langle N(Ax, Bx, Cx), \eta(v, Fx)\rangle \\ &\quad + \rho b(g(Fx), Fx) - \rho b(g(Fx), v) + \rho f(Fx, v), \quad \forall v \in H, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\langle S(Fy), \eta(v, Fy)\rangle \\ &\geq \langle Sy, \eta(v, Fy)\rangle - \rho\langle N(Ay, By, Cy), \eta(v, Fy)\rangle \\ &\quad + \rho b(g(Fy), Fy) - \rho b(g(Fy), v) + \rho f(Fy, v), \quad \forall v \in H. \end{aligned} \tag{3.15}$$

Taking $v = Fy$ in (3.14) and $v = Fx$ in (3.15), respectively, and adding the inequalities, we obtain that

$$\begin{aligned} &\langle S(Fx) - S(Fy), \eta(Fx, Fy)\rangle \\ &\leq \langle Sx - Sy, \eta(Fx, Fy)\rangle \\ &\quad - \rho\langle N(Ax, Bx, Cx) - N(Ay, By, Cy), \eta(Fx, Fy)\rangle \\ &\quad + \rho b(g(Fx) - g(Fy), Fy - Fx). \end{aligned}$$

From the condition (c3), Lipschitz continuity of g and η and strong monotonicity of S, we have

$$\begin{aligned} &(\delta - \rho\alpha\gamma)\|Fx - Fx\|^2 \\ &\leq \langle Sx - Sy - \rho[N(Ay, Bx, Cx) - N(Ay, By, Cy)], \eta(Fx, Fy)\rangle \\ &\quad - \rho\langle N(Ax, Bx, Cx) - N(Ay, Bx, Cx), \eta(Fx, Fy)\rangle \\ &\leq \xi\|Sx - Sy - \rho[N(Ay, Bx, Cx) - N(Ay, By, Cy)]\|\|Fx - Fy\| \\ &\quad + \rho\xi rl\|x - y\|\|Fx - Fy\|. \end{aligned} \tag{3.16}$$

Since $N : H \times H \times H \to H$ is s-Lipschitz continuous in the second argument and t-Lipschitz continuous in the third argument, B, $C : H \to H$ are m-Lipschitz continuous and p-Lipschitz continuous, respectively, and are S-τ-strongly monotone and S-ς-relaxed strongly monotone with respect to the second and the third arguments of N, respectively, $S : H \to H$ is η-δ-strongly monotone and σ-Lipschitz continuous, and $\eta : H \times H \to H$ is ξ-Lipschitz continuous, it follows that

$$\begin{aligned}
&\|Sx - Sy - \rho[N(Ay, Bx, Cx) - N(Ay, By, Cy)]\|^2 \\
&= \|Sx - Sy\|^2 - 2\rho\langle N(Ay, Bx, Cx) - N(Ay, By, Cy), Sx - Sy\rangle \\
&\quad + \rho^2\|N(Ay, Bx, Cx) - N(Ay, By, Cy)\|^2 \\
&\leq [\sigma^2 - 2\rho(\tau - \varsigma) + (sm + tp)^2\rho^2]\|x - y\|^2.
\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17), we can easily infer that

$$\|Fx - Fy\| \leq \theta\|x - y\|,$$

where

$$\theta = \xi(\delta - \rho\gamma\alpha)^{-1}\left[\sqrt{\sigma^2 + (sm + tp)^2\rho^2 - 2(\tau - \varsigma)\rho} + \rho rl\right].$$

It follows from (3.11) and one of (3.12) and (3.13) that $0 < \theta < 1$ and F is a contraction mapping. Hence F has a unique fixed point $u \in H$, which is a solution of the generalized mixed variational-like inequality (2.1).

Now we show that $\lim_{n\to\infty} u_n = u$. Since u is a solution of the generalized mixed variational-like inequality (2.1), it follows that

$$\begin{aligned}
\langle Su, \eta(v, u)\rangle \geq \langle Su, \eta(v, u)\rangle &- \rho\langle N(Au, Bu, Cu), \eta(v, u)\rangle \\
&- \rho b(g(u), v) + \rho b(g(u), u) + \rho f(u, v), \quad \forall v \in H.
\end{aligned} \tag{3.18}$$

Taking $v = u$ in (3.8) and $v = u_{n+1}$ in (3.18) and adding them, we can easily obtain that $\|u_{n+1} - u\| \leq \theta\|z_n - u\|$. Similarly, taking $v = u$ in (3.7) and $v = z_n$ in (3.18) and adding them, we have that $\|z_n - u\| \leq \theta\|u_n - u\|$. Hence $\|u_{n+1} - u\| \leq \theta^2\|u_n - u\|$ for $n \leq 0$. From (3.11) and one of (3,12) and (3.13) we know that $\theta^2 < 1$ and $u_n \to u$ as $n \to \infty$.

Next we prove that (b) holds. As the proof of (a), using (3.9), (3.10) and (3.18), we see that $\|x_{n+1} - u\| \leq \theta^2\|y_n - u\|$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Since

$$\begin{aligned}
\varepsilon_n = \|y_{n+1} - x_{n+1}\| &\geq \|y_{n+1} - u\| - \|x_{n+1} - u\| \\
&\geq \|y_{n+1} - u\| - \theta^2\|y_n - u\|,
\end{aligned}$$

that is,

$$\|y_{n+1} - u\| \le \varepsilon_n + \theta^2 \|y_n - u\|.$$

It follows from Lemma 2.2 that $y_n \to u$ as $n \to \infty$.

Conversely, suppose that $\lim_{n\to\infty} y_n = u$. It follows that

$$\varepsilon_n \le \|y_{n+1} - u\| + \|x_{n+1} - u\| \le \|y_{n+1} - u\| + \theta^2 \|y_n - u\| \to 0$$

as $n \to \infty$. This completes the proof. □

Theorem 3.3. *Let S, N, g, A, B, C, η, $\{y_n\}_{n\ge0}$, $\{\varepsilon_n\}_{n\ge0}$ and c be as in Theorem 3.2. Suppose that $k = rl\sigma + rl\sqrt{\sigma^2 - 2\delta + \xi^2}$, $a = (sm + tp)^2\xi^2 - (\alpha\gamma + k)^2$, $b = (\tau - \varsigma)\xi^2 - (\gamma\alpha + k)\delta$. If Assumption 2.1 holds and there exists a constant $\rho > 0$ satisfying $\rho < \delta(\gamma\alpha + k)^{-1}$ and one of* (3.12) *and* (3.13)*, then the generalized mixed variational-like inequality* (2.1) *has a solution $u \in H$ and the sequence $\{u_n\}_{n\ge0}$ defined by Algorithm 3.1 converges strongly to u. Moreover, $\lim_{n\to\infty} y_n = u$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.*

Proof. Since S is η-δ-strongly monotone and σ-Lipschitz continuous and η is ξ-Lipschitz continuous, we have

$$\|S(Fx) - S(Fy) - \eta(Fx, Fy)\|^2 \le (\sigma^2 - 2\delta + \xi^2)\|Fx - Fy\|^2. \tag{3.19}$$

As the proof of Theorem 3.2 and from (3.19), we obtain that

$$\begin{aligned}
&(\delta - \rho\gamma\alpha)\|Fx - Fy\|^2 \\
&\le \langle Sx - Sy, \eta(Fx, Fy)\rangle - \rho\langle N(Ax, Bx, Cx) - N(Ay, By, Cy), \eta(Fx, Fy)\rangle \\
&= \langle Sx - Sy - \rho[N(Ay, Bx, Cx) - N(Ay, By, Cy)], \eta(Fx, Fy)\rangle \\
&\quad - \rho\langle N(Ax, Bx, Cx) - N(Ay, Bx, Cx), S(Fx) - S(Fy)\rangle \\
&\quad + \rho\langle N(Ax, Bx, Cx) - N(Ay, Bx, Cx), S(Fx) - S(Fy) - \eta(Fx, Fy)\rangle \\
&\le \left[\xi\sqrt{\sigma^2 - 2(\tau - \varsigma)\rho + \rho^2(sm + tp)^2} + \rho k\right]\|x - y\|\|Fx - Fy\|,
\end{aligned}$$

that is,

$$\|Fx - Fy\| \le \theta\|x - y\|,$$

where

$$\theta = (\delta - \rho\gamma\alpha)^{-1}\left[\xi\sqrt{\sigma^2 - 2(\tau - \varsigma)\rho + \rho^2(sm + tp)^2} + \rho k\right].$$

The rest of the argument is just as the proof of Theorem 3.2 and is therefore omitted. This completes the proof. □

Remark 3.2. Theorem 3.2 and Theorem 3.3 improve and generalize Theorem 3.1 in [9] and Theorem 2.2 in [11].

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