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CHARACTERIZATIONS OF A PARTITION TOPOLOGY ON A SET

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Abstract: In this paper three characterizations of a partition topology on a set X are obtained respectively in terms of clopen sets, interior of any subset of X and order of every element of X.

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Definition 1. The *partition topology* is a topology that can be induced on any set X by partitioning X into disjoint subsets P; these subsets form the basis for the topology.

Definition 2. A *clopen set* in a topological space is a set which is both open and closed.

Definition 3. Let (X, τ) be a topological space and $a \in X$. Then *order of a relative to topology* τ, $O_\tau(a)$ is the number of τ-open sets containing a.

Definition 4. Let (X, τ) be a topological space and $A \subset X$. Then *order of A relative to* topology τ, $O_\tau(A)$ is the number of τ-open sets containing A.

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Theorem 1. *Let X have a partition topology τ. Then every open set is closed.*

Proof. We refer to [7].

We now prove the converse of Theorem 1.

Theorem 2. *Let X have a topology τ such that every open set is closed. Then τ is a partition topology on X.*

Proof. If τ is indiscrete topology on X then τ is generated by a trivial partition of X.

Let τ be different from indiscrete topology on X.

Let $\delta = \{A_\alpha \ : \alpha \in \Lambda\ \}$ be the largest subfamily of τ containing mutually disjoint non – empty subsets.

or $X - \bigcup_{\alpha\in\Lambda} A_\alpha$ is a non-empty open set and $\left\{\ X - \ \bigcup_{\alpha\in\Lambda} A_\alpha\ \right\} \bigcup \delta$ is the largest subfamily of τ with mutually disjoint non-empty sets, a contradiction.

Hence δ is a partition of X.

Let $B \in\ \tau$ be arbitrary. Then B is contained in $\bigcup\{\ A_i\ : A_i\ \bigcap B \neq \emptyset,\ i \in \Lambda\}$.

If B is properly contained in $\bigcup\{\ A_i\ : A_i\ \bigcap B \neq \emptyset,\ i \in \Lambda\}$ then

$$\{B\cap A_i\ : A_i\cap B\neq\emptyset,\ i\in\Lambda\}\bigcup\{(X\sim B)\cap A_i\ :\ A_i\cap B\neq\emptyset,\ \ i\in\Lambda\}$$
$$\bigcup\{A_i\ : A_i\cap B=\emptyset,\ i\in\Lambda\}$$

is a subfamily of τ formed by mutually disjoint non-empty subsets of X, which is a contradiction.

Hence $B = \bigcup\{\ A_i\ : A_i \cap B \neq \emptyset,\ i \in \Lambda\}$, or τ is generated by partition δ of X.

Another characterization of a partition topology on a set is presented below.

Theorem 3. *Let (X, τ) be a topological space. Then τ is a partition topology on X if and only if τ - interior of any set is a closed set.*

Proof. Let τ be a partition topology on X and $A \subset\ X$.

As $A^o \in \tau$, A^o is a closed set by Theorem 1.

Conversely let τ - interior of any set be closed. Suppose $A \in\ \tau$, then $A = A^o$.

But $A = \overline{A^o}$ or $A = \overline{A}$.

Hence, by Theorem 2, τ is a partition topology on X.

The proofs of next three theorems are omitted as these are direct consequences of Theorem 1 and Theorem 2.

Theorem 4. *Intersection of two partition topologies on a set is a partition topology.*

Theorem 5. *The relative topology on a subspace of a topological space with a partition topology, is a partition topology.*

Theorem 6. *Homeomorphic image of a topological space with a partition topology, is a topological space with a partition topology.*

Theorem 7. *Let τ be a partition topology on a set X generated by a finite partition. Then order of each element of X relative to τ is same and one-half the number of members of τ.*

Proof. Let τ be generated by a partition $\mathcal{P} = \{A_1,\ A_2,\ \cdots,\ A_k\}$ of X.
Number of members of τ is ${}^kC_0 + {}^kC_1 + \cdots + {}^kC_k = 2^k$.
Let $x \in X$ then $x \in A_i$ for some i ; $1 \le i \le k$

$$O_\tau(x) = O_\tau(A_i) = {}^{k-1}C_0 + {}^{k-1}C_1 + \cdots + {}^{k-1}C_{k-1} = 2^{k-1}.$$

Hence the theorem is proved.

The following example provides an illustration of the above theorem.

Example 1. Let $X = \{a, b, c, d, e\}$. Let $A_1 = \{a,\ b\}$, $A_2 = \{c\}$, $A_3 = \{d,\ e\}$.
Then $\mathcal{P} = \{A_1,\ A_2,\ A_3\}$ is a partition of X. The topology τ on X generated by $\mathcal{P}$ is

$$\tau = \Big\{\emptyset,\ \{a,b\},\ \{c\},\ \{d,e\},\ \{a,b,c\},\ \{c,d,e\},\ \{a,b,d,e\},\ X\Big\}.$$

Here $O_\tau(x) = 4$ for all $x \in X$ and τ has 8 elements.

Theorem 8. *Let X be a topological space with topology τ such that order of each element of X relative to τ is finite and same then τ is a partition topology on X.*

Proof. Let order of each element of X relative to τ be k.
Let $a \in X$ and A be the smallest open set containing a. Then $O_\tau(A) = O_\tau(a) = k$.
Let $b \in A$ such that $b \neq a$.
Let B be the smallest τ-open set containing b. Then $O_\tau(b) = O_\tau(B) = k$.
If B is properly contained in A then

$$O_\tau(B) > O_\tau(A) \quad \text{or} \quad O_\tau(B) > k,$$

a contradiction.

Hence $B = A$ or A is smallest τ-open set for each element contained in it.

Let $A = X$. Then τ is generated by trivial partition of X and order of each element of X relative to τ is unity.

Next, let $A \neq X$ then there exists an element $b \notin B$ and the smallest τ-open set B containing b such that $O_\tau(b) = O_\tau(B) = k$.

Moreover $A \cap B = \emptyset$ otherwise A is not the smallest τ-open set for some elements of A.

If A and B form a partition of X then $\tau = \{\emptyset, X, A, B\}$ is a partition topology on X and order of each element of X relative to τ is $2^{2-1} = 2$.

If A and B do not form a partition of X then there exists an element c of X such that c is neither contained in A nor in B. Again there exists a smallest τ-open set C containing c and disjoint from each of the sets A and B.

Next if $\mathcal{P} = \{A, B, C\}$ is a partition of X then it generates the given topology τ on X as follows:

Let D be any τ-open set.

Case (i) Suppose D does not intersect with any of the sets A, B or C. Hence D does not intersect with X or D is empty set.

Case (ii) Let D intersect with each of the sets A,B and C. Since A is the smallest τ-open set for elements contained in it, $D \cap A = A$ or $A \subset D$.

Likewise $B \subset D$ and $C \subset D$ or $D = X$.

Case (iii) Let D intersects with two of sets A, B, C say

$$D \cap B \neq \emptyset, \quad D \cap C \neq \emptyset \quad \text{but} \quad D \cap A = \emptyset.$$

Then $B \subset D$ *and* $C \subset D$ *or* $B \cup C \subset D$.

However $D \cap A = \emptyset$ leads to $D \subset B \cup C$.

Hence $D = B \cup C$.

Case (iv) Finally, let D intersects one of the three sets A,B and C say $D \cap C \neq \emptyset$ but $D \cap A = \emptyset$, $D \cap B = \emptyset$.

Then $C \subset D$ and $D \subset B \cup C$, $D \subset A \cup C$ or $D \subset (A \cup C) \cap (B \cup C)$ or $D \subset C$. Hence $D = C$.

Thus τ is generated by the partition $\mathcal{P} = \{A, B, C\}$ of X and order of each element of X relative to this partition topology is $2^{3-1} = 4$.

Continuing the process and as order of each element of X relative to τ is finite and same, after a finite number of steps, a partition of X is obtained which generates τ.

The example given below exhibits that in the above theorem, order of each element of X relative to τ is finite is necessary.

Example 2. Let $X = \{1,\ 2,\ 3,\ \cdots\}$ and topology τ on X be

$$\tau = \Big\{ \{1\},\ \{1,2\},\ \{1,2,3\},\ \cdots \Big\}.$$

Here order of each element of X relative to τ is same but infinite and τ is not a partition topology.

We recall the definition of identification topological space from [1].

Definition 5. Let X be a topological space and let P be the family of disjoint non-empty subsets of X such that $\bigcup \mathcal{P} = X$. Such a family is usually called a partition of X. We form a new space Y, called an *identification space*, as follows. The points of Y are members of P and, if $\pi\ : X \to Y$ sends each point of X to the subset of P containing it, the topology of Y is the largest for which π is continuous. Therefore a subset O of Y is open if and only if $\pi^{-1}(O)$ is open in X. This topology is called the *identification topology* on Y.

Theorem 9. *Let τ be a partition topology on set X generated by a partition P of X. Let Y be the identification space corresponding to partition P and let $\pi\ : X \to Y$ be identification map. Then identification topology $\mathcal{U}$ on Y is discrete and $O_\tau(a) = O_{\mathcal{U}}(\pi(a))$ for each $a \in X$.*

Proof. Let $\mathcal{P} = \{A_1,\ A_2,\ \cdots,\ A_k\}$ and let $\pi\ : X \to Y$ send each point of X to the subset of P containing it. Let $A \subset Y$.

Then $A = \bigcup \{A_{\alpha_i}\ : \alpha_1 \le \alpha_i \le \alpha_k\}$ where $\{\alpha_1, \cdots, \alpha_k\} \subset \{1,\ 2, \cdots, k\}$. Now $\pi(x) = A_{\alpha_i}$ for all $x \in A_{\alpha_i}$; $\quad 1 \le i \le k$;

or $\pi^{-1}(A_{\alpha_i}) = A_{\alpha_i}$; $\quad 1 \le i \le k$;

or $\pi^{-1}(A) = \bigcup \left\{\pi^{-1}(A_{\alpha_i})\ : \alpha_1 \le \alpha_i \le \alpha_k\right\} = \bigcup \{A_{\alpha_i}\ : \alpha_1 \le \alpha_i \le \alpha_k\}$ or $\pi^{-1}(A) \in \tau$, or A is open in identification space Y.

Hence identification space Y is discrete.

Next let $a \in X$. Then $O_\tau(a) = 2^{k-1}$ by Theorem 7.

Also $O_{\mathcal{U}}(\pi(a)) = 2^{k-1}$ as $\mathcal{U}$ is discrete topology and Y has k members.

Hence $O_\tau(a) = O_{\mathcal{U}}(\pi(a))$; $\ a \in X$

The following example is an illustration of the above theorem.

Example 3. Let $X = \{a, b, c, d, e\}$ and partition $\mathcal{P} = \Big\{ \{a\}, \{b, c\}, \{d, e\} \Big\}$.

Then the topology τ on X induced by partition P is given by

$$\tau = \Big\{\emptyset, \{a\}, \{b, c\}, \{d, e\}, \{a, b, c\}, \{a, d, e\},\ \{b, c, d, e\}, X\Big\},$$

$$Y = \Big\{ \{a\}, \{b, c\}, \{d, e\} \Big\}.$$

And identification topology on Y is given by

$$\mathcal{U}=\Big\{\emptyset,\{\{a\}\},\{\{b,c\}\},\{\{d,e\}\},\{\{a\},\{b,c\}\},\{\{a\},\{d,e\}\},\{\{b,c\},\{d,e\}\},Y\Big\},$$

$\mathcal{U}$ is discrete topology on Y.

Theorem 10. *Let (X,τ) be a topological space and $(Y,\mathcal{U})$ be identification space corresponding to a partition $\mathcal{P}$ of X. If $\mathcal{U}$ is discrete then the partition topology on X induced by $\mathcal{P}$ is coarser than τ.*

Proof. Let $\mathcal{P} = \{A_1, A_2, \cdots, A_k\}$.
Since $\mathcal{U}$ is discrete, $\{A_i\} \in \mathcal{U} \quad for\ 1 \le i \le k$
or $\pi^{-1}(A_i) \in \tau \quad for\ 1 \le i \le k$
or $A_i \in \tau \quad for \quad 1 \le i \le k$
or the partition topology $\tau_{\mathcal{P}}$ on X induced by $\mathcal{P}$ is coarser than τ.

Corollary 1. *If there is a ono-to-one correspondence between τ and then τ is a partition topology on X.*

Proof. Now $\mathcal{P}$ has k members and hence $\tau_{\mathcal{P}}$ has 2^k members.
$\mathcal{U}$ being discrete topology on Y having k elements, has 2^k members.
Hence τ and $\tau_{\mathcal{P}}$ both have 2^k members and so $\tau = \tau_{\mathcal{P}}$
or τ is a partition topology on X.
Example 4 is in support of above Theorem 10.

Example 4. Let $X = \{a,b,c,d,e\}$ and $Y = \Big\{\{a\},\{b,c\},\{d,e\}\Big\}$.
Let

$$\tau=\Big\{\emptyset,\ \{a\},\{b,c\},\{d,e\},\ \{a,b\},\{a,b,c\},\ \{a,d,e\},\ \{b,d,e\},\{a,b,d,e\},\{b,c,d,e\},X\Big\}.$$

be any topology on X.
The identification topology $\mathcal{U}$ on Y is discrete topology.
The topology $\tau_{\mathcal{P}}$ on X induced by $\mathcal{P}$ is

$$\tau_{\mathcal{P}}=\Big\{\emptyset,\ \{a\},\{b,c\},\{d,e\},\ \{a,b,c\},\ \ \{a,d,e\},\{b,c,d,e\},X\Big\}.$$

Hence $\tau_{\mathcal{P}} \subset \tau$.

Association. Every boolean algebra is associated with a number of partition topologies on the set of atoms of the boolean algebra. These partition

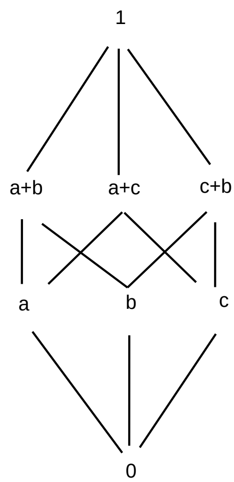

Figure 1

topologies have generators as disjoint sets consisting of all the atoms of the boolean algebra.The biggest partition topology thus obtained is the discrete topology on the set of atoms. For instance if boolean algebra $\mathcal{B}$ is represented as given in Figure 1.

Then $S = \{a, b, c\}$ is the set of atoms of $\mathcal{B}$. The various associated partition topologies are generated by the sets

$$\Big\{ \{a\}, \{b\}, \{c\} \Big\},\ \Big\{ \{a, b\}, \{c\} \Big\},\ \Big\{ \{a\}, \{b, c\} \Big\}, S.$$

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