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LINE BUNDLES ON $X_1 \cup X_2$ WITH $X_2 \cong \mathbb{P}^1$

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we study the h^0 of line bundles on a reducible curve $X_1 \cup X_2$ with $X_2 \cong \mathbb{P}^1$ using deformation theory for reducible rational curves in projective spaces.

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1. Introduction

In this note we look at a few particular cases of the following question and applications of it (with $t \in \{x-3, x-2, x-1\}$).

Question 1. Fix integers t, m, x such that $x > m > 1$. Let $S \subset \mathbb{P}^m$ be a finite subset such that $\sharp(S) = x$ and S spans $\mathbb{P}^m$. Find necessary and/or sufficient conditions on S for the existence of an irreducible curve $C \subset \mathbb{P}^m$ such that $S \subset C$, the normalization of C is rational and $\deg(C) = t$.

In the case $m \geq 3$ we also find as C a smooth rational curve, but the main point is to find an irreducible one.

We want to answer in a few very particular cases the following question. Fix a reducible curve $X = X_1 \cup X_2$ with, say, $X_2 \cong \mathbb{P}^1$. Fix $R_1 \in \mathrm{Pic}(X_1)$. Under what assumption on t and $x := \deg(X_1 \cap X_2)$ there is a line bundle L on X such that $L|X_1$, $\deg(L|X_2) = t$ and $h^0(X, L) = h^0(X_1, R_1) + \max\{0, t+1-x\}$.

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For all integers g, r, d set $\rho(g, r, d) := (r+1)d - rg - r(r+1)$. Quite often reducible curves have Brill-Noether theory very far from the one of a general smooth curve with the same genus (see [2]). In our case $X = X_1 \cup X_2$ with X_1 and X_2 smooth and connected and X nodal we prove the following result.

Theorem 1. *Fix integers $g_1 \geq 2$, $g_2 \geq 0$, $r \geq 3$, $x \geq 2$ and $d > t > 0$, $s > 0$. Set $g := g_1 + g_2 + x - 1$. Let X_1 be any smooth curve of genus g_1 and $S \subset X_1$ such that $\sharp(S) = x$. Let $X = X_1 \cup X_2$ be the semistable nodal curve with X_2 a general smooth curve of genus g_2 and $X_1 \cap X_2 = S$. If $t \geq x - 1$ and $\rho(g_1, d-t, s) \geq 0$, then there is an $(x-1+\rho(d-t, g_1, s)+g_2)$-dimensional family $\mathcal{L}$ of line bundles on X such that $h^0(X, L) = s+2+t-x-g_2$, $h^0(X_1, L|X_1) = s+1$, $\deg(L|X_1) = d - t$ and $\deg(L|X_2) = t$ for all $L \in \mathcal{L}$.*

Since $x - 1 + \rho(d-t, g_1, s) + g_2 \gg \rho(g, d, s + t - g_2 + x)$ when $t \gg x + g_2$, it is clear that not all these line bundles are preserved by general smoothings of X. Nevertheless, even these line bundles are interesting. We believe that the main problem is that the associated linear system $|L|$ is not strongly r-regular in the sense of [4], but only r-regular. There is a non-existence range if we take $t < x$, i.e. we prove the following result

Proposition 1. *Let $X = X_1 \cup X_2$ be a nodal projective curve with X_1, X_2 smooth. Set $x := \sharp(X_1 \cap X_2)$. Fix an integer t such that $0 < t < x$ and $g^r_{d-t}(X_1) = \emptyset$. Then $h^0(X, L) \leq r$ for every $L \in \mathrm{Pic}(X)$ such that $\deg(L|X_1) = d - t$ and $\deg(L|X_2) = t$.*

The gap between Theorem 1 and Proposition 1 is smaller when $g_2 = 0$. An immediate corollary of Proposition 1 is the following result.

Corollary 1. *Let $X = X_1 \cup X_2$ be a nodal projective curve with X_1 smooth and $X_2 \cong \mathbb{P}^1$. Set $x := \sharp(X_1 \cap X_2)$ and assume $x > 0$. Fix any $R_1 \in \mathrm{Pic}(X_1)$. Then there exists $L \in \mathrm{Pic}(X)$ such that $L|X_1 = R_1$, $\deg(L|X_2) = x - 1$ and $h^0(X, L) = h^0(X, R_1)$.*

The case $t = x-2$ is more interesting. Indeed, we prove the following result.

Theorem 2. *Let $X = X_1 \cup X_2$ be a nodal projective curve with X_1 smooth and $X_2 \cong \mathbb{P}^1$. Set $x := \sharp(X_1 \cap X_2)$ and assume $x \geq 3$. Fix any $R_1 \in \mathrm{Pic}(X_1)$. Set $A := X_1 \cap X_2$. Assume that no point of A is a base point of R_1. There exists $L \in \mathrm{Pic}(X)$ such that $L|X_1 = R_1$, $\deg(L|X_2) = x-2$ and $h^0(X, L) = h^0(X, R_1)$ if and only if there is no $A_1 \subset A$ such that $h^0(X_1, R(-A_1)) > h^0(X_1, R(-A))$ and $\sharp(A_1) = x - 1$.*

A finite set $S \subset \mathbb{P}^m$ is said to be in linearly general position if every $S' \subseteq S$ spans a linear subspace of dimension $\min\{m, \sharp(S') - 1\}$.

Proposition 2. *Let $X = X_1 \cup X_2$ be a nodal projective curve with X_1 smooth and $X_2 \cong \mathbb{P}^1$. Set $x := \sharp(X_1 \cap X_2)$ and assume $x \geq 3$. Fix any $R_1 \in \mathrm{Pic}(X_1)$. Set $A := X_1 \cap X_2$. Assume that no point of A is a base point of R_1 and call $\psi : X_1 \dashrightarrow \mathbb{P}^k$, $k := h^0(X_1, R_1) - 1$, the rational map induced by $|R_1|$. Let m be the dimension of the linear span of $\psi(A)$.*

(a) If there is $L \in \mathrm{Pic}(X)$ such that $L|X_1 = R_1$ and $\deg(L|X_2) = x - 3$, then $m \leq x - 3$.

(b) Assume $m = x - 3$. There is $L \in \mathrm{Pic}(X)$ such that $L|X_1 = R_1$ and $\deg(L|X_2) = x - 3$ if and only if $\psi(A)$ is in linearly general position in its linear span.

2. The Proofs

Lemma 1. *Fix a nodal projective curve $X = X_1 \cup X_2$ with each X_i a smooth curve. Set $x := \sharp(X_1 \cap X_2)$ and assume $x > 0$. Set $g_2 := p_a(X)$. Fix an integer $d_2 \geq 2g_2 - 1 + x$ and any $R_i \in \mathrm{Pic}(X_i)$, $i = 1, 2$, such that $\deg(R_2) = d_2$. There is a non-empty and $(x-1)$-dimensional family $\mathcal{L}$ of line bundles on X such that $L|X_i \cong R_i$ for all i. We have $h^0(X, L) = h^0(X_1, R_1) + d_2 + 1 - g_2 + x$ for every $L \in \mathcal{L}$.*

Proof. Notice that $\mathrm{Pic}^0(X)$ is isomorphic to an extension of the abelian variety $\mathrm{Pic}^0(X_1) \times \mathrm{Pic}^0(X_2)$ by the $(x-1)$-dimensional torus $(\mathbb{C}^*)^{x-1})$. Hence $\mathcal{L}$ is isomorphic to an $(x-1)$dimensional affine variety. Fix any $L \in \mathcal{L}$ and look at the Mayer -Vietoris exact sequence

$$0 \to L \to L|X_1 \oplus L|X_2 \to L|(X_1 \cap X_2) \to 0 \tag{1}$$

Since $L|X_2 \cong R_2$ and $d_2 \geq 2g_2 - 1 + x$, we have $h^0(X_2, R_2(-(X_1 \cap X_2)) = 0$. Hence the restriction map $H^0(X_2, L|X_2) \to H^0(X_1 \cap X_2, L|(X_1 \cap X_2))$ is surjective. Hence the cohomology exact sequence of (1) gives $h^0(X, L) = h^0(X_1, R_1) + h^0(X_2, R_2) - x = h^0(X_1, R_1) + d_2 + 1 - g_2 + x$. □

Lemma 2. *Fix a nodal projective curve $X = X_1 \cup X_2$ with each X_i a smooth curve. Set $x := \sharp(X_1 \cap X_2)$ and assume $x > 0$. Set $g_2 := p_a(X)$. Fix an integer $d_2 < x$ and any $R_i \in \mathrm{Pic}(X_i)$, $i = 1, 2$, such that $\deg(R_2) = d_2$. There is a non-empty and $(x-1)$-dimensional family $\mathcal{L}$ of line bundles on X such that $L|X_i \cong R_i$ for all i. We have $h^0(X, L) \leq h^0(X_1, R_1)$ for every $L \in \mathcal{L}$.*

Proof. Since $x > d_2$ we have $h^0(X_2, R_2(-X_1 \cap X_2)$, i.e. the restriction map $H^0(X_2, L|X_2) \to H^0(X_1 \cap X_2, L|(X_1 \cap X_2))$ is injective. The exact sequence (1) gives the injectivity of the restriction map $H^0(X, L) \to H^0(X_1, R_1)$. □

Proof of Theorem 1. Since $\rho(g_1, d-t, s) \geq 0$ and X_1 has general moduli, the scheme $G^s_{r-t}(X_1)$ is non-empty, of pure dimension $\rho(g_1, d-t, s)$ and a dense open subset $G^s_{r-t}(X_1) \setminus G^{s+1}_{r-t}(X_1)$ is formed by complete linear systems. Fix any $R_1 \in \mathrm{Pic}^{d-t}(X_1)$ such that $|R_1| \in G^s_{r-t}(X_1) \setminus G^{s+1}_{r-t}(X_1)$ and apply Lemma 1 to R_1. □

Remark 1. Take the set-up of Theorem 1, but drop the assumption that X_1 has general moduli. If $g_1 \leq 2$, then no modification must be done to get the same statement (although the bounds make it useless). If g_1 is arbitrary, then the existence theorem for g^s_{d-t} on X_1 gives that $g^s_{d-t}(X_1)$ is non-empty and at least of dimension $\rho(g_1, d-t, s)$. From the line bundle R_1 associated to any such g^s_{d-t} (with say $h^0(X_1, R_1) = y \geq s+1$, we find an $(x-1)$-dimensional family $\mathcal{C}(R_1)$ of line bundles on X such that $h^0(X, L) = y + t + 1 - g_2 - x$ for each $L \in \mathcal{C}(R_1)$. Varying $R_1 \in W^s_{d-t}(X_1)$ we get an $(x-1+\rho(d-t, g_1, s)+g_2)$-dimensional family of g^r_d's on X with bidegree $(d-t, t)$, but not necessarily complete.

Proof of Proposition 1. Use Lemma 2. □

Lemma 3. *Fix a set $S \subset \mathbb{P}^m$, $m \geq 2$, such that $\sharp(S) = m+3$ and S spans $\mathbb{P}^m$. There is a smooth rational curve $C \subset \mathbb{P}^m$ such that* $\deg(C) = m+1$ *and $S \subset C$ if and only if there in no $S_1 \subset S$ such that $\sharp(S_1) = m+2$ and S_1 does not span $\mathbb{P}^m$.*

Proof. The " only if " part follows from Bezout theorem. Now assume that any $S_1 \subset S$ such that $\sharp(S_1) = m+2$ spans $\mathbb{P}^m$.

Fix $A \subset S$ such that $\sharp(A) = m+1$ and A spans $\mathbb{P}^m$. Let $D \subset \mathbb{P}^m$ be the line spanned by the two points of $S \setminus A$. Take any hyperplane $H \subset \mathbb{P}^m$ spanned by m points of A. D is not contained in H, because we assume $\sharp(S \cap M) \leq m+1$ for any hyperplane M of $\mathbb{P}^m$. Hence a general $Q \in D$ is not contained in H. Fix a general $Q \in D$. Since there are only finitely many hyperplanes spanned by m points of A, the set $A \cup \{Q\}$ is in linearly general position. Since $\sharp(A \cup \{Q\}) = m+2$, there is an $(m-1)$-dimensional family Γ of rational normal curves of $\mathbb{P}^m$ containing $A \cup \{Q\}$ and a general $E \in \Gamma$ is not tangent to D at Q and is not secant to D, unless D contains at least 3 points of S. Assume for the moment that this is not the case. Hence $Y := E \cup D$ is a connected nodal curve of degree $m+1$ and arithmetic genus 0. For any nodal curve $T \subset \mathbb{P}^m$ let

N_T denote its normal bundle. Since $S \subset D \setminus \{Q\}$, we have the following exact sequence

$$0 \to N_Y(-S) \to N_Y(-S)|E \oplus N_Y(-S)|D \to N_Y(-S)|\{Q\} \to 0 \qquad (2)$$

The vector bundle N_E is a direct sum of $m-1$ line bundle of degree $m+2$ (see [8], [7]). Hence $N_Y|E$ is a direct sum of a line bundle of degree $m+3$ and $m-2$ line bundle of degree $m+2$ (see [5], Corollary 3.2, or [9]). Hence $h^1(E, N_Y(-S)|E) = 0$ and the restriction map $H^0(D, N_Y(-S)|D) \to H^0(\{Q\}, N_Y(-S)|\{Q\})$ is surjective. Since $N_Y(-S)|D$ is a direct sum of a line bundle of degree 0 and $m-2$ line bundles of degree -1 (see [5], Corollary 3.2, or [9]), we have $h^1(D, N_Y(_s)|D) = 0$. Hence (2) gives $H^1(Y, N_Y(-S))$. Hence the subscheme Γ of $\mathrm{Hilb}(\mathbb{P}^m)$ parametrizing the degree m curves with arithmetic genus 0 containing S is smooth at Y and of dimension $h^0(Y, N_Y(-S)) = m+1$. To conclude it is sufficient to prove that a general $Y' \in \Gamma$ is not of the form $E' \cup D'$ with E' rational normal curve containing A, D' a line containing $S \setminus A$ and $D' \cap E' \neq \emptyset$. The line D' must be D. Hence E' is one of the rational normal curves containing A and a point $Q_1 \in D$. For fixed Q_1 the set of all rational normal curves containing $A \cup \{Q_1\}$ has dimension $m-1$. Varying Q_1 inside D we get that the set of all $E' \cup D'$ has dimension m, a contradiction. Now assume that D contains at least 3 points of S. Since no hyperplane contains $m+2$ points of S we easily get $\sharp(D \cap S) = 3$. Fix two of these 3 point, say O, O', and call O_1 the third one. Fix $O_2 \in S \setminus S \cap D$. Instead of A we use the set $S \setminus \{O_1, O_2\}$ and instead of D the line D' spanned by $\{O_1, O_2\}$. We need to check that we may find O_1 and O_2 so that A' spans $\mathbb{P}^m$ and $\sharp(S \cap D') = 2$. Assume that A' is contained in a hyperplane M. Since $\{O_1, O_2\} \subset A' \subset M$, we have $D \subset M$ and hence $O_1 \in M$. Hence $A' \cup \{O_1\} \subset H$. Hence M contains $m+2$ points of S, a contradiction. Now assume $\sharp(D' \cap S) \geq 3$. As above we get $\sharp(S \cap D') = 3$. This must be true for any choice of a point of $D \cap S$ and for any choice of a point of $S \setminus S \cap D$. Then this must be true taking D' instead of D. And so on. Since $\sharp(S)$ is small, we get a contradiction. □

We need the following classical result.

Lemma 4. *Let $B \subset \mathbb{P}^k$ be a finite subset such that $\sharp(B) \leq k+3$. There is a rational normal curve $C \subset \mathbb{P}^k$ if and only if B is in linearly general position. If $\sharp(B) = k+3$ and B is in linearly general position, then C is unique.*

Proof of Theorem 2. Let R_2 denote the degree $x-2$ line bundle on X_2. Fix any $A_1 \subset A$ such that $\sharp(A_1) = x-1$. Since the restriction map $H^0(X_2, R_2)) \to H^0(A_1, R_2|A_1)$ is bijective, we immediately get the " only if " part. Now we get the “ if ” part. Set $m := h^0(X_1, R_1) - h^0(X_1, R(-A)) - 1$. The rational map ψ is associated to a morphism $\phi : X_1 \to \mathbb{P}^k$ associated to the line bundle $R_1(-B)$ obtained from R_1 deleting its base locus B. Since $\psi|A = \phi|A$, we may apply Lemma 3 and get the existence of a morphism $u : \mathbb{P}^1 \to \mathbb{P}^m := \langle\psi(A)\rangle$ such that $u^*(\mathcal{O}_{\mathbb{P}^m}(1))$ has degree $x-2$. The pair (ϕ, u) induces a morphism: $v : X_1 \sqcup X_2 \to \mathbb{P}^k$ such that $v|X_1 = \phi$ and $v|X_2 = u$. Since $u(P) = v(P)$ for each $P \in A$ and X is nodal, v induces a unique morphism $w : X \to \mathbb{P}^k$ such that $w|X_1 = \phi$ and $v'|X_2 = u$. Since $B \cap A = \emptyset$, B is a Cartier divisor of X. Hence $L := w^*(\mathcal{O}_{\mathbb{P}^k}(1))(B)$ is a line bundle on X such that $L|X \cong R_1$ and $\deg(L|X_2) = t$. Since $B \geq 0$, we have $h^0(X, L) \geq h^0(X, w^*(\mathcal{O}_{\mathbb{P}^k}(1)) \geq k+1 = h^0(X_1, R_1)$. The inequality $h^0(X, L) \leq h^0(X_1, R_1)$ follows from Lemma 2. □

Proof of Proposition 2. Part (a) follows from Lemma 2. For part (b) use Lemma 4 and that any finite subset of a rational normal curve is in linearly general position. □

3. Balanced Line Bundles

In this section we look at our line bundles on a two-component nodal curve from the point of view of balanced line bundles is the sense of [1], [2], [3]. We check that the line bundles obtained in the statement of the introduction are balanced and hence cannot be ruled out restricting us to balanced line bundles. Let X be a nodal and connected projective curve with exactly two irreducible components, say $X = X_1 \cup X_2$. Set $x := \sharp(X_1 \cap X_2)$, $g := p_a(X)$ and $g_i := p_a(X_i)$. Set $w_i := \deg(\omega_X|X_i)$. We have $g = g_1 + g_2 + x - 1$ and $w_i = 2g_i - 2 + x$. Fix any $L \in \mathrm{Pic}(X)$ and set $d := \deg(L)$ and $d_i := \deg(L|X_i)$. We have $d = d_1 + d_2$. The line bundle L is balanced if it satisfies the following inequalities for all $i = 1, 2$.

$$dw_i/(2g-2) - x/2 \leq d_i \leq dw_i/(2g-2) + x/2 \qquad (3)$$

It is sufficient to check both inequalities for $i = 2$. We have $d > 0$ and $d_2 > 0$. In Corollary 1 (resp. Theorem 2, resp. Proposition 2) we have $d_2 = x-1$ (resp. $d_2 = x-2$, resp. $d_2 = x-3$) and $w_2 = x-2$. In the set-up of Theorem 1 we have $v_2 = 2g_2 - 2 + x$; in the only cases which are important for Brill-Noether theory we have $0 < d(2g-2) \leq 1$.

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