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A CLASS OF VERTEX TRANSITIVE GRAPHS INDUCED BY RIGHT SOLVABLE WARD GROUPOIDS

Anil Kumar V.

Department of Mathematics
University of Calicut
Malappuram, Kerala, 673 635, INDIA

Abstract: In this paper, we introduce a class of vertex transitive graphs induced by right solvable ward groupoids whose vertices are cosets. Also, many graph properties are expressed in terms of algebraic properties. This did not attract much attention in the literature.

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Key Words: vertex-transitive graphs, hasse-diagram, right solvable Ward groupoid

1. Introduction

A *binary relation* on a set G is a subset R of $G \times G$. A *graph* is a pair (G, R) where G is a non-empty set (called vertex set) and R is a binary relation on G. The elements of R are the *edges* of the graph. An edge of the from (x, x) is called a *loop*. If $u \in G$, the *out-degree* of u is $|\{v \in G : (u, v) \in R\}|$ and *in-degree* of u is $|\{v \in G : (v, u) \in R\}|$. A graph in which each vertex has the same out-degree r is called an *out-regular graph* with index of out-regularity r. *In-regular graphs* are similarly defined. A graph which is both out-and in-regular with index r, is a *regular graph*. A graph (G, R) is called a *vertex-transitive graph* if, given any two vertices a and b of G, there is some graph automorphism $f : V \longrightarrow V$ such that $f(a) = b$. In other words, a graph is *vertex-transitive* if its automorphism group acts transitively upon its vertices [4].

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A non-empty set G, together with a mapping $* : G \times G \longrightarrow G$ is called a *groupoid*. The mapping $*$ is called a *binary operation* on the set G. If $a, b \in G$, we use the symbol ab to denote $*(a, b)$. A groupoid $(G, *)$ is called a *quasigroup*, if for every $a, b \in G$, the equations, $ax = b$ and $ya = b$ are uniquely solvable in G. This implies both left and right cancelation laws [7]. In the terminology of [9], an algebraic structure $(G, *)$ is said to be a *right solvable Ward groupoid* if and only if for any $a, b \in G$ there is an element $x \in G$ such that $ax = b$ and the following identity holds:

$$(ac)(bc) = ab$$

In [9] it is proved that a right solvable Ward groupoid is a quasigroup.

A non-empty subset H of a right solvable Ward groupoid is said to be a *right solvable subward groupoid* if, under the induced operation from G, H itself form a right solvable Ward groupoid. Observe that a right solvable Ward groupoid is a weaker algebraic structure than a group.

Here we need the following result due to Vladimir Volenec [10]

Lemma 1.1. *In any right solvable Ward groupoid G there is the uniquely determined element o such that the following identities hold:*

$$aa = o,$$
$$ao = a.$$

2. The Main Theorem

Vertex transitive graphs have very rich structure and have been studied by many authors [1, 2, 3, 5, 6, 8]. In this section we introduce a class of vertex transitive graphs induced by right solvable Ward groupoids and then study various graph properties in terms of algebraic properties. These graphs can be considered as the generalization of Cayley graphs defined in [4].

Definition 2.1. Let A be a subset of a right solvable Ward groupoid G. We say that A is $\mathscr{R}$ associative, if

$$(xy)A = x(yA) \quad \text{for every } x, y \in G$$

Here we have the following:

Lemma 2.2. *Let A and B be $\mathscr{R}$ associative subsets of a right solvable Ward groupoid G. Then AB is also $\mathscr{R}$ associative.*

Proof. Let z be any arbitrary element in $(xy)AB$. Then $z = (xy)(ab)$ for some $a \in A$, $b \in B$. Note that

$$\begin{aligned}(xy)(ab) &= [(xy)a]b_1 \quad \text{for some } \; b_1 \in B \;\; (\text{since } B \text{ is } \mathscr{R} \text{ associative})\\ &= [x(ya_1)]b_1 \quad \text{for some } \; a_1 \in A \;\; (\text{since } A \text{ is } \mathscr{R} \text{ associative})\\ &= x[(ya_1)b_2] \quad \text{for some } \; b_2 \in B \;\; (\text{since } B \text{ is } \mathscr{R} \text{ associative})\\ &= x[y(a_1b_3)] \quad \text{for some } \; b_3 \in B \;\; (\text{since } B \text{ is } \mathscr{R} \text{ associative})\end{aligned}$$

This implies that $z \in x(yAB)$. Hence $xy(AB) \subseteq x(yAB)$. Similarly, one can prove that $x(yAB) \subseteq xy(AB)$. This completes the proof of the lemma. □

Theorem 2.3. *Let G be a right solvable Ward groupoid, H a $\mathscr{R}$ associative right solvable subward groupoid of G and A is a $\mathscr{R}$ associative subset of G. Let G/H denotes the collection of all left cosets of H in G. Let*

$$R_A = \{(xH, yH) \in G/H \times G/H : z \in HAH\}$$

where z denotes the solution of the equation $y = xz$. Then we have the following:

(a) $(G/H, R_A)$ is a vertex transitive graph.

(b) $(G/H, R_A)$ is a symmetric graph.

(c) $(G/H, R_A)$ is a regular graph with index $|\frac{HAH}{H}|$.

Proof. (a) We first prove that R_A is well defined. Let $xH, yH, x'H$ and $y'H$ be any four elements in G/H such that $xH = x'H$ and $yH = y'H$. Then we have

$$x'o = xh_1 \;\text{ and }\; y'o = yh_2 \text{ for some } h_1, h_2 \in H$$

Then by lemma 1.1, we have

$$x' = xh_1 \text{ and } y' = yh_2$$

Consider the equation $y = xz$. Since a right solvable Ward groupoid is a quasigroup, the equation $y = xz$ has a unique solution z in G. If $(xH, yH) \in R_A$, then by the definition of R_A, $z \in HAH$. But then, $z = (h_3a)h_4$ for some $h_3, h_4 \in H$ and $a \in A$.

Observe that

$y' = (xz)h_2$

$$
\begin{aligned}
&= (x((h_3a)h_4))h_2 \\
&= ((x(h_3a))h_5)h_2 \quad \text{(because } H \ \textit{is} \ \mathscr{R} \text{ associative)} \\
&= (x(h_3a))(h_5h_6) \ \text{ for some } h_6 \in H \quad \text{(because } H \text{ is } \mathscr{R} \text{ associative)} \\
&= (x(h_3a))(h_7) \ \text{ where } h_7 = h_5h_6 \in H \\
&= ((xh_3)a_1)h_7 \ \text{ for some } a_1 \in A \quad \text{(because } A \text{ is } \mathscr{R} \text{ associative)} \\
&= (xh_3)(a_1h_8) \ \text{ for some } h_8 \in A \quad \text{(because } H \text{ is } \mathscr{R} \text{ associative)} \\
&= (x(h_1h_9))(a_1h_8) \text{ where } h_3 = h_1h_9 \in H \ \text{ (because } H \text{ is a quasigroup)} \\
&= ((xh_1)h_{10})(a_1h_8) \text{ for some } h_{10} \in H \quad \text{(because } H \ \textit{is} \ \mathscr{R} \text{ associative)} \\
&= (x'h_{10})(a_1h_8) \\
&= x'(h_{10}(a_2h_{11})) \text{ for some } a_2 \in A, h_{11} \in H \quad \text{(because } AH \ \textit{is} \ \mathscr{R} \text{ associative)} \\
&= x'((h_{10}a_2)h_{12}) \text{ for some } h_{12} \in H \quad \text{(because } H \ \textit{is} \ \mathscr{R} \text{ associative)} \\
&= x'z',
\end{aligned}
$$

where $z' = (h_{10}a_2)h_{12}) \in HAH$. This implies that $(x'H, y'H) \in R_A$. Similarly, if $(x'H, y'H) \in R_A$, one can prove that $(xH, yH) \in R_A$. Hence R_A is well defined and hence $(G/H, R_A)$ is a graph. Next, we will show that the graph $(G/H, R_A)$ is a vertex transitive graph.

Let aH and bH be any two arbitrary elements in G/H. Define a mapping $\theta : G/H \longrightarrow G/H$ by

$$\theta(xH) = b/a(xH)$$

where b/a denote the solution z of the equation $b = za$. One can easily verify that the map θ is bijective. Furthermore, for every $xH, yH \in G/H$ with $y = xz$, if $(xH, yH) \in R_A$, then $z \in HAH$. This implies that

$$(b/a)y = (b/a)(xz') \ \text{ for some } z' \in HAH.$$

This equation tells us that $(\theta(xH), \theta(yH)) \in R_A$. Conversely, if $(\theta(xH), \theta(yH)) \in R_A$, one can prove that $(xH, yH) \in R_A$. Hence, the map θ is a graph automorphism. Finally,

$$
\begin{aligned}
\theta(aH) &= (b/a)(aH) \\
&= ((b/a)a)H = bH
\end{aligned}
$$

(b) Let $y = xz$. If $(xH, yH) \in R_A$, then $z \in HAH$. Consider the equation $x = yz'$. We will show that the solution $z' \in HAH$. Observe that

$$
\begin{aligned}
y &= (yz')z \\
&= y(z'z''), \ \text{for some } z'' \in HAH.
\end{aligned}
$$

That is,

$$yo = y(z'z'')$$

Then by lemma 1.1, we have

$$y(z'z') = y(z'z'')$$

This implies that $z' = z''$ for some $z'' \in HAH$ and hence $(yH, xH) \in R_A$. Hence the graph $(G/H, R_A)$ is symmetric.

(c) Since the graph is vertex transitive, it suffices to consider the out-degree and in-degree of the vertex oH.

Let

$$\rho(oH) = \{xH \in G/H : \{oH, xH\} \in R_A\}$$

$$\begin{aligned} xH \in \rho(oH) &\Leftrightarrow (oH, xH) \in R_A \\ &\Leftrightarrow x = oz \text{ for some } z \in HAH. \\ &\Leftrightarrow x = z' \text{ for some } z' \in HAH. \end{aligned}$$

This implies that

$$|\rho(H)| = \left|\frac{HAH}{H}\right|$$

Similarly, let

$$\sigma(oH) = \{xH \in G/H : (xH, oH) \in R_A\}$$

Observe that

$$\begin{aligned} xH \in \sigma(oH) &\Leftrightarrow (xH, oH) \in R_A \\ &\Leftrightarrow o = xz \text{ for some } z \in HAH. \\ &\Leftrightarrow xx = xz \text{ for some } z \in HAH \\ &\Leftrightarrow x = z \text{ for some } z \in HAH. \end{aligned}$$

Obviously,

$$|\sigma(H)| = \left|\frac{HAH}{H}\right|$$

□

Corollary 2.4. *$(G/H, R_A)$ is empty $\Leftrightarrow A = \emptyset$.*

Proof. Observe that

$$\begin{aligned}(G/H, R_A) \text{ is empty } &\Leftrightarrow R_A = \emptyset \\ &\Leftrightarrow A = \emptyset \quad (\text{because } o \in H)\end{aligned}$$

□

Corollary 2.5. *$(G/H, R_A)$ is a reflexive graph (each vertex has a loop) $\Leftrightarrow o \in HAH$.*

Proof. Assume that $(G/H, R_A)$ is a reflexive graph and let $x \in G$. Then $(xH, xH) \in R_A$. This implies that $x = xz$ for some $z \in HAH$. That is, $xo = xz$ for some $z \in HAH$. Consequently, $z = 0$.

Conversely, if $o \in HAH$, one can prove that $(G/H, R_A)$ is reflexive. □

Corollary 2.6. *$(G/H, R_A)$ is a complete graph $(R_A = G/H \times G/H) \Leftrightarrow G = HAH$.*

Proof. Assume that $(G/H, R_A)$ is a complete graph and let $x \in G$. Then $(oH, xH) \in R_A$. This implies that $x = oz$ for some $z \in HAH$. Since H and A are $\mathscr{R}$ associative, it follows that $x = z'$ for some $z' \in HAH$. Hence $G = HAH$.

Conversely, if $G = HAH$, then for every $xH, yH \in G/H$ with $y = xz$, we have $z \in G = HAH$. This implies that $(xH, yH) \in R_A$ for all x and y in G. Hence the graph is complete. □

Corollary 2.7. *$(G/H, R_A)$ is a transitive graph $(R_A o R_A \subseteq R_A) \Leftrightarrow HAHAH \subseteq HAH$.*

Proof. Suppose that $(G/H, R_A)$ is a transitive graph. If $x \in HAH$, then $(oH, xH) \in R_A$. Also, if $y \in HAH$, then $(xH, xyH) \in R_A$. Since $(G/H, R_A)$ is transitive, therefore $(oH, xyH) \in R_A$. This implies that $xy = oz$ for some $z \in HAH$. That is, $xy = z'$ for some $z' \in HAH$. Hence $HAHAH \subseteq HAH$.

Conversely, assume that $(G/H, R_A)$ is a transitive graph and let aH, bH, $cH \in G/H$. If $(aH, bH), (bH, cH) \in R_A$, then $b = az_1$ and $c = bz_2$ for some $z_1, z_2 \in HAH$. Note that

$$\begin{aligned}c &= (az_1)z_2 \\ &= a(z_1 z_3) \quad \text{for some } z_3 \in HAH \text{ (since } HAH \text{ is } \mathscr{R} \text{ associative)}\end{aligned}$$

Since $z_1 z_3 \in HAHAH \subseteq HAH$, $(aH, cH) \in R_A$. Hence $(G/H, R_A)$ is a transitive graph. □

Corollary 2.8. *$(G/H, R_A)$ is a connected graph if and only if $G = [HAH]$, where $[HAH]$ denotes the set of all finite products $z_1 z_2 z_3 \cdots z_n$ of elements of HAH.*

Proof. Assume that $(G/H, R_A)$ is connected and let $x \in G$. Then there is a path from oH to xH, say

$$(oH, x_1H, x_2H, \ldots, x_nH, xH)$$

This implies that

$$\begin{aligned} x_1 &= oz_1, \\ x_2 &= x_1z_2, \\ x_3 &= x_2z_3, \\ &\vdots \\ x_n &= x_{n-1}z_n, \\ x &= x_nz_{n+1} \end{aligned}$$

for some $z_1, z_2, z_3, \ldots, z_{n+1} \in HAH$. That is,

$$x = z_1' z_2, z_3, \ldots, z_{n+1}$$

for some $z_1', z_2, z_3, \ldots, z_{n+1} \in HAH$. That is,

$$x \in [HAH]$$

Since x is an arbitrary element of G, it follows that $G = [HAH]$.

Conversely, assume that $G = [HAH]$. Then for every $xH, yH \in G/H$ with $y = xz$, we have $z \in G = [HAH]$. This implies that

$$z = z_1z_2z_3 \ldots z_3$$

for some $z_1, z_2, z_3, \ldots, z_n \in HAH$. Let

$$x_1 = z_1, x_2 = x_1z_2, x_3 = x_2z_3, \ldots, x_n = x_{n-1}z_n$$

Then

$$(oH, x_1H, x_2H, \ldots, x_nH)$$

is a path from oH to zH. This implies that

$$(xH, xx_1H, x_2H, \ldots, xzH)$$

is a path from xH to yH. Hence $(G/H, R_A)$ is connected. □

Corollary 2.9. *$(G/H, R_A)$ is a Hasse-diagram* $\Leftrightarrow (HAH)^n \cap (HAH) = \emptyset$.

Proof. Assume that $(G/H, R_A)$ is a Hasse-diagram. Then for every x_0H, $x_1H, \cdots, x_nH \in G/H$ with $(x_iH, x_{i+1}H) \in R_A$ for $i = 0, 1, 2, \ldots, n-1$ implies $(x_0H, x_nH) \notin R_A$. Then by the definition of R_A, we have

$$x_1 = x_0z_1, x_2 = x_1z_2, \ldots, x_n = x_{n-1}z_n$$

for some $z_i \in HAH, i = 1, 2, \ldots, n$. That is,

$$x_n = x_0z'_1z'_2 \ldots z'_n$$

for some $z'_i \in HAH$, $i = 1, 2, \cdots, n$. That is, $x_n = x_0z$, where $z = z'_1z'_2 \ldots z'_n \in (HAH)^n$. Since, $(x_0H, x_nH) \notin R_A$, therefore, $(HAH)^n \cap (HAH) = \emptyset$.

Conversely, assume that $(HAH)^n \cap (HAH) = \emptyset$. for $n \geq 2$. We will show that $(G/H, R_A)$ is a Hasse-diagram. Let $x_0H, x_1H, \cdots, x_nH$ any $(n+1)$ elements of G/H with $n \geq 2$ and $(x_iH, x_{i+1}H) \in R_A$ for $i = 0, 1, \cdots, n-1$. Then we have

$$x_n = x_0z_1z_2 \ldots z_n$$

for some $z_i \in HAH$, $i = 1, 2, \ldots, n$. Since, $(HAH)^n \cap (HAH) = \emptyset$, therefore, $(x_0H, x_nH) \notin R_A$. Hence $(G/H, R_A)$ is a Hasse- diagram. □

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