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# EMBEDDINGS OF GENERAL CURVES IN PROJECTIVE SPACES: PARTS OF THE RANGE OF THE CUBICS

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**Abstract:** Here we prove the existence of linearly normal smooth curves $C \subset \mathbb{P}^r$, $r \ge 5$, with maximal rank, $h^1(X, \mathcal{O}_C(1)) \le r/2$, any genus at most of order $r^3/12$ and general moduli. In this range of degrees and genera we prove the surjectivity of the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(3)) \to H^0(C, \mathcal{O}_C(3))$.

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## 1. Introduction

Let $C \subset \mathbb{P}^r$ be any projective curve. The curve $C$ is said to have *maximal rank* if for every integer $x > 0$ the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ has maximal rank, i.e. either it is injective or it is surjective. For any smooth curve $X$ and any spanned line bundle $L$ on $X$ let $h_L : X \to \mathbb{P}^n$, $n := h^0(X, L) - 1$, be the morphism induced by the complete linear system $|L|$. For all integers $c \ge 0$, and $r \ge 3$ set $g_{2,c,r} := \binom{r+2}{2} - 2r - 1 + 2c$ and $q_{3,r} := \lfloor (\binom{r+1}{3} - 3r + 5)/2 \rfloor$. Here we prove the following result.

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**Theorem 1.** *Fix integers $r, c, g$ such that $r \geq 5$, $0 < c \leq \lfloor r/2 \rfloor$, and $c(r+1) \leq g \leq g_{2,c,r} + q_{3,r}$. Let $X$ be a general smooth curve of genus $g$. Set $d := g + r - c$. Then $W^r_d(X) \neq \emptyset$, a general $L \in W^r_d(X)$ is very ample, $h^0(X, L) = r + 1$ and the curve $h_L(X) \subset \mathbb{P}^r$ has maximal rank.*

In the set-up of Theorem 1 we have $h^1(\mathbb{P}^r, \mathcal{I}_{h_L(X)}(3)) = 0$. We have $h^1(\mathbb{P}^r, \mathcal{I}_{h_L(X)}(2)) = 0$ (i.e. $h_L(X)$ is projectively normal) if and only if $g \leq g_{2,c,r}$ ([6], Theorem 1).

*Proof of Theorem 1.* For any integer $g, r, d$ set $\rho(g, r, d) := (r+1)d - rg - r(r+1)$ (the Brill-Noether number). Notice that $\rho(g, r, d) = (r+1)(g+r-c) - rg - r(r+1)g - c(r+1) \geq 0$. Brill-Noether theory gives $W^r_d(X) \neq \emptyset$, that $W^r_d(X)$ has pure dimension $\rho(g, r, d)$ (it is also irreducible if $\rho(g, r, d) > 0$) and that $W^r_d(X) \neq W^{r+1}_d(X)$, i.e. $h^0(X, L) = r + 1$ for a general $L \in W^r_d(X)$ ([1], Ch. V) (in the case $\rho(g, r, d) = 0$ we have $W^{r+1}_d(X) = \emptyset$). Hence $h^1(X, L) = r+1$ for a general $L \in W^r_d(X)$ (or for all $L \in W^r_d(X)$ if $\rho(g, r, d) = 0$). For this range of triples $(g, r, d)$ it is very easy to prove that a general $L \in W^r_d(X)$ is very ample (e.g., see the proof of [7], Theorem at pages 26-27). If $g \leq g_{2,r,c}$ then $h_L(X)$ is projectively normal ([6], Theorem 1). Hence we may assume $g > g_{2,r,c}$. By [2] we have $h^0(\mathbb{P}^r, \mathcal{I}_{h_L(X)}(2)) = 0$ for a general $X$ and a general $L$. Hence it is sufficient to prove $h^1(\mathbb{P}^r, \mathcal{I}_{h_L(X)}(t)) = 0$ for all $t \geq 3$. Gieseker-Petri theory gives $h^1(X, L^{\otimes 2}) = 0$. Hence by Castelnuovo-Mumford's lemma it is sufficient to prove $h^1(\mathbb{P}^r, \mathcal{I}_{h_L(X)}(3)) = 0$. Set $x := g - g_{2,r,c}$. By [6], Theorem 1, a general smooth curve of genus $g_{2,r,c}$ has a linearly normal embedding $C \subset \mathbb{P}^r$ such that $\deg(C) = g_{2,r,c} + r - c$, $h^1(C, \mathcal{O}_C(1)) = c$ and $h^1(\mathbb{P}^r, \mathcal{I}_C(2) = 0$. Fix a general hyperplane $H \subset \mathbb{P}^r$. In particular we assume that $H$ is transversal to $C$ and that the set $H \cap C$ is in uniform position ([1], p. 113). We have $h^2(\mathbb{P}^r, \mathcal{I}_C(1)) = h^1(C, \mathcal{O}_C(1)) = c$. Since $C$ is linearly normal, we have From the exact sequence

$$0 \to \mathcal{O}_C(1) \to \mathcal{O}_C(2) \to \mathcal{I}_{C \cap H, H}(2) \to 0 \tag{1}$$

we get $h^1(H, \mathcal{I}_{C \cap H, H}(2)) \leq c$. Since $C \cap H$ is in uniform position, we get $h^1(H, \mathcal{I}_{S,H}(2)) = 0$ for every $S \subset C \cap H$ such that $\sharp(S) \leq \deg(C) - c$. Fix a hyperplane $M \subset H$ spanned by points of $C \cap H$. Since $C \cap H$ is in uniform position, then $\sharp(M \cap C) = r - 1$. Notice that $\dim(M) \geq 3$. Let $T \subset M$ be a general non-special embedding of degree $x + r - 1$ of a general smooth curve of genus $x$. The curve $T$ has maximal rank in $M$ ([3], [4]). Look at the exact sequence of coherent sheaves on $H$ (the residual sequence of $(C \cap H) \cup T$ with respect to $M$):

$$0 \to \mathcal{I}_{C \cap H \setminus C \cap M}(2) \to \mathcal{I}_{C \cup T, H}(3) \to \mathcal{I}_{T,M}(3) \to 0 \tag{2}$$

Since $\sharp(C \cap M) = r - 1 \geq c$, we have $h^1(H, \mathcal{I}_{C\cap H\setminus C\cap M}(2)) = 0$. Since $x \leq q_{3,r,c}$, we have $3(x + r - 2) + 1 - x \leq \binom{r+1}{3}$. Since $T$ has maximal rank, we get $h^1(M, \mathcal{I}_{T,M}(3)) = 0$. Hence (2) gives $h^1(H, \mathcal{I}_{C\cup T,H}(3)) = 0$. From the exact sequence of coherent sheaves on $\mathbb{P}^r$ (the residual sequence of $C \cup T$ with respect to $H$)

$$0 \to \mathcal{I}_C(2) \to \mathcal{I}_{C\cup T}(3) \to \mathcal{I}_{(C\cap H)\cup T,H}(3) \to 0 \qquad (3)$$

we get $h^1(\mathbb{P}^r, \mathcal{I}_{C\cup T}(3)) = 0$. We claim that the curve $C \cup T$ is in the irreducible component $W(g + r - c, g; r)$ of the Hilbert scheme of $\mathbb{P}^r$ defined in [5], §1. To check the claim first degenerate $T$ to a union of a rational normal curve $Y$ of $M$ at $x$ general secant lines of $M$. Then degenerate $Y$ to a loop of $r - 2$ lines containing $C \cap M$. Then apply $x + r - 2$ times [5], Lemma 2.2. Since $\rho(g, r, g + r - c) \geq 0$, a general element of $W(g + r - c, g; r)$ has general moduli ([5], Proposition 3.1). Hence Theorem 1 follows from semicontinuity. □

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