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SOME RESULTS ON t-BEST APPROXIMATION IN FUZZY ANTI-NORMED LINEAR SPACES

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Abstract: The aim of this paper to give the set of all t-best approximations on fuzzy anti-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [12].

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1. Introduction

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The theory of fuzzy sets was introduced by Zadeh [14] in 1965, since then many mathematicians have studied from several angles [11,5]. The idea of fuzzy norm was initiated by Katsaras in [9]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [8]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [10].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [10]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and

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Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [9], Felbin [6], and Bag and Samanta [1]. Veeramani [13] introduced the concept of t-best approximations in fuzzy metric spaces. Recently, Vaezpour and Karimi [12], studied on the set of all t-best approximations on fuzzy normed spaces and proved several theorems pertaining to this set.

In [7] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigated their important properties.

In this paper, we give the set of all t-best approximations on fuzzy anti-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [12].

2. Preliminaries

Definition 1. [1] Let X be a linear space over a real field F. A fuzzy subset N of $X \times R$ is called a fuzzy norm on X if the following conditions are satisfied for all $x, y \in X$.

(N_1): For all $t \in R$ with $t \leq 0$, $N(x,t) = 0$,

(N_2): For all $t \in R$ with $t > 0$, $N(x,t) = 1$ if and only if $x = \bar{0}$,

(N_3): For all $t \in R$ with $t > 0$, $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$

(N_4): For all $s, t \in R$, $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$

(N_5): $N(x,t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x,t) = 1$.

Then the pair (X, N) is called a fuzzy normed linear space (briefly F-NLS).

Definition 2. [7] Let X be a linear space over a real field F. A fuzzy subset N of $X \times R$ is called a fuzzy anti-norm on X if the following conditions are satisfied for all $x, y \in X$

$(a - N_1)$: For all $t \in R$ with $t \leq 0$, $N(x,t) = 1$,

$(a - N_2)$: For all $t \in R$ with $t > 0$, $N(x,t) = 0$ if and only if $x = \bar{0}$,

$(a - N_3)$: For all $t \in R$ with $t > 0$, $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(a - N_4)$: For all $s, t \in R$, $N(x+y, s+t) \leq \max\{N(x,s), N(y,t)\}$

$(a - N_5)$: $N(x,t)$ is a non-increasing function of $t \in R$ and $\lim_{t\to\infty} N(x,t) = 0$.

Then the pair (X, N) is called a fuzzy anti-normed linear space (briefly Fa-NLS).

Example 3. Let $(X, \| \bullet \|)$ be a normed linear space. Define

$$
\begin{aligned}
N(x,t) &= \frac{\|x\|}{t+\|x\|}, \quad \text{if} \quad t>0, \quad t\in R, \quad x\in X \\
&= 1, \quad \text{if} \quad t\le 0, \quad t\in R, \quad x\in X.
\end{aligned}
$$

Then (X, N) is a Fuzzy anti-normed linear space.

Definition 4. A sequence $\{x_k\}$ in a fuzzy anti-normed linear space (X, N) is said to be converges to $x \in X$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_k - x, t) < r$, for all $k \geq n_0$.

Theorem 5. *In a fuzzy anti-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only $\lim_{k\to\infty} N(x_k - x, t) = 0$, $\forall\, t > 0$.*

3. Main Results

Definition 6. Let (X, N) be a fuzzy anti-normed space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x - y, t) < r\}$$
$$B[x, r, t] = \{y \in X : N(x - y, t) \leq r\}$$

Definition 7. Let (X, N) be a fuzzy anti-normed space. A subset A of X is said to be open if there exists $r \in (0, 1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 8. Let (X, N) be a fuzzy anti-normed space. A subset A of X is said to be closed if for any sequence $\{x_k\}$ in A converges to $x \in A$.

i.e., $\lim_{k\to\infty} N(x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Definition 9. Let (X, N) be a fuzzy anti-normed space. A subset B of X is said to be closure of $A \subset B$ if for any $x \in B$, there exists a sequence $\{x_k\}$ in A such that $\lim_{k\to\infty} N(x_k - x, t) = 0$, $\forall\, t > 0$. We denote the set B by $\overline{A}$.

Definition 10. Let (X, N) be a fuzzy anti-normed space. A subset A of X is said to be compact if for any sequence $\{x_k\}$ in A has a sequence converging to an element of A.

Lemma 11. *If (X, N) be a fuzzy anti-normed space then*
(i) the function $(x, y) \longrightarrow x + y$ is continuous
(ii) the function $(\alpha, x) \longrightarrow \alpha x$ is continuous

Proof. (i) If $x_k \longrightarrow x$ and $y_k \longrightarrow y$, then as $k \longrightarrow \infty$,
$N((x_k + y_k) - (x + y), t) \le \max\{N(x_k - x, \frac{t}{2}), N(y_k - y, \frac{t}{2})\} \longrightarrow 0$
(ii) If $x_k \longrightarrow x$, $\alpha_k \longrightarrow \alpha$ and $\alpha_k \neq 0$ then

$$\begin{aligned} N(\alpha_k x_k - \alpha x, t) &= N(\alpha_k(x_k - x) + x(\alpha_k - \alpha), t) \\ &\le \max\{N(\alpha_k(x_k - x), \tfrac{t}{2}), N(x(\alpha_k - \alpha), \tfrac{t}{2})\} \\ &= \max\{N(x_k - x, \tfrac{t}{2\alpha_k}), N(x, \tfrac{t}{2(\alpha_k - \alpha)})\} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$ □

Definition 12. Let (X, N) be a fuzzy anti-normed space and A is a nonempty subset of X. Let $d(A, x, t) = \inf\{N(x - y, t) : y \in A\}$, where $x \in X$, $t > 0$. An element $y_0 \in A$ is said to be a t-best approximation of x from A if $N(y_0 - x, t) = d(A, x, t)$.

Definition 13. Let (X, N) be a fuzzy anti-normed space and A is a nonempty subset of X. For $x \in X$, $t > 0$, we shall denote the set of all elements of t-best approximation of x from A by $P_A^t(x)$; i.e.,

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(y - x, t)\}.$$

If each $x \in X$ has at least (respectively exactly) one t-best approximation in A then A is called a t-proximinal (respectively t-chebyshev) set.

Definition 14. Let (X, N) be a fuzzy anti-normed space and A is a nonempty subset of X. For $t > 0$, A is said to be t-boundedly compact if for each $x \in X$ and $0 < r < 1$, $B[x, r, t] \cap A$ is a compact subset of X.

Theorem 15. *Let (X, N) be a fuzzy anti-normed space and A is a nonempty subset of X then*

(i) $d(A + y, x + y, t) = d(A, x, t)$, for all $x, y \in X$ and $t > 0$,

(ii) $P_A^t(x + y) = P_A^t(x) + y$, for all $x, y \in X$ and $t > 0$,

(iii) $d(\alpha A, \alpha x, t) = d(A, x, \frac{t}{|\alpha|})$, for all $x \in X$, $t > 0$ and $\alpha \in R\backslash\{0\}$,

(iv) $P_{\alpha A}^{|\alpha| t}(\alpha x) = \alpha P_A^t(x)$, for all $x \in X$, $t > 0$ and $\alpha \in R\backslash\{0\}$,

(v) A is t-proximinal (respectively t-chebyshev) if and only if $A + y$ is t-proximinal (respectively t-chebyshev) for any given $y \in X$,

(vi) A is t-proximinal (respectively t-chebyshev) if and only if αA is $|\alpha| t$-proximinal (respectively $|\alpha| t$-chebyshev) for any given $\alpha \in R\backslash\{0\}$.

Proof. (i) For $x, y \in X$ and $t > 0$,

$$\begin{aligned} d(A + y, x + y, t) &= \inf\{N((z + y) - (x + y), t) : z \in A\} \\ &= \inf\{N(z - x, t) : z \in A\} = d(A, x, t). \end{aligned}$$

(ii) On using (i), it follows that, $y_0 \in P_{A+y}^t(x + y)$ if and only if $y_0 \in A + y$ and $d(A+y, x+y, t) = N(x+y-y_0, t)$ if and only if $y_0 - y \in A$ and $d(A, x, t) = N(x - (y_0 - y), t)$ if and only if $y_0 - y \in P_A^t(x)$, i.e., $y_0 \in P_A^t(x) + y$

(iii) We have $d(\alpha A, \alpha x, t) = \inf\{N(\alpha x - \alpha z, t) : z \in A\}$
$= \inf\{N(\alpha(x - z), t) : z \in A\}$
$= \inf\{N(x - z, \frac{t}{|\alpha|}) : z \in A\} = d(A, x, \frac{t}{|\alpha|})$.

(iv) On using (iii), it follows that $y_0 \in P^{|\alpha|t}_{\alpha A}(\alpha x)$ if and only if $y_0 \in \alpha A$ and $d(\alpha A, \alpha x, |\alpha|t) = N(\alpha x - y_0, |\alpha|t)$ if and only if $\frac{y_0}{\alpha} \in A$ and $N(x - \frac{y_0}{\alpha}, t) = d(A, x, t)$. However, this is equivalent to $\frac{y_0}{\alpha} \in P^t_A(x)$; i.e., $y_0 \in \alpha P^t_A(x)$.

(v) The proof of (v) is an immediate consequence of (ii).

(vi) The proof of (vi) follows from (iv). □

Corollary 16. *Let M be a nonempty subspace of X then*
(i) $d(M, x + y, t) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $y \in M$,
(ii) $P^t_M(x + y) = P^t_M(x) + y$, for all $t > 0$, $x \in X$ and $y \in M$,
(iii) $d(M, \alpha x, |\alpha t|) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $\alpha \in R\backslash\{0\}$,
(iv) $P^{|\alpha|t}_M(\alpha x) = \alpha P^t_M(x)$, for all $t > 0$, $x \in X$ and $\alpha \in R\backslash\{0\}$.

Proof. The proof of (i) and (ii) follows from theorem 15(i) and 15(ii) and the fact that if M is a subspace and $y \in M$ then $M + y = M$.

The proof of (iii) and (iv) follows from theorem 15(iii) and 15(iv) and the fact that if M is a subspace and $\alpha \neq 0$ then $\alpha M = M$ □

Definition 17. For $x \in X$, $0 < r < 1$, $t > 0$,

$$S[x, r, t] = \{y \in X : N(x - y, t) = r\} \quad \text{and} \quad e^t_A(x) = d(A, x, t).$$

Theorem 18. *Let (X, N) be a fuzzy anti-normed space, $A \subset X$, $x \in X\backslash\overline{A}$ and $t > 0$ then we have*

$$P^t_A(x) = A \cap B[x, e^t_A(x), t] = A \cap S[x, e^t_A(x), t]. \quad (1)$$

Proof. The inclusions;

$$P^t_A(x) \subseteq A \cap S[x, e^t_A(x), t] \subseteq A \cap B[x, e^t_A(x), t]. \quad (2)$$

are obvious by the definitions of $P^t_A(x)$ and $e^t_A(x)$.

Conversely, let $y \in A \cap B[x, e^t_A(x), t]$, then we have $y \in A$ and
$N(y - x, t) \leq e^t_A(x) = d(A, x, t) \leq N(y - x, t)$.
Therefore $y \in A$ and $N(y - x, t) = d(A, x, t)$, which implies that $y \in P^t_A(x)$. So, $A \cap B[x, e^t_A(x), t] \subset P^t_A(x)$. Hence by (2) we have (1) which completes the proof. □

Remark 19. Let (X,N) be a fuzzy anti-normed linear space and A is a nonempty subset of X, $x \in X\backslash\overline{A}$ and $t>0$ then we have

$$A \cap B(x, e^t_A(x), t) = \emptyset, \tag{3}$$

because, if $y_0 \in A \cap B(x, e^t_A(x), t)$ then $d(A,x,t) \le N(x-y_0,t) < d(A,x,t)$ which is impossible.

Corollary 20. *Let (X,N) be a fuzzy anti-normed space and A is a nonempty subset of X, $x \in X\backslash\overline{A}$ with $P^t_A(x) \neq \emptyset$ and $0<r<1$ such that,*

$$\emptyset \neq A \cap B[x,r,t] \subseteq S[x,r,t] \tag{4}$$

then we have $r = e^t_A(x)$, and we can write $A \cap B[x,r,t] = P^t_A(x)$.

Proof. If $r < e^t_A(x)$ then by the definition of $e^t_A(x)$ we have $A \cap B[x,r,t] = \emptyset$, which contradicts (4). If $r > e^t_A(x)$, since $P^t_A(x) \neq \emptyset$, then by (1) we have $\emptyset \neq P^t_A(x) = A \cap B[x, e^t_A(x), t] \subseteq A \cap B(x,r,t)$, which contradicts (4), and this completes the proof. □

Definition 21. Let (X,N) be a fuzzy anti-normed space, $0<r<1$ and $t>0$. We shall say that a set $A \subset X$ supports the cell $B[x,r,t]$, or that A is a support set of the cell $B[x,r,t]$, if we have $d(A, B[x,r,t], t) = 1$ and $A \cap B(x,r,t) = \emptyset$.

Theorem 22. *Let (X,N) be a fuzzy anti-normed space and A is a nonempty subset of X, $x \in X\backslash\overline{A}$, $a_0 \in A$ and $t>0$. We have $a_0 \in P^t_A(x)$ if and only if the set A supports the cell $B = B[x, N(a_0-x,t), t]$.*

Proof. Assume that $a_0 \in P^t_A(x)$. Hence $N(a_0-x,t) = d(A,x,t)$. Then by (3), we have $A \cap B(x, N(a_0-x,t), t) = \emptyset$, on the other hand, since $a_0 \in A \cap B[x, N(a_0-x,t), t]$, we have $d(A,B,t) = 1$. Consequently, the set A supports the cell B. Conversely, suppose $a_0 \notin P^t_A(x)$, hence
$N(a_0-x,t) > d(A,x,t)$, and let $0<\varepsilon<1$ such that $N(a_0-x,t) > d(A,x,t)+\varepsilon$. Then there exists an $a \in A$ such that $N(a_0-x,t) > d(A,x,t)+\varepsilon > N(a-x,t)$, hence $a \in B(x, N(a_0-x,t), t)$. Consequently, A does not support the cell B. □

Remark 23. We recall that a set A in a topological space τ is said to be countably compact, if every countable open cover of A has a finite subcover, or, which is equivalent, if for every decreasing sequence $A_1 \supset A_2 \supset \ldots$ of non-void closed subset of A we have $\bigcap\limits_{n=1}^{\infty} A_n \neq \emptyset$

Theorem 24. *Let (X, N) be a fuzzy anti-normed space, τ be an arbitrary topology on X and $t > 0$. If A is a nonempty subset of X such that for $A \cap B[x, r, t]$ is $\tau-$countably compact, then A is t-proximinal.*

Proof. For all $n \in N$, $0 < 1 - d(A, x, t) + \frac{d(A,x,t)}{n+1} < 1$. put

$$A_n^t = A \cap B\left[x, 1 - d(A, x, t) + \frac{d(A, x, t)}{n+1}, t\right], \quad (n = 1, 2, \dots).$$

Since for every $n \in N$, $d(A, x, t)\left(1 - \frac{1}{n+1}\right) > d(A, x, t)$, obviously $A_1^t \supset A_2^t \supset \dots$ and each $A_n^t \neq \emptyset$. Hence there exists $a_n^t \in A$ such that

$$d(A, x, t)\left(1 - \frac{1}{n+1}\right) > N(a_n^t - x, t).$$

It follows that $a_n^t \in A_n^t$. Now, since each A_n^t is $\tau-$countably compact and $\tau-$closed, we conclude that there exists an $a_0 \in \bigcap_{n=1}^{\infty} A_n^t$. Then we have

$$d(A, x, t) \leq N(a_0 - x, t) \leq d(A, x, t)\left(1 - \frac{1}{n+1}\right), \quad (n = 1, 2, \dots),$$

hence $a_0 \in P_A^t(x)$ which completes the proof. □

Definition 25. Let (X, N) be a fuzzy anti-normed space and A is a nonempty subset of X. An element $y_0 \in A$ is said to be an F-best approximation of $x \in X$ from A if it is a t-best approximation of x from A, for every $t > 0$, i.e.,

$$y_0 \in \bigcap_{t \in (0,\infty)} P_A^t(x).$$

The set of all elements of F-best approximations of $x \in X$ from A is denoted by $FP_A(x)$, i.e.,

$$FP_A(x) = \bigcap_{t \in (0,\infty)} P_A^t(x).$$

If each $x \in X$ has at least (respectively exactly) one F-best approximation in A then A is called a F-proximinal (respectively F-chebyshev) set.

Example 26. Let $X = R$. Define $N : X \times R \longrightarrow [0, 1]$ by

$$N(x, t) = \frac{\|x\|}{t + \|x\|}, \quad \text{if } t > 0, \quad t \in R, \quad x \in X$$

$$= \quad 1, \quad \text{if} \quad t \leq 0, \quad t \in R, \quad x \in X.$$

Then (X, N) is a fuzzy anti-normed space. Let $A = [0, 1]$. Then, for every $x < 1$, we have 1 is an F-best approximation of x from A and for every $x > 0$, we have 0 is an F-best approximation of x from A. So A is an F-proximinal set.

Remark 27. For an arbitrary set $A \subset X$ we shall denote by ∂A the boundary of A, and by $\mathcal{M}_A$ the set of all elements of the F-best approximation of the elements $x \in X$ from A. i.e.,

$$\mathcal{M}_A = \bigcup_{x \in X} FP_A(x).$$

Theorem 28. *Let (X, N) be a fuzzy anti-normed space, A is a nonempty subset of X and A be a F-best proximinal set in X then $\partial A \subset \overline{\mathcal{M}_A}$.*

Proof. If $\partial A = \emptyset$, the proof is obvious. If $\partial A \neq \emptyset$, let $a_0 \in \partial A$, $0 < \varepsilon < 1$ and $t > 0$ be arbitrary. Then there exists $0 < \varepsilon' < 1$ such that $\varepsilon' < \varepsilon$ and the cell $B(a_0, \varepsilon', \frac{t}{2})$ contains at least one element $x \in X \backslash A$. Let $\pi_A(x) \in FP_A(x)$ (it exists, since by hypothesis, A is F-proximinal). Then we have,

$$\begin{aligned} N(a_0 - \pi_A(x), t) &\leq \max\left\{N(a_0 - x, \tfrac{t}{2}), N(x - \pi_A(x), \tfrac{t}{2})\right\} \\ &= \max\left\{N(a_0 - x, \tfrac{t}{2}), N(A - x, \tfrac{t}{2})\right\} \\ &\leq \max\left\{N(a_0 - x, \tfrac{t}{2}), N(a_0 - x, \tfrac{t}{2})\right\} \\ &\leq \max\{\varepsilon', \varepsilon'\} = \varepsilon' \\ &< \varepsilon \end{aligned}$$

So, $B(a_0, \varepsilon, t) \cap \mathcal{M}_A \neq \emptyset$ and since $\varepsilon > 0$ is arbitrary, we obtain $a_0 \in \overline{\mathcal{M}_A}$ which completes the proof. □

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