SOME RESULTS ON \( t \)-BEST APPROXIMATION IN Fuzzy Anti-Normed Linear Spaces

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Abstract: The aim of this paper is to give the set of all \( t \)-best approximations on fuzzy anti-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [12].

AMS Subject Classification: 46A30, 46S40, 46A70, 54A40

Key Words: fuzzy anti-norms, \( t \)-best approximation, \( t \)-proximinal set, \( t \)-chebyshev set

1. Introduction

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The theory of fuzzy sets was introduced by Zadeh [14] in 1965, since then many mathematicians have studied from several angles [11,5]. The idea of fuzzy norm was initiated by Katsaras in [9]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [8]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [10].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [10]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and
Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [9], Felbin [6], and Bag and Samanta [1]. Veeramani [13] introduced the concept of $t$-best approximations in fuzzy metric spaces. Recently, Vaezpour and Karimi [12], studied on the set of all $t$-best approximations on fuzzy normed spaces and proved several theorems pertaining to this set.

In [7] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigated their important properties.

In this paper, we give the set of all $t$-best approximations on fuzzy anti-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [12].

2. Preliminaries

**Definition 1.** [1] Let $X$ be a linear space over a real field $F$. A fuzzy subset $N$ of $X \times \mathbb{R}$ is called a fuzzy norm on $X$ if the following conditions are satisfied for all $x, y \in X$.

(N1): For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x, t) = 0$,
(N2): For all $t \in \mathbb{R}$ with $t > 0$, $N(x, t) = 1$ if and only if $x = \mathbf{0}$,
(N3): For all $t \in \mathbb{R}$ with $t > 0$, $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$
(N4): For all $s, t \in \mathbb{R}$, $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$
(N5): $N(x, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$.

Then the pair $(X, N)$ is called a fuzzy normed linear space (briefly F-NLS).

**Definition 2.** [7] Let $X$ be a linear space over a real field $F$. A fuzzy subset $N$ of $X \times \mathbb{R}$ is called a fuzzy anti-norm on $X$ if the following conditions are satisfied for all $x, y \in X$.

(a $- N_1$): For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x, t) = 1$,
(a $- N_2$): For all $t \in \mathbb{R}$ with $t > 0$, $N(x, t) = 0$ if and only if $x = \mathbf{0}$,
(a $- N_3$): For all $t \in \mathbb{R}$ with $t > 0$, $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$
(a $- N_4$): For all $s, t \in \mathbb{R}$, $N(x + y, s + t) \leq \max\{N(x, s), N(y, t)\}$
(a $- N_5$): $N(x, t)$ is a non-increasing function of $t \in \mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 0$.

Then the pair $(X, N)$ is called a fuzzy anti-normed linear space (briefly Fa-NLS).
Example 3. Let \((X, \| \cdot \|)\) be a normed linear space. Define
\[
N(x, t) = \begin{cases} 
\frac{\|x\|}{t + \|x\|}, & \text{if } t > 0, \ t \in \mathbb{R}, \ x \in X \\
1, & \text{if } t \leq 0, \ t \in \mathbb{R}, \ x \in X.
\end{cases}
\]
Then \((X, N)\) is a Fuzzy anti-normed linear space.

Definition 4. A sequence \(\{x_k\}\) in a fuzzy anti-normed linear space \((X, N)\) is said to be converges to \(x \in X\) if given \(t > 0, 0 < r < 1\), there exists an integer \(n_0 \in \mathbb{N}\) such that \(N(x_k - x, t) < r\), for all \(k \geq n_0\).

Theorem 5. In a fuzzy anti-normed linear space \((X, N)\), a sequence \(\{x_k\}\) converges to \(x \in X\) if and only \(\lim_{k \to \infty} N(x_k - x, t) = 0, \ \forall \ t > 0\).

3. Main Results

Definition 6. Let \((X, N)\) be a fuzzy anti-normed space. The open ball \(B(x, r, t)\) and the closed ball \(B[x, r, t]\) with the center \(x \in X\) and radius \(0 < r < 1, t > 0\) are defined as follows:
\[
B(x, r, t) = \{ y \in X : N(x - y, t) < r \} \\
B[x, r, t] = \{ y \in X : N(x - y, t) \leq r \}
\]

Definition 7. Let \((X, N)\) be a fuzzy anti-normed space. A subset \(A\) of \(X\) is said to be open if there exists \(r \in (0, 1)\) such that \(B(x, r, t) \subset A\) for all \(x \in A\) and \(t > 0\).

Definition 8. Let \((X, N)\) be a fuzzy anti-normed space. A subset \(A\) of \(X\) is said to be closed if for any sequence \(\{x_k\}\) in \(A\) converges to \(x \in A\).
\[\text{i.e., } \lim_{k \to \infty} N(x_k - x, t) = 0, \ \forall \ t > 0\] implies that \(x \in A\).

Definition 9. Let \((X, N)\) be a fuzzy anti-normed space. A subset \(B\) of \(X\) is said to be closure of \(A \subset B\) if for any \(x \in B\), there exists a sequence \(\{x_k\}\) in \(A\) such that \(\lim_{k \to \infty} N(x_k - x, t) = 0, \ \forall \ t > 0\). We denote the set \(B\) by \(\overline{A}\).

Definition 10. Let \((X, N)\) be a fuzzy anti-normed space. A subset \(A\) of \(X\) is said to be compact if for any sequence \(\{x_k\}\) in \(A\) has a sequence converging to an element of \(A\).

Lemma 11. If \((X, N)\) be a fuzzy anti-normed space then
(i) the function \((x, y) \mapsto x + y\) is continuous
(ii) the function \((\alpha, x) \mapsto \alpha x\) is continuous
Proof. (i) If \( x_k \to x \) and \( y_k \to y \), then as \( k \to \infty \),
\[ N((x_k + y_k) - (x + y), t) \leq \max\{N(x_k - x, \frac{t}{2}), N(y_k - y, \frac{t}{2})\} \to 0 \]
(ii) If \( x_k \to x \), \( \alpha_k \to \alpha \) and \( \alpha_k \neq 0 \) then
\[ N(\alpha_k x_k - \alpha x, t) = N(\alpha_k (x_k - x), t) \]
\[ \leq \max\{N(\alpha_k (x_k - x), \frac{t}{2}), N(x(\alpha_k - \alpha), \frac{t}{2})\} \]
\[ = \max\{N(x_k - x, \frac{t}{2\alpha_k}), N(x, \frac{t}{2(\alpha_k - \alpha)})\} \to 0 \text{ as } k \to \infty. \quad \square \]

Definition 12. Let \((X, N)\) be a fuzzy anti-normed space and \( A \) is a nonempty subset of \( X \). Let \( d(A, x, t) = \inf\{N(x - y, t) : y \in A\} \), where \( x \in X \), \( t > 0 \). An element \( y_0 \in A \) is said to be a \( t \)-best approximation of \( x \) from \( A \) if \( N(y_0 - x, t) = d(A, x, t) \).

Definition 13. Let \((X, N)\) be a fuzzy anti-normed space and \( A \) is a nonempty subset of \( X \). For \( x \in X \), \( t > 0 \), we shall denote the set of all elements of \( t \)-best approximation of \( x \) from \( A \) by \( P^t_A(x) \); i.e.,
\[ P^t_A(x) = \{ y \in A : d(A, x, t) = N(y - x, t) \}. \]

If each \( x \in X \) has at least (respectively exactly) one \( t \)-best approximation in \( A \) then \( A \) is called a \( t \)-proximinal (respectively \( t \)-chebyshev) set.

Definition 14. Let \((X, N)\) be a fuzzy anti-normed space and \( A \) is a nonempty subset of \( X \). For \( t > 0 \), \( A \) is said to be \( t \)-boundedly compact if for each \( x \in X \) and \( 0 < r < 1 \), \( B[x, r, t] \cap A \) is a compact subset of \( X \).

Theorem 15. Let \((X, N)\) be a fuzzy anti-normed space and \( A \) is a nonempty subset of \( X \) then
(i) \( d(A + y, x + y, t) = d(A, x, t) \), for all \( x, y \in X \) and \( t > 0 \),
(ii) \( P^t_A(x + y) = P^t_A(x) + y \), for all \( x, y \in X \) and \( t > 0 \),
(iii) \( d(\alpha A, x, t) = d(A, x, \frac{t}{|\alpha|}) \), for all \( x \in X \), \( t > 0 \) and \( \alpha \in R \setminus \{0\} \),
(iv) \( P^{\alpha |t|}_A(\alpha x) = \alpha P^t_A(x) \), for all \( x \in X \), \( t > 0 \) and \( \alpha \in R \setminus \{0\} \),
(v) \( A \) is \( t \)-proximinal (respectively \( t \)-chebyshev) if and only if \( A + y \) is \( t \)-proximinal (respectively \( t \)-chebyshev) for any given \( y \in X \),
(vi) \( A \) is \( t \)-proximal (respectively \( t \)-chebyshev) if and only if \( \alpha A \) is \( |\alpha|t \)-proximinal (respectively \( |\alpha|t \)-chebyshev) for any given \( \alpha \in R \setminus \{0\} \).

Proof. (i) For \( x, y \in X \) and \( t > 0 \),
\[ d(A + y, x + y, t) = \inf\{N((z + y) - (x + y), t) : z \in A\} \]
\[ = \inf\{N(z - x, t) : z \in A\} = d(A, x, t). \]
(ii) On using (i), it follows that, \( y_0 \in P^t_{A + y}(x + y) \) if and only if \( y_0 \in A + y \)
and \( d(A + y, x + y, t) = N(x + y - y_0, t) \) if and only if \( y_0 - y \in A \) and \( d(A, x, t) = N(x - (y_0 - y), t) \) if and only if \( y_0 - y \in P^t_A(x) \), i.e., \( y_0 \in P^t_A(x) + y \).
(iii) We have \(d(\alpha A, \alpha x, t) = \inf \{N(\alpha x - \alpha z, t) : z \in A\}\)
\[= \inf \{N(\alpha (x - z), t) : z \in A\}\]
\[= \inf \{N(x - z, \frac{t}{|\alpha|}) : z \in A\} = d(A, x, \frac{t}{|\alpha|}).\]

(iv) On using (iii), it follows that \(y_0 \in P^t_{\alpha A}(\alpha x)\) if and only if \(y_0 \in \alpha A\) and \(d(\alpha A, \alpha x, |\alpha|t) = N(\alpha x - y_0, |\alpha|t)\) if and only if \(\frac{y_0}{\alpha} \in A\) and \(N(x - \frac{y_0}{\alpha}, t) = d(A, x, t)\). However, this is equivalent to \(\frac{y_0}{\alpha} \in P^t_A(x)\); i.e., \(y_0 \in \alpha P^t_A(x)\).

(v) The proof of (v) is an immediate consequence of (ii).

(vi) The proof of (vi) follows from (iv). \(\square\)

**Corollary 16.** Let \(M\) be a nonempty subspace of \(X\) then
(i) \(d(M, x + y, t) = d(M, x, t)\), for all \(t > 0\), \(x \in X\) and \(y \in M\),
(ii) \(P^t_M(x + y) = P^t_M(x) + y\), for all \(t > 0\), \(x \in X\) and \(y \in M\),
(iii) \(d(M, \alpha x, |\alpha|t) = d(M, x, t)\), for all \(t > 0\), \(x \in X\) and \(\alpha \in R\{0\}\),
(iv) \(P^{|\alpha|t}_M(\alpha x) = \alpha P^t_M(x)\), for all \(t > 0\), \(x \in X\) and \(\alpha \in R\{0\}\).

**Proof.** The proof of (i) and (ii) follows from theorem 15(i) and 15(ii) and the fact that if \(M\) is a subspace and \(y \in M\) then \(M + y = M\).

The proof of (iii) and (iv) follows from theorem 15(iii) and 15(iv) and the fact that if \(M\) is a subspace and \(\alpha \neq 0\) then \(\alpha M = M\). \(\square\)

**Definition 17.** For \(x \in X\), \(0 < r < 1\), \(t > 0\),
\[S[x, r, t] = \{y \in X : N(x - y, t) = r\}\] and \(e^t_A(x) = d(A, x, t)\).

**Theorem 18.** Let \((X, N)\) be a fuzzy anti-normed space, \(A \subset X\), \(x \in X\setminus \overline{A}\) and \(t > 0\) then we have
\[P^t_A(x) = A \cap B[x, e^t_A(x), t] = A \cap S[x, e^t_A(x), t].\]  \(\text{(1)}\)

**Proof.** The inclusions;
\[P^t_A(x) \subseteq A \cap S[x, e^t_A(x), t] \subseteq A \cap B[x, e^t_A(x), t].\]  \(\text{(2)}\)
are obvious by the definitions of \(P^t_A(x)\) and \(e^t_A(x)\).

Conversely, let \(y \in A \cap B[x, e^t_A(x), t]\), then we have \(y \in A\) and \(N(y - x, t) \leq e^t_A(x) = d(A, x, t) \leq N(y - x, t)\).

Therefore \(y \in A\) and \(N(y - x, t) = d(A, x, t)\), which implies that \(y \in P^t_A(x)\).

So, \(A \cap B[x, e^t_A(x), t] \subset P^t_A(x)\). Hence by (2) we have (1) which completes the proof. \(\square\)
Remark 19. Let \((X, N)\) be a fuzzy anti-normed linear space and \(A\) is a nonempty subset of \(X\), \(x \in X\setminus A\) and \(t > 0\) then we have
\[ A \cap B(x, e^t_A(x), t) = \emptyset, \tag{3} \]
because, if \(y_0 \in A \cap B(x, e^t_A(x), t)\) then \(d(A, x, t) \leq N(x - y_0, t) < d(A, x, t)\) which is impossible.

Corollary 20. Let \((X, N)\) be a fuzzy anti-normed space and \(A\) is a nonempty subset of \(X\), \(x \in X\setminus A\) with \(P^t_A(x) \neq \emptyset\) and \(0 < r < 1\) such that,
\[ \emptyset \neq A \cap B(x, r, t) \subseteq S[x, r, t] \tag{4} \]
then we have \(r = e^t_A(x)\), and we can write \(A \cap B(x, r, t) = P^t_A(x)\).

Proof. If \(r < e^t_A(x)\) then by the definition of \(e^t_A(x)\) we have \(A \cap B[x, r, t] = \emptyset\), which contradicts (4). If \(r = e^t_A(x)\), since \(P^t_A(x) \neq \emptyset\), then by (1) we have \(\emptyset \neq P^t_A(x) = A \cap B[x, e^t_A(x), t] \subseteq A \cap B(x, r, t)\), which contradicts (4), and this completes the proof. \(\square\)

Definition 21. Let \((X, N)\) be a fuzzy anti-normed space, \(0 < r < 1\) and \(t > 0\). We shall say that a set \(A \subset X\) supports the cell \(B[x, r, t]\), or that \(A\) is a support set of the cell \(B[x, r, t]\), if we have \(d(A, B[x, r, t], t) = 1\) and \(A \cap B(x, r, t) = \emptyset\).

Theorem 22. Let \((X, N)\) be a fuzzy anti-normed space and \(A\) is a nonempty subset of \(X\), \(x \in X\setminus A\), \(a_0 \in A\) and \(t > 0\). We have \(a_0 \in P^t_A(x)\) if and only if the set \(A\) supports the cell \(B = B[x, N(a_0 - x, t), t]\).

Proof. Assume that \(a_0 \in P^t_A(x)\). Hence \(N(a_0 - x, t) = d(A, x, t)\). Then by (3), we have \(A \cap B(x, N(a_0 - x, t), t) = \emptyset\), on the other hand, since \(a_0 \in A \cap B[x, N(a_0 - x, t), t]\), we have \(d(A, B, t) = 1\). Consequently, the set \(A\) supports the cell \(B\). Conversely, suppose \(a_0 \notin P^t_A(x)\), hence \(N(a_0 - x, t) > d(A, x, t)\), and let \(0 < \varepsilon < 1\) such that \(N(a_0 - x, t) > d(A, x, t) + \varepsilon\). Then there exists an \(a \in A\) such that \(N(a_0 - x, t) > d(A, x, t) + \varepsilon > N(a - x, t)\), hence \(a \in B(x, N(a_0 - x, t), t)\). Consequently, \(A\) does not support the cell \(B\). \(\square\)

Remark 23. We recall that a set \(A\) in a topological space \(\tau\) is said to be countably compact, if every countable open cover of \(A\) has a finite subcover, or, which is equivalent, if for every decreasing sequence \(A_1 \supset A_2 \supset \ldots\) of non-void closed subset of \(A\) we have \(\bigcap_{n=1}^{\infty} A_n \neq \emptyset\).
**Theorem 24.** Let \((X, N)\) be a fuzzy anti-normed space, \(\tau\) be an arbitrary topology on \(X\) and \(t > 0\). If \(A\) is a nonempty subset of \(X\) such that for \(A \cap B[x, r, t]\) is \(\tau\)-countably compact, then \(A\) is \(t\)-proximinal.

**Proof.** For all \(n \in \mathbb{N}, 0 < 1 - d(A, x, t) + \frac{d(A, x, t)}{n+1} < 1\). put

\[
A^t_n = A \cap B \left[ x, 1 - d(A, x, t) + \frac{d(A, x, t)}{n+1}, t \right], \quad (n = 1, 2, \ldots).
\]

Since for every \(n \in \mathbb{N}, d(A, x, t) \left(1 - \frac{1}{n+1}\right) > d(A, x, t)\), obviously \(A^t_1 \supset A^t_2 \supset \ldots\) and each \(A^t_n \neq \emptyset\). Hence there exists \(a^t_n \in A\) such that

\[
d(A, x, t) \left(1 - \frac{1}{n+1}\right) > N(a^t_n - x, t).
\]

It follows that \(a^t_n \in A^t_n\). Now, since each \(A^t_n\) is \(\tau\)-countably compact and \(\tau\)-closed, we conclude that there exists an \(a_0 \in \bigcap_{n=1}^{\infty} A^t_n\). Then we have

\[
d(A, x, t) \leq N(a_0 - x, t) \leq d(A, x, t) \left(1 - \frac{1}{n+1}\right), \quad (n = 1, 2, \ldots),
\]

hence \(a_0 \in P^t_A(x)\) which completes the proof. 

**Definition 25.** Let \((X, N)\) be a fuzzy anti-normed space and \(A\) is a nonempty subset of \(X\). An element \(y_0 \in A\) is said to be an \(F\)-best approximation of \(x \in X\) from \(A\) if it is a \(t\)-best approximation of \(x\) from \(A\), for every \(t > 0\), i.e.,

\[
y_0 \in \bigcap_{t \in (0, \infty)} P^t_A(x).
\]

The set of all elements of \(F\)-best approximations of \(x \in X\) from \(A\) is denoted by \(FP_A(x)\), i.e.,

\[
FP_A(x) = \bigcap_{t \in (0, \infty)} P^t_A(x).
\]

If each \(x \in X\) has at least (respectively exactly) one \(F\)-best approximation in \(A\) then \(A\) is called a \(F\)-proximinal (respectively \(F\)-chebyshev) set.

**Example 26.** Let \(X = \mathbb{R}\). Define \(N : X \times \mathbb{R} \longrightarrow [0, 1]\) by

\[
N(x, t) = \frac{\|x\|}{t + \|x\|}, \quad \text{if}\ t > 0, \ t \in \mathbb{R}, \ x \in X
\]
Then \((X, N)\) is a fuzzy anti-normed space. Let \(A = [0, 1]\). Then, for every \(x < 1\), we have 1 is an \(F\)-best approximation of \(x\) from \(A\) and for every \(x > 0\), we have 0 is an \(F\)-best approximation of \(x\) from \(A\). So \(A\) is an \(F\)-proximinal set.

**Remark 27.** For an arbitrary set \(A \subset X\) we shall denote by \(\partial A\) the boundary of \(A\), and by \(\mathcal{M}_A\) the set of all elements of the \(F\)-best approximation of the elements \(x \in X\) from \(A\). i.e.,

\[
\mathcal{M}_A = \bigcup_{x \in X} F\mathcal{P}_A(x).
\]

**Theorem 28.** Let \((X, N)\) be a fuzzy anti-normed space, \(A\) is a nonempty subset of \(X\) and \(A\) be a \(F\)-best proximinal set in \(X\) then \(\partial A \subset \overline{\mathcal{M}_A}\).

**Proof.** If \(\partial A = \emptyset\), the proof is obvious. If \(\partial A \neq \emptyset\), let \(a_0 \in \partial A\), \(0 < \varepsilon < 1\) and \(t > 0\) be arbitrary. Then there exists \(0 < \varepsilon' < 1\) such that \(\varepsilon' < \varepsilon\) and the cell \(B(a_0, \varepsilon', \frac{t}{2})\) contains at least one element \(x \in X\setminus A\). Let \(\pi_A(x) \in F\mathcal{P}_A(x)\) (it exists, since by hypothesis, \(A\) is \(F\)-proximinal). Then we have,

\[
N(a_0 - \pi_A(x), t) \leq \max \left\{ N(a_0 - x, \frac{t}{2}), N(x - \pi_A(x), \frac{t}{2}) \right\} \\
= \max \left\{ N(a_0 - x, \frac{t}{2}), N(A - x, \frac{t}{2}) \right\} \\
\leq \max \left\{ N(a_0 - x, \frac{t}{2}), N(a_0 - x, \frac{t}{2}) \right\} \\
\leq \max \{\varepsilon', \varepsilon'\} = \varepsilon' \\
< \varepsilon
\]

So, \(B(a_0, \varepsilon, t) \cap \mathcal{M}_A \neq \emptyset\) and since \(\varepsilon > 0\) is arbitrary, we obtain \(a_0 \in \overline{\mathcal{M}_A}\) which completes the proof. \(\Box\)

**Acknowledgments**

The author is grateful to the referee for careful reading of the article and suggested improvements.

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