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A COMMON FIXED POINT THEOREM FOR TWO PAIRS OF (IT)-COMMUITING ON CONE METRIC SPACES

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Abstract: In this paper, we prove existence of coincidence points and a common fixed point theorem for two pairs of (IT)-Commuting in cone metric spaces. These result extends and improve several well known comparable results in the existing literature.

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1. Introduction and Preliminaries

In 2007 Huang and Zhang [3] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3,4] and the references mentioned therein). Recently, Abbas and Jungck [1] have obtained coincidence points and common fixed point theorems for two mappings in cone metric spaces. The purpus of this paper is to extend and improves the fixed point theorem of Abbas and Jungck [1].

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Throughout this paper, E is a real Banach space, $\mathbb{N} = 1, 2, 3, \ldots$ the set of all natural numbers. For the mappings $f, g : X \to X$, let $C(f, g)$ denotes set of coincidence points of f, g, i.e. $C(f, g) := \{z \in X : fz = gz\}$.

We recall some definitions of cone metric spaces and some of their properties (see [3]).

Definition 1.1. Let E be a real Banach Space and P be a subset in E. The set P is called *cone* if and only if:

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;

(c) $x \in P$ and $x \in P$ implies $x = 0$.

Definition 1.2. Let P be a cone in a Banach space E, define *partial ordering* "$\leq$" on E with respect to P by $x \leq y$ if and only if $y - x \in P$.

We shall write $x < y$ to indicate $x \leq y$, but $x \neq y$, while $x \ll y$ will stand for $y - x \in IntP$ (here $IntP$ denotes the interior of the set P). This $ConeP$ is called *order cone*.

Definition 1.3. Let E be a Banach space and $P \subset E$ be an order cone. The order cone P is called *normal* if there exists $L > 0$ such that for all $x, y \in E$

$$0 \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number L satisfying the above inequality is called the *normal constant* of P.

Definition 1.4. Let X be a nonempty set of E. Suppose that the map $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called *cone metric* on X and (X, d) is called a *cone metric space.*

It is obvious that the cone metric spaces generalize metric spaces.

Example 1.1. (see [3]) Let $E = \mathbb{R}^2$,

$$P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2,$$

$X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. ?

Definition 1.5. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(a) *Cauchy sequence*, if for every c in E with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(b) *convergent sequence*, if for any $0 \ll c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed $x \in X$.

A cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

Lemma 1.1. (see [3]) *Let (X,d) be a cone metric space, and let P be a normal cone with normal constant L. Let $\{x_n\}$ be a sequence in X. Then:*

(i) $\{x_n\}$ converges to x, if and only if $d(x_n, x) \to 0$ $(n \to \infty)$;

(ii) $\{x_n\}$ is a cauchy sequence, if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

Definition 1.6. (see [5]) Let $f, g : X \to X$. Then the pair (f, g) is said to be *(IT)-commuting* at z if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

2. Common Fixed Point Theorem

In this section we obtain existence of coincidence points and a common fixed point theorem for two pairs of (IT)-Commuting defined on a cone metric space.

The following theorem is extends and improves Theorem 2.1 from [1].

Theorem 2.2. *Let (X, d) be a cone metric space and P be a normal cone with normal constant L. Suppose that the mappings S, T, I and J are four selfmaps on X such that $T(X) \subset I(X)$, $S(X) \subset J(X)$, and $T(X) = S(X)$ and satisfy the condition*

$$d(Sx, Ty) \leq kd(Ix, Jy) \text{ for all } x, y \in X, \tag{2.1}$$

where $k \in [0, 1)$ is a constant.

If $S(X) = T(X)$ is a complete subspace of X, then $\{S, I\}$ and $\{T, J\}$ have a coincidence point in X. Moreover, if $\{S, I\}$ and $\{T, J\}$ are (IT)-commuting then, S, T, I and J have a unique common fixed point.

Proof. For any arbitrary point x_0 in X, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Jx_{2n+1} \text{ and } y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}, \text{ for all } n = 0, 1, 2, \ldots$$

By (2.1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) =& d(Sx_{2n}, Tx_{2n+1}) \\ \leq& kd(Ix_{2n}, Jx_{2n+1}) \\ \leq& kd(y_{2n-1}, y_{2n}). \end{aligned}$$

Similarly, it can be show that

$$d(y_{2n+1}, y_{2n+2}) \leq kd(y_{2n}, y_{2n+1}).$$

Therefore, for all n,

$$d(y_{n+1}, y_{n+2}) \leq kd(y_n, y_{n+1}) \leq \cdots \leq k^{n+1} d(y_0, y_1).$$

Now, for any $m > n$,

$$\begin{aligned} d(y_n, y_m) \leq& d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ \leq& \left[k^n + k^{n+1} + \cdots + k^{m-1}\right] d(y_1, y_0) \\ \leq& \frac{k^n}{1-k} d(y_1, y_0). \end{aligned}$$

From (1.3), we have

$$kn \left\| d(y_n, y_m) \right\| \leq \frac{k^n}{1-k} L \left\| d(y_1, y_0) \right\|.$$

Which implies that $d(y_n, y_m) \to 0$ as $n, m \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence.

Let us suppose that $S(X)$ is complete subspace of X. Completeness of $S(X)$ implies the existence of $z \in S(X)$ such that $\lim_{n\to\infty} y_{2n} = Sx_{2n} = z$,

$$\lim_{n\to\infty} Jx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ix_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z. \tag{2.2}$$

That is, for any $0 \ll c$, for sufficiently large n, we have $d(y_n y) \ll c$.

Since $z \in T(X) \subset I(X)$, then there exists a point $u \in X$ such that $z = Iu$.

We will prove that $z = Su$. By the triangle inequality, we have:

$$\begin{aligned} d(Su, z) \leq& d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z) \\ \leq& kd(Iu, Jx_{2n+1}) + d(Tx_{2n+1}, z). \end{aligned}$$

Letting $n \to \infty$, we get

$$d(Su, z) \leq kd(z, z) + d(z, z)$$

$$\leq k(0) + 0 = 0.$$

Hence $Su = z$.
Therefore $z = Su = Iu$, and

$$u \text{ is a coincidence point of } S \text{ and } I. \tag{2.3}$$

Since, $z = S(X)J(X)$, there exists a point vsuch that $z = Jv$. We will show that $Tv = z$. We have

$$\begin{aligned} d(Tv, z) \leq & d(Su, Tv) \\ \leq & kd(Iu, Jv) \\ \leq & kd(z, z) \\ = & 0. \end{aligned}$$

Hence $Tv = z$, and $z = Tv = Jv$, that is

$$v \text{ is a coincidence point of Tand } J. \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$Su = Iu = Tv = Jv (= z). \tag{2.5}$$

Since (S, I) and (T, J) are (IT)commuting, we have

$$\begin{aligned} d(SSu, Su) = & d(SSu, Iu) \\ = & d(SSu, Tv) \\ \leq & kd(ISu, Jv) \\ = & kd(SIu, Su) \\ = & kd(SSu, Su) \\ < & d(SSu, Su) \quad (\text{since } k < 1) \end{aligned}$$

A contradiction, therefore $SSu = Su(= z)$,

$$Su = SSu = SIu = ISu.$$

That is $SSu = ISu = Su(= z)$.
Therefore

$$Su = z \text{ is a common fixed point of } S \text{ and } I. \tag{2.6}$$

Similarly, $Tv = TTv = TJv = JTv$ implies $TTv = JTv = Tv(= z)$. Therefore

$$Tv(= z) \text{ is a common fixed point of } T \text{ and } J. \tag{2.7}$$

In view of (2.6) and (2.7) it follows S, T, I and J have a common fixed point, namely z.

Uniqueness. Let z_1 be another common fixed point of S, T, I and J. Then

$$\begin{aligned} d(z, z1) &= d(Sz, Tz) \\ &\le kd(Iz, Jz1) \\ &\le kd(z, z1) \\ &< d(z, z1) \quad (\text{since } k < 1). \end{aligned}$$

This contradiction implies $z = z1$.
Therefore S, T, I and J have a unique common fixed point. □

Remark 2.1. If $S = T$ and $I = J$, then the theorem reduces to the Theorem 2.1 of Abbas and Jungck [1], with $I(X)$ complete, which is an improvement of Theorem 2.1 in [1]. Since in this paper $J(X)$ is complete which is a super space of $I(X)$.

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