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OPTIMAL PARAMETERS FOR DAMPED SINE-GORDON EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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Abstract: In this paper we study an identification problem for physical parameters associated with damped sine-Gordon equation with Neumann boundary conditions. The existence, uniqueness, and continuous dependence of weak solution of sine-Gordon equations are established. The method of transposition is used to prove the Gâteaux differentiability of the solution map. The Gâteaux differential of the solution map is characterized. The optimal parameters are established. Computational algorithm and numerical results are presented.

AMS Subject Classification: 35B30, 49J50
Key Words: identification problem, Neumann boundary conditions, Gâteaux differentiability, optimal parameters

1. Introduction

Let Ω be an open bounded set of $\mathbb{R}^n$ with C^1 boundary. Let us consider the following sine-Gordon equation with Neumann boundary condition.

$$\begin{aligned}
&u_{tt}(t,x) + \alpha u_t(t,x) - \beta \triangle u(t,x) + \delta \sin u(x,t) = f(x,t);\ (t,x) \in Q,\\
&\frac{\partial u}{\partial n}(t,x)|_{x\in\Gamma} = 0,\ t \in\ (0,T),\\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega,
\end{aligned} \tag{1}$$

where $T > 0,\ Q = (0,T) \times \Omega,\ f \in L^2(Q),\ u_0 \in V = H^1(\Omega)$ and $u_1 \in H = L^2(\Omega)$.

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Equation (1) describes the dynamics of a Josephson junction driven by current source taking into account the damping effect [3]. The sine-Gordon model has attracted an additional interest because it is known to exhibit chaotic behavior for certain values of governing parameters α, β and δ. For details see [7] and [8]. For the sine-Gordon model with Dirichlet boundary condition, Gutman and Ha estimated the governing parameters. For details see [5]. In this paper we estimate the governing parameters of the sine-Gordon equation for Neumann boundary condition such that the solution of (1) exhibits the desired behavior.

More precisely, let

$$\mathcal{P} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]\}, \tag{2}$$

where $\beta_{min} > 0$. Define the cost functional $J(q)$ by

$$J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|^2_{L^2(0,T;H)} \tag{3}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \geq 0$ for $i = 1, 2$ with $k_1 + k_2 > 0$. The data z_d^1 and z_d^2 can be thought of as the targeted behavior of (1). The parameter identification problem for (1) with the objective function (3) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}_{ad}$ satisfying

$$J(q^*) = \inf_{q \in \mathcal{P}_{ad}} J(q). \tag{4}$$

In this paper we proved existence and uniqueness of the weak solution of damped sine-Gordon equation with Neumann boundary condition. Weak Gâteaux differentiability of the solution is established by using the method of transposition by Lions and Magenes [8]. Weak Gâteaux differentiability of the solution map is used to establish the Gâteaux differentiability of the cost functional J. An adjoint system is established and used to represent the Gâteaux derivative of the cost functional J. We proved that the partial derivatives $\frac{\partial J}{\partial \alpha}$, $\frac{\partial J}{\partial \beta}$, and $\frac{\partial J}{\partial \delta}$ are 0 when optimal parameter $q^* \in int\mathcal{P}$. Continuity of partial derivatives with respect to α, β, and δ is used to prove differentiability of cost functional J on the admissible set of parameters $\mathcal{P}_{ad}$. In addition, we developed a computational algorithm for approximate solutions of the adjoint system. A Fourier series method is used to compute numerical solution of the sine-Gordon equation (1). MATLAB function *fminicon* is used for the minimization of the cost functional J. Our experiments showed that the identification algorithm is successful for data without noise, whereas the precision of identification decreases with the increasing noise level. In addition, our experiments revealed that for

$\epsilon = 0$, identification algorithm is successful for any k_1. For $\epsilon = 0.001$, the best identification is achieved for $k_1 = 1$, and for $\epsilon = 0.01$, the best identification is achieved for $k_1 = 2$. This paper is organized as follows. In section 2 we introduce appropriate function spaces with their respective inner products and norms. In addition, we show the existence of eigenvalues and eigenfunctions of the operator $-\beta\Delta + I$. In general, equation (1) does not have a classical solution. To overcome such a problem, we define weak solution of (1). In addition, we prove existence, uniqueness, and continuity of weak solutions of (1). In Chapter 3 we show that the weak solution of (1), as a function of q, is weakly Gâteaux differentiable by using the method of transposition by Lions and Magenes [8]. In Chapter 4 we show that the cost functional (3) is Gâteaux differentiable on $\mathcal{P}$. We derive the optimal parameters and finally we show that the cost functional (3) is differentiable. In Chapter 5 we develop a computational algorithm. In Chapter 6 we present numerical results.

2. Existence, Uniqueness, and Continuity of Weak Solution

Let $H = L^2(\Omega)$ be a Hilbert space with following inner product and norm

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx, \qquad |\phi| = (\phi, \phi)^{\frac{1}{2}} \tag{5}$$

for all ϕ , $\psi \in L^2(\Omega)$. Let $V = H^1(\Omega)$ be a Hilbert space with following inner product and norm

$$((\phi, \psi)) = (\phi, \psi) + (\nabla\phi, \nabla\psi), \qquad \|\phi\| = ((\phi, \phi))^{\frac{1}{2}} \tag{6}$$

for all ϕ , $\psi \in H^1(\Omega)$. The dual H' is identified with H leading to $V \subset H \subset V'$ with compact, continuous, and dense injections [9]. Hence there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1\|w\| \quad \text{for any} \quad w \in V. \tag{7}$$

Let $\langle u, v\rangle_{V,\, V'}$ denote the duality pairing between V and V'. To use the variational formulation let us define the following bilinear form on $V \times V$

$$a_\beta(u, v) = \int_\Omega u\ v dx + \beta \int_\Omega \nabla u \nabla v dx \tag{8}$$

for any $u, v \in H^1(\Omega)$ and diffusion coefficient β. For $\beta > 0$, $a_\beta(u, v)$ is bounded and coercive in V. Define a linear operator $A_\beta : D(A_\beta) = \{u : u \in V, A_\beta u \in H\}$

into H by $a_\beta(u,v) = (A_\beta u, v)$ for all $u \in D(A_\beta)$ and for all $v \in V$. Let the norm on $D(A_\beta)$ be $\|u\|_\beta^2 = \int_\Omega |u|^2 dx + \beta \int_\Omega |\nabla u|^2 dx$. Then A_β is an isomorphism between $D(A_\beta)$ and H. The operator $A_\beta : D(A_\beta) \subset H$ into H is self-adjoint and bounded so A_β^{-1} exists. In addition, A_β^{-1} is bounded, self-adjoint compact operator. Let $\{\lambda_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ respectively be the nonzero eigenvalues and eigenfunctions for the operator $-\Delta + I$ defined in V such that $\{w_k\}_{k=1}^\infty$ forms an orthonormal basis in H. Then functions $\{\frac{w_k}{\sqrt{\mu_k}}\}_{k=1}^\infty$ form an orthonormal basis in V.

From now on the dependency on x is suppressed, and $'$ and $''$ stand for the time derivatives. Let

$$W(0,T) = \{u \ : \ u \in L^2(0,T;V), u' \in L^2(0,T;H), u'' \in L^2(0,T;V')\} \tag{9}$$

in the sense of distributions with the values in V'. The weak solution of (1) is a function $u \in W(0,T)$ satisfying

$$\begin{aligned} &\langle u'', w_j\rangle + \alpha(u', w_j) + a_\beta(u, w_j) + \delta(\sin(u), w_j) = (f, w_j) + (u, w_j), \ \forall j \in \mathbb{N}, \\ &u(0) = u_0 \in V, \quad u'(0) = u_1 \in H, \end{aligned} \tag{10}$$

where the equations in t are satisfied in the distributional sense. Since the span $\{w_1, w_2, w_3, ...\}$ is dense in V, (10) is satisfied for any $v \in V$

$$\langle u'' + \alpha u' + A_\beta u + \delta \sin u, v\rangle = \langle f + u, v\rangle, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H. \tag{11}$$

Thus

$$u'' + \alpha u' + A_\beta u + \delta \sin u = f + u, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H \tag{12}$$

which is understood in the sense of distributions on $(0,T)$ with the values in V'. For more details see [4].

The following two Lemmas are of critical importance for the existence and uniqueness of weak solutions.

Lemma 1. *Let $w \in L^2(0,T;V)$, $w' \in L^2(0,T;H)$ and $w'' + A_\beta w \in L^2(0,T;H)$. Then, after a modification on the set of measure zero, $w \in C([0,T];V)$, $w' \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$ one has*

$$(w'' + A_\beta w, w') = \frac{1}{2}\frac{d}{dt}\{|w'|^2 + a_\beta(w,w)\}. \tag{13}$$

For proof see [4].

Lemma 2. *(Gronwall's Lemma) Let $\xi(t)$ be a nonnegative, summable function on [0,T] which satisfies the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2 \;\; for\ constants\ C_1\ , C_2 \geq 0 \tag{14}$$

almost everywhere $t \in$ [0,T]. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}) \ a.e.\ on\ 0 \leq t \leq T. \tag{15}$$

In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds \;\; a.e.\ on\ 0 \leq t \leq T,\ then\ \ \xi(t) \ = \ 0 \ \ a.e.\ on\ [0,\ T] \tag{16}$$

For proof see [11].

Lemma 3. *The solution of equation (12) is unique.*

Proof. Let z_1 and z_2 be two solutions of (12). Then $w = (z_2 - z_1)$ satisfies

$$w'' + \alpha w' + A_\beta w + \delta(\sin z_2 - \sin z_1) = w, \quad w(0) = 0 \in V, \quad w'(0) = 0 \in H, \tag{17}$$

By lemma(1) and (2) $|w'(t)|^2 + \|w(t)\|^2 = 0$. Therefore $w = 0$ a.e. in $W(0,T)$ Hence $z_1 = z_2$ a.e. in $W(0,T)$.

□

Fix $m \in \mathbb{N}$ and let $V_m = span\{w_1, w_2, ..., w_m\}$. Let $P_m : H \to V_m$ be the projection operator defined by $P_m v = \sum_{k=1}^{m}(v, w_k)w_k$ for any $v \in H$.

The approximate solution of (10) is a function $u_m(t) \in W(0,T)$ that satisfies

$$\begin{aligned} u''_m + \alpha u'_m + A_\beta u_m + \delta P_m \sin(u_m) &= P_m f + u_m \\ u_m(0) = P_m u_0 \quad u'_m(0) &= P_m u_1. \end{aligned} \tag{18}$$

Lemma 4. *The solution of equation* (18) *is unique.*

Proof. Assume z_1 and z_2 be two solutions of (18). Then their difference $w = z_1 - z_2$ satisfies

$$w'' + A_\beta(w) = w - \alpha w' - \delta P_m((\sin(z_2) - (\sin(z_1)) \in L^2(0,T;H) \tag{19}$$

with zero initial conditions. The fact $|P_m u| \leq |u|$ for any $u \in H$ and lemma (3) provides the result. □

Let

$$z_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j(x) \tag{20}$$

satisfy

$$\begin{aligned}\frac{d^2}{dt^2}(z_m, w_j) + \alpha\frac{d}{dt}(z_m, w_j) + a_\beta(z_m, w_j) + \delta(P_m \sin z_m, w_j),\\ = (P_m f, w_j) + (z_m, w_j)\\ z_m(0) = P_m z_0 \quad \text{and} \quad \frac{d}{dt}z_m(0) = P_m z_1 \quad \text{for} \quad \text{any} \quad j \in \mathbb{N}.\end{aligned} \tag{21}$$

Theorem 5. *For each integer $m = 1, 2, ...,$ there exist a unique function $z_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j(x)$ satisfying (21).*

Proof. Let $P_m : H \to V_m$ be the projection operator defined by $P_m v = \sum_{k=1}^{m}(v, w_k)w_k$ for any $v \in H$. We can write equation (21) as the vector differential equation

$$\frac{d^2}{dt^2}\vec{g}_m(t) + \alpha\frac{d}{dt}\vec{g}_m(t) + \beta\Lambda\vec{g}_m(t) = \vec{F}(t, \vec{z}_m) \tag{22}$$

with the initial values

$$\vec{g}_m(0) = \vec{u}_1,$$
$$\frac{\vec{d}}{dx}g_m(0) = \vec{u}_2$$

where u_1, u_2, $\vec{F}(t, \vec{z}_m)$ and Λ respectively are $n \times 1$ $n \times n$ matrices.

Definition 6. Carathéodory Condition: $\vec{f}(x, \vec{y})$ is continuous as a function of $\vec{y}$ for fixed x and measurable as a function of x for each fixed $\vec{y}$.

Theorem 7. *Let $J = [\xi, \xi + a]$, $S = J \times \mathbb{R}^n$, and assume that the function $\vec{f} : S \to \mathbb{R}^n$ satisfies the Carathéodory condition in S. Let $\vec{f}$ satisfy $\vec{f}(x, \vec{y}) \in L(J)$, the class of functions that are integrable and measurable over J for each fixed $\vec{y}$, and satisfying the generalized Lipschitz condition*

$$|\vec{f}(x, \vec{y}) - \vec{f}(x, \vec{y}_1)| \leq l(x)|\vec{y} - \vec{y}_1| \text{ in } S \tag{23}$$

where $l(x) \in L(J)$. Then there exists a unique solution of $\vec{y}' = \vec{f}(x, \vec{y})$, $\vec{y}(\xi) = \vec{\eta}$ in J. For details see [16].

Hence the system of m second order vector differential equations admits a unique solution $\vec{g}_m(t)$ on $[0, T]$. This is shown by reducing it into a system of first order vector differential equations and by applying Carathéodory type extension Theorem 7.

Lemma 8. *Function* $z_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j(x)$ *satisfies*

$$\frac{d^2}{dt^2}(z_m, w_j) + \alpha\frac{d}{dt}(z_m, w_j) + a_\beta(z_m, w_j) + \delta(P_m \sin z_m, w_j) \\ = (P_m f, w_j) + (z_m, w_j), \\ z_m(0) = P_m z_0 \quad \text{and} \quad \frac{d}{dt}z_m(0) = P_m z_1 \tag{24}$$

for $j > m$.

Proof. It suffices to show that $(A_\beta z_m, w_j) = a_\beta(z_m, w_j) = 0$ for $j > m$. Since $\{w_j\}_{j=1}^{\infty}$ are the eigenfunctions of the operator A_β, we have $(z_m, w_j)+\beta(\nabla z_m, \nabla w_j) = \lambda_j(z_m, w_j)$. This implies $\beta(\nabla z_m, \nabla w_j) = \lambda_j(z_m, w_j)-(z_m, w_j) = (\lambda_j-1)(z_m, w_j)$. For $j > m$, $\beta(\nabla z_m, \nabla w_j) = 0$. Hence, $(A_\beta z_m, w_j) = 0$ for $j > m$.

□

Hence z_m is a weak solution of the sine-Gordon equation. Furthermore, z_m also satisfies (18). By Lemma 4 the approximate solution u_m is in fact a weak solution of ($ode4$)

□

Theorem 9. *Let* $q = (\alpha, \beta, \delta) \in \mathcal{P}$, $u_0 \in V$, $u_1 \in H$ *and* $f \in L^2(0, T; H)$. *Then*

(i). There exists a unique weak solution $u(t; q)$ *of (1). This solution satisfies* $u \in C([0, T]; V) \cap W(0, T)$, $u' \in C([0, T]; H)$, *and*

$$\max_{0\le t\le T}(\|u(t)\|^2 + |u'(t)|^2) + \|u''(t)\|^2_{L^2(0,T;V')} \le C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right], \tag{25}$$

where C *is a constant independent of* $q \in \mathcal{P}$. *The approximate solutions* $u_m(t; q)$ *also satisfy the energy estimate (25) with the same constant* C.

(ii). The solution $u(t; q)$ *and its approximations* $u_m(t; q)$ *satisfy the following convergence estimate*

$$|u'(t) - u'_m(t)|^2 + \|u(t) - u_m(t)\|^2 \le C_2(|u_1 - P_m u_1|^2 + \|u_0 - P_m u_0\|^2 \\ +\|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |\sin u(s; q) - P_m \sin u(s; q)|^2 ds) \tag{26}$$

where C_2 *is a constant independent of* $q \in \mathcal{P}$.

(iii). Furthermore, $u_m \to u$ *in* $C([0, T]; V)$ *and* $u'_m \to u'$ *in* $C([0, T]; H)$ *as* $m \to \infty$.

For proof see [18]

Theorem 10. *Let $q \in \mathcal{P}$. Then the solution maps $q \to u(q)$ from $\mathcal{P}$ into $C([0,T];V)$ and $q \to u'(q)$ from $\mathcal{P}$ into $C([0,T];H)$ are continuous.*

For proof see [18].

3. Weak Gâteaux Differentiability of the Solution Map

Let

$$\mathcal{H} = \left\{ G = \begin{pmatrix} \xi \\ g \end{pmatrix} : \ \xi \in H \quad \text{and} \quad g \in L^2(0,T;H) \right\}. \tag{27}$$

Then H is a Hilbert space with the following inner product and the norm

$$(G_1, G_2)_{\mathcal{H}} = (\xi_1, \xi_2)_H + (g_1, g_2)_{L^2(0,T;H)}, \quad \|G\|_{\mathcal{H}} = (G,G)_{\mathcal{H}}^{\frac{1}{2}}, \tag{28}$$

where $G_1 = \begin{pmatrix} \xi_1 \\ g_1 \end{pmatrix} \in \mathcal{H}$ and $G_2 = \begin{pmatrix} \xi_2 \\ g_2 \end{pmatrix} \in \mathcal{H}$.
To show the weak Gâteaux differentiability of $J(q)$ at $q^* \in \mathcal{P}$ we have to estimate the quotient

$$z_\lambda = \frac{u(q_\lambda) - u(q^*)}{\lambda}, \tag{29}$$

where $q_\lambda = q^* + \lambda(q - q^*)$, $\lambda \in (0,1]$. Generally it is desirable to estimate z_λ in the solution space $W(0,T)$. Since the second order evolution equations for z_λ in (50) have the forcing term containing a diffusion operator, it is not easy or impossible to solve equation (50) by standard variational manner as in [7]. Hence we will restrict ourselves to an estimate of $\begin{pmatrix} z_\lambda(T) \\ z_\lambda(t) \end{pmatrix} \in H \times L^2(0,T;H)$ as $\lambda \to 0$ based on the method of transposition presented in [8].

Now we show the Gâteaux differentiability of the solution map

$$q \to \begin{pmatrix} u(q;T) \\ u(q;t) \end{pmatrix}$$

of $\mathcal{P}$ into $H \times L^2(0,T;H)$ via the method of transposition and characterize its Gâteaux derivative.

Fix $q = (\alpha, \beta, \delta) \in \mathcal{P}$ and $h \in L^2(0,T;H)$. Let $G = \begin{pmatrix} \xi \\ g \end{pmatrix} \in \mathcal{H}$.
Let us consider the following linear terminal value problem

$$\phi'' - \alpha\phi' + A_\beta \phi + (\delta h - 1)\phi = g \quad \text{in} \quad (0,T)$$

$$\phi(T) = 0, \quad \phi'(T) = \xi. \tag{30}$$

Let $\phi(T-s,x) = w(s,x)$ for any $x \in (0,1)$. Then we have $\phi_t(T-s,x) = -w_s(s,x)$ and $\phi_{tt}(T-s,x) = w_{ss}(s,x)$. Then (30) can be written as

$$\begin{aligned} & w'' + \alpha w' + A_\beta w + (\delta h - 1)w = g \quad \text{in} \quad (0,T) \\ & w(0) = 0, \quad w'(0) = -\xi. \end{aligned} \tag{31}$$

Arguing as in section 2, we can conclude that (31) has a unique weak solution. Hence (30) has a unique weak solution $\phi = \phi(\xi, g) \in W(0,T)$ that satisfies the energy estimate

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \le c(|\xi|^2 + \|g\|^2_{L^2(0,T;H)}), \quad t \in [0,T]. \tag{32}$$

Definition 11. Solution map: Given $G \in \mathcal{H}$ define the solution map from $\mathcal{H}$ into $W(0,T)$ by $\tau(G) = \phi$, where ϕ is the weak solution of (30).

Definition 12. Fix $q = (\alpha, \beta, \delta) \in \mathcal{P}$ and $h \in L^2(0,T;H)$. Let the solution space $\mathcal{X}(q;h) = \tau(\mathcal{H})$ be defined by

$$\mathcal{X}(q,h) = \{\phi : \phi \text{ is solution of (30) for each G} \in \mathcal{H}\}.$$

Let the linear operator $\mathcal{L}(q;h)$ from $\mathcal{X}(q;h)$ into $\mathcal{H}$ be defined by

$$\mathcal{L}(q;h)\phi = \begin{pmatrix} \phi'(T) \\ \phi'' - \alpha\phi' + A_\beta\phi + (\delta h - 1)\phi. \end{pmatrix} = \begin{pmatrix} \phi'(T) \\ g \end{pmatrix}. \tag{33}$$

Let the inner product $(.\ ,.)$ in $\mathcal{X}(q;h)$ be defined by

$$(\phi, \psi)_{\mathcal{X}(q;h)} = (\mathcal{L}(q;h)\phi, \mathcal{L}(q;h)\psi)_{\mathcal{H}}. \tag{34}$$

In terms of the operator $\mathcal{L}(q;h)$ the energy estimate (32) can be written as

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \le c(\|\mathcal{L}(q;h)\phi\|^2_{\mathcal{H}}) = c\|\phi\|^2_{\mathcal{X}(q,h)}. \tag{35}$$

Definition 13. Given $q \in \mathcal{P}$, $h \in L^2(0,T;H)$, and $f \in L^2(0,T;V')$, the element $\bar{z} = \begin{pmatrix} z_1 \\ z \end{pmatrix} \in \mathcal{H}$, $z_1 \in H$, $z \in L^2(0,T;H)$ is called a weakened solution of the problem

$$\begin{aligned} & z''(t) + \alpha z'(t) + A_\beta z(t) + (\delta h(t) - 1)z(t) = f(t) \\ & z(0) = 0\ ,\ z'(0) = 0, \quad t \in (0,T), \end{aligned} \tag{36}$$

if

$$(\bar{z}, \mathcal{L}(q;h)\phi)_{\mathcal{H}} = \int_0^T \langle f(t), \phi(t)\rangle dt \tag{37}$$

for any $\phi \in \mathcal{X}(q;h)$. That is,

$$(z_1, \xi)_H + \int_0^T (z(t), g(t))dt = \int_0^T \langle f(t), \phi(t)\rangle dt \tag{38}$$

for all $\phi \in \mathcal{X}(q,h)$.

Remark 14. If $f \in L^2(0,T;H)$ and $z(t)$ is the weak solution (in the sense of section 2) of the problem (36), then the integration by parts shows that $\bar{z} = \begin{pmatrix} z'(T) \\ z(t) \end{pmatrix}$ also is its weakened solution.

Lemma 15. *If $f \in L^2(0,T;V')$, then there exists a unique weakened solution of the problem* (36).

Proof. By the method of transposition of Lions for details see [6], if F is a bounded linear functional on $\mathcal{X}(q;h)$, then there exists a unique $\bar{\xi} \in \mathcal{H}$ such that

$$F(\phi) = (\bar{\xi}(t), \mathcal{L}(q;h)(\phi)(t))_{\mathcal{H}} \quad \text{for} \quad \text{any} \quad \phi \in \mathcal{X}(q;h). \tag{39}$$

Let

$$F(\phi) = \int_0^T \langle f(t), \phi(t)\rangle dt, \;\; \phi \in \mathcal{X}(q,h).$$

Using the energy estimate (35) we get

$$\begin{aligned} |F(\phi)| &\leq \|f\|_{L^2(0,T:V')}\|\phi\|_{L^2(0,T;V)} = \|f\|_{L^2(0,T:V')}\sqrt{\int_0^T \|\phi(t)\|_V^2 dt} \\ &\leq \|f\|_{L^2(0,T:V')}\sqrt{c}\int_0^T \|\phi(t)\|^2_{\mathcal{X}(q,h)} dt \\ &\leq \sqrt{cT}\|f\|_{L^2(0,T:V')}\|\phi(t)\|_{\mathcal{X}(q,h)} \end{aligned} \tag{40}$$

and the result follows. □

Let $\hat{u}$ and $\hat{v}$ be two measurable functions on Ω. Define function $B(\hat{u},\hat{v})(x)$ for $x \in \Omega$ by

$$B(\hat{u},\hat{v})(x) = \begin{cases} \frac{\sin(\hat{u}(x)) - \sin(\hat{v}(x))}{\hat{u}(x)-\hat{v}(x)}, & \hat{u}(x) \neq \hat{v}(x), \\ \cos(\hat{v}(x)), & \hat{u}(x) = \hat{v}(x). \end{cases} \tag{41}$$

Then B is an integrable function on Ω with $|B(\hat{u},\hat{v})(x)| \le 1$ for any $x \in \Omega$.

If $\hat{u}_1 = \hat{u}$ a.e. on Ω , and $\hat{v}_1 = \hat{v}$ a.e. on Ω, then $B(\hat{u}_1,\hat{v}_1) = B(\hat{u},\hat{v})$ a.e. on Ω. Thus $B(u,v) : H \times H \to H$ is well defined by (41).

Furthermore, the inequality

$$|\cos(b) - \frac{\sin(a) - \sin(b)}{a-b}| \le |a-b| \tag{42}$$

for $a, b \in \mathbb{R}$, $a \ne b$ implies that

$$|\cos(b) - B(u,v)|_H \le |u-v|_H \tag{43}$$

for any $u, v \in H$.

Definition 16. Let $q, q^* \in \mathcal{P}$. Let $q_\lambda = q^* + \lambda(q - q^*)$ for $\lambda \in (0,1]$. The solution map $q \to \bar{u}(q) = \begin{pmatrix} u'(T;q) \\ u(t;q) \end{pmatrix}$ of $\mathcal{P}$ into $\mathcal{H}$ is said to be weakly Gateaux differentiable at q^* in the direction $q - q^*$ if there exist $\bar{z} \in \mathcal{H}$ such that

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}(\bar{u}(q_\lambda) - \bar{u}(q^*), \bar{v})_{\mathcal{H}} = (\bar{z}, \bar{v})_{\mathcal{H}} \tag{44}$$

for any $\bar{v} \in \mathcal{H}$.

Theorem 17. *Let $q = (\alpha, \beta, \delta), q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}$. Then the weak Gâteaux derivative $\bar{z} \in \mathcal{H}$ at $q^* \in \mathcal{P}$ in the direction $q - q^*$ is the unique weakened solution of the problem*

$$\begin{aligned} &z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + (\delta^* \cos\, u(t;q^*) - 1) z(t) = f_0(t), \\ &z(0) = 0,\ z'(0) = 0,\ t \in (0,T), \end{aligned} \tag{45}$$

where $f_0(t) = (\alpha^* - \alpha) u'(t;q^*) + (A_{\beta^*} - A_\beta) u(t;q^*) + (\delta^* - \delta)\sin(u(t;q^*))$.

Remark 18. For $\mathcal{X}$ and $\mathcal{L}$ defined by (34) and (33) respectively with q^* and $h = \cos(u(q^*))$ the solution $\bar{z} = \begin{pmatrix} z(T) \\ z(t) \end{pmatrix}$ satisfies

$$(\bar{z}(t), \mathcal{L}(q^*; \cos u(t;q^*)\phi(t))_{\mathcal{H}} = \int_0^T \langle f_0(t), \phi(t) \rangle dt \tag{46}$$

for any $\phi \in \mathcal{X}(q^*; \cos(u(q^*)))$.

Proof. Let $q_\lambda = q^* + \lambda(q - q^*) = (\alpha_\lambda, \beta_\lambda, \delta_\lambda)$ and denote $A_\lambda = A_{\beta_\lambda}$. Then $A_0 = A_{\beta^*}$. By (12) functions $u(q_\lambda)$ and $u(q^*)$ are the weak solutions of the equations

$$u''(q_\lambda) + \alpha_\lambda u'(q_\lambda) + A_\lambda u(q_\lambda) + \delta_\lambda \sin(u(q_\lambda)) = f + u(q_\lambda)$$

$$u_\lambda(0,q) = u_0,\ u'_\lambda(0;q) = u_1 \tag{47}$$

and

$$\begin{aligned} &u''(q^*) + \alpha^* u'(q^*) + A_{\beta^*} u(q^*) + \delta^* \sin(u(q^*)) = f + u(q^*) \\ &u(0,q^*) = u_0,\ u'(0;q^*) = u_1 \end{aligned} \tag{48}$$

correspondingly.

Then the quotient $z_\lambda = (u(q_\lambda) - u(q^*))/\lambda$ satisfies

$$\begin{aligned} &z''_\lambda + \alpha^* z'_\lambda + A_{\beta^*} z_\lambda + \delta^* \frac{\sin(u(q_\lambda)) - \sin(u(q^*))}{\lambda} - z_\lambda \\ &= (\alpha^* - \alpha) u'(q_\lambda) + (A_{\beta^*} - A_\beta) u(q_\lambda) + (\delta^* - \delta) \sin(u(q_\lambda)), \\ &z_\lambda(0) = 0, \quad z'_\lambda(0) = 0. \end{aligned} \tag{49}$$

Let

$$f_\lambda(t) = (\alpha^* - \alpha) u'(t;q_\lambda) + (A_{\beta^*} - A_\beta) u(t;q_\lambda) + (\delta^* - \delta) \sin(u(t;q_\lambda)).$$

Using the notation (41) we let $B_\lambda(t) = B(u(t;q_\lambda), u(t;q^*)) \in H$ for $0 \le t \le T$.

Then

$$\begin{aligned} &z''_\lambda + \alpha^* z'_\lambda + A_{\beta^*} z_\lambda + (\delta^* B_\lambda(t) - 1) z_\lambda = f_\lambda, \\ &z_\lambda(0) = 0, \quad z'_\lambda(0) = 0. \end{aligned} \tag{50}$$

Since H is continuously embedded in V' there exists a constant $K_2 = K_2(\Omega)$ such that $\|v\|_{V'} \le K_2 |v|$ for any $v \in H$. Therefore one can estimate

$$\|f_\lambda(t)\|_{V'} \le K_2 (|\alpha^* - \alpha| |u'(t;q_\lambda)| + 2\mu K_1 \|u(t;q_\lambda)\| + K_1 |\delta^* - \delta| \|u(t;q_\lambda)\|). \tag{51}$$

Now the energy estimate (25) shows that there exists $C_2 > 0$ independent of $q \in \mathcal{P}$ such that

$$\|f_\lambda\|_{L^2(0,T;V')} \le C_2 \tag{52}$$

for all $\lambda \in (0,1]$.

Since z_λ is a weak solution of (50) it is also its weakened solution, i.e.

$$(\bar{z}_\lambda, \mathcal{L}(q^*;B_\lambda)\phi)_{\mathcal{H}} = \int_0^T \langle f_\lambda(t), \phi(t) \rangle dt \tag{53}$$

for any $\phi \in \mathcal{X}(q^*;B_\lambda)$.

Since $\bar{z}_\lambda \in \mathcal{H}$ and $\mathcal{L}(q^*;B_\lambda)$ from $\mathcal{X}(q^*;B_\lambda) \to \mathcal{H}$ is surjective, there exists $\phi_\lambda \in \mathcal{X}(q^*;B_\lambda)$ such that $\mathcal{L}(q^*;B_\lambda)\phi_\lambda = \bar{z}_\lambda$.

For such a function ϕ_λ one gets from (53)

$$\|\bar{z}_\lambda\|_{\mathcal{H}}^2 \leq \|f_\lambda\|_{L^2(0,T;V')}\|\phi_\lambda\|_{L^2(0,T;V)}. \tag{54}$$

This inequality and estimates (35) and (52) give

$$\|\bar{z}_\lambda\|_{\mathcal{H}}^2 \leq C_2\|\bar{z}_\lambda\|_{\mathcal{H}}.$$

Thus $\|\bar{z}_\lambda\|_{\mathcal{H}} \leq C_2$ for some constant C_2 independent of $\lambda \in (0,1]$. Here we used the fact that $|B_\lambda(t)| \leq 1$ for any t, λ and $q, q^* \in \mathcal{P}$. Therefore one can extract a subsequence $\bar{z}_{\lambda_k},\ \lambda_k \to 0+$, such that $\bar{z}_{\lambda_k} \rightharpoonup \bar{z}$ weakly in $\mathcal{H}$. Now we would like to pass to the limit in (53) as $\lambda_k \to 0$ to obtain (58). However, the domains of the operators $\mathcal{L}(q^*; B_\lambda)$ depend on λ, so one has to proceed differently. Let

$$f_0(t) = (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*) + (\delta^* - \delta)\sin u(t;q^*). \tag{55}$$

From Lemma **??** we get $u(q_\lambda) \to u(q^*)$ in $L^2(0,T;V)$, and $u'(q_\lambda) \to u'(q^*)$ in $L^2(0,T;H)$. Therefore $f_\lambda \rightharpoonup f_0$ weakly in $L^2(0,T;V')$. In fact, Theorem 10 shows that this is a strong convergence. Thus $\|f_0\|_{L^2(0,T;V')} \leq C_2$.

Write $\mathcal{L}_0 = \mathcal{L}(q^*; \cos\, u(q^*))$ and $\mathcal{L}_k = \mathcal{L}(q^*; B_{\lambda_k})$ to simplify the notation. Let $\phi \in \mathcal{X}(q^*; \cos\, u(q^*))$. Then $\mathcal{L}_0\phi \in \mathcal{H}$. Therefore

$$(\bar{z}_{\lambda_k}, \mathcal{L}_0\phi(t))_{\mathcal{H}} \to (\bar{z}(t), \mathcal{L}_0\phi(t))_{\mathcal{H}}, \quad \text{and}$$

$$\int_0^T \langle f_{\lambda_k}(t), \phi(t)\rangle dt \to \int_0^T \langle f_0(t), \phi(t)\rangle dt \tag{56}$$

as $\lambda_k \to 0+$.

On the other hand,

$$\begin{aligned}
(\bar{z}_{\lambda_k}(t), \mathcal{L}_0\phi(t))_{\mathcal{H}} &= (z_{1\lambda_k}, \xi)_H + \int_0^T (z''_{\lambda_k}(t) + \alpha^* z'_{\lambda_k}(t) + A_{\beta^*} z_{\lambda_k}(t), \phi(t))dt \\
&+ \int_0^T (\delta^* \cos\, u(t;q^*) - 1)z_{\lambda_k}(t), \phi(t))dt \\
&= \int_0^T (z''_{\lambda_k}(t) + \alpha^* z'_{\lambda_k}(t) + A_{\beta^*} z_{\lambda_k}(t), \phi(t))dt \\
&+ (z_{1\lambda_k}, \xi)_H + \int_0^T ((\delta^* B_{\lambda_k}(t) - 1)z_{\lambda_k}(t), \phi(t))dt \\
&+ \delta^* \int_0^T ((\cos\, u(t;q^*) - B_{\lambda_k}(t)))z_{\lambda_k}(t), \phi(t))dt \\
&= (z_{1\lambda_k}, \xi)_H + \int_0^T \langle f_{\lambda_k}(t), \phi(t)\rangle dt
\end{aligned}$$

$$+\delta^* \int_0^T ((\cos\ u(t;q^*) - B_{\lambda_k}(t))z_{\lambda_k}(t), \phi(t))dt. \tag{57}$$

Using $\|\bar{z}_\lambda\|_{\mathcal{H}} \le C_2, \phi \in W(0,T)$ and the estimate (43), the last term in (57) can be estimated by $c\|u(q_{\lambda_k}) - u(q^*)\|_{L^2(0,T;H)}\|\phi\|_{L^\infty(0,T;H)}$. Since the mapping $q \to u(q)$ from $\mathcal{P}$ into $L^2(0,T;H)$ is continuous, then the last term of (57) tends to 0 as $\lambda_k \to 0+$.

Now we can pass to the limit as $\lambda_k \to 0+$ in (57), and conclude that

$$(\bar{z}, \mathcal{L}(q^*; \cos\ u(t;q^*))\phi)_{\mathcal{H}} = \int_0^T \langle f_0, \phi(t)\rangle dt \tag{58}$$

for any $\phi \in \mathcal{X}(q^*; \cos\ u(q^*))$. Since $\|f_0\|_{L^2(0,T;V')} \le C_2$, Lemma (15) shows that that $\bar{z}$ is the unique weakened solution of (45). Hence $\bar{z}_\lambda \rightharpoonup \bar{z}$ as $\lambda \to 0+$ weakly in $\mathcal{H}$ by Definition 16. This proves that the $\bar{z}$ is the weak Gâteaux derivative of the map $q \to \bar{u}(q)$. □

4. Optimal Parameters

From Theorem 17 the map $q \to \bar{u}(q)$ is weakly Gâteaux differentiable at $q = q^* \in \mathcal{P}$ in any direction of $q - q^*$, and its weak Gâteaux derivative $\bar{z}(t,x) = D\bar{u}(q^*; q-q^*)(t,x)$ can be described by (46).

Let us consider the functional

$$J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|^2_{L^2(0,T;H)} \tag{59}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \ge 0$ for $i = 1,2$ with $k_1 + k_2 > 0$.

Lemma 19. *$J(q)$ is Gâteaux differentiable, and its Gâteaux derivative is given by*

$$DJ(q^*; q-q^*) = 2k_1((u(q^*;T) - z_d^1), z_1) + 2k_2 \int_0^T (u(q^*;t) - z_d^2), z)dt, \tag{60}$$

where $\bar{z}$ is the solution of integral equation (46).

Proof. In the previous section we have shown that the weak solution $u(q;t)$ is weakly Gâteaux differentiable in the admissible set of parameters $\mathcal{P}$. Hence the following limits exist

$$\lim_{\lambda\to 0+} \left(\frac{u(q^* + \lambda(q-q^*);T) - u(q^*;T)}{\lambda}, v_1\right)_H = (z_1, v_1) \tag{61}$$

for any $v_1 \in H$ and

$$\lim_{\lambda \to 0+} \left(\frac{u(q^* + \lambda(q - q^*); t) - u(q^*; t)}{\lambda}, v_2 \right)_{L^2(0,T;H)} = (z, v_2) \tag{62}$$

for any $v_2 \in L^2(0, T; H)$.

To show that the cost functional $J(q)$ is Gâteaux differentiable at q^*, it suffices to show that the following limit exists

$$\lim_{\lambda \to 0+} \left(\frac{J(q^* + \lambda(q - q^*)) - J(q^*)}{\lambda} \right) = DJ(q^*; q - q^*). \tag{63}$$

Evaluating the limit in (63)

$$\begin{aligned}
&\lim_{\lambda \to 0+} \left(\frac{J(q^* + \lambda(q - q^*)) - J(q^*)}{\lambda} \right) \\
&= k_1 \lim_{\lambda \to 0+} \frac{1}{\lambda} \big([(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^* + \lambda(q - q^*); T) - z_d^1) \\
&-(u(q^*; T) - z_d^1, u(q^*; T) - z_d^1)] \\
&+k_2 \lim_{\lambda \to 0+} \frac{1}{\lambda} [(u(q^* + \lambda(q - q^*); t) - z_d^2, u(q^* + \lambda(q - q^*); t) - z_d^2)_{L^2(0,T;H)} \\
&-(u(q^*; t) - z_d^2, u(q^*; t) - z_d^2)_{L^2(0,T;H)}].
\end{aligned} \tag{64}$$

Consider the first part of the limit from (64)

$$\begin{aligned}
&k_1 \lim_{\lambda \to 0+} \frac{1}{\lambda} [(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^* + \lambda(q - q^*); T) - z_d^1) \\
&-(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^*; T) - z_d^1) \\
&+(u(q^* + \lambda(q - q^*); T) - z_d^1, u(q^*; T) - z_d^1) - (u(q^*; T) - z_d^1, u(q^*; T) - z_d^1)] \\
&k_1 \lim_{\lambda \to 0+} \frac{1}{\lambda} [(u(q^* + \lambda(q - q^*); T) - z_d^1 - u(q^*; T) \\
&+z_d^1, u(q^* + \lambda(q - q^*); T) - z_d^1) \\
&+(u(q^*; T) - z_d^1), u(q^* + \lambda(q - q^*); T) - z_d^1 - u(q^*; T) + z_d^1)] \\
&= 2k_1 (u(q^*; T) - z_d^1, z_1).
\end{aligned} \tag{65}$$

Similarly,

$$\begin{aligned}
&k_2 \lim_{\lambda \to 0+} \frac{1}{\lambda} [(u(q^* + \lambda(q - q^*); t) - z_d^2, u(q^* + \lambda(q - q^*); t) - z_d^2)_{L^2(0,T;H)} \\
&-(u(q^*; t) - z_d^2, u(q^*; t) - z_d^2)_{L^2(0,T;H)}] \\
&= 2k_2 (u(q^*; t) - z_d^2, z)_{L^2(0,T;H)}.
\end{aligned} \tag{66}$$

Using (65) and (66) we get

$$DJ(q^*;q-q^*) = 2k_1((u(q^*;T)-z_d^1),z_1) + 2k_2\int_0^T (u(q^*;t)-z_d^2),z)dt. \quad (67)$$

□

Since $\mathcal{P} = \{q=(\alpha,\beta,\delta)\in[\alpha_{min},\alpha_{max}]\times[\beta_{min},\beta_{max}]\times[\delta_{min},\delta_{max}]\}$ is a closed and convex subset of $\mathbb{R}^3$, then we have the following optimality condition

$$2k_1((u(q^*;T)-z_d^1),z_1) + 2k_2\int_0^T (u(q^*;t)-z_d^2),z)dt \geq 0 \quad \text{for} \quad q\in\mathcal{P}, \quad (68)$$

where $\begin{pmatrix} z_1 \\ z \end{pmatrix}$ is a solution of the integral equation (46).

Let us introduce the adjoint state p defined to be the weak solution of the following adjoint system

$$\begin{aligned} &p'' - \alpha^* p' + A^*_\beta p + (\delta^*\cos(u(q^*)-1))p = k_2(u(q*;t)-z_d^2) \\ &p(T) = 0 \quad p'(T) = k_1(u(q^*;T)-z_d^1). \end{aligned} \quad (69)$$

System (69) can be written as

$$\begin{aligned} &\mathcal{L}(q^*;\cos(u(q^*))p(q^*) = \begin{pmatrix} k_1u(q^*;T)-z_d^1 \\ k_2u(q^*;t)-z_d^2 \end{pmatrix} \in \mathcal{H} \\ &p(T)=0, \quad p'(T) = k_1(u(q^*;T)-z_d^1). \end{aligned} \quad (70)$$

Since $k_2(u(q^*;t)-z_d^2)\in L^2(0,T;H)$, as shown in section 2 problem in (69) has a unique weak solution. Using $p(q^*)$ in place of ϕ in (46) equation (60) can be written as

$$\begin{aligned} DJ(q^*;q-q^*) = 2\int_0^T &\langle(\alpha^*-\alpha)u'(t;q^*) + (A_{\beta^*}-A_\beta)u(t;q^*) \\ &+(\delta^*-\delta)\sin u(t;q^*),p(q^*)\rangle dt. \end{aligned} \quad (71)$$

Thus we obtain the following result.

Theorem 20. *The Gâteaux derivative of the objective function $J(q)$ has the following representation*

$$DJ(q^*;q-q^*) = (\alpha^*-\alpha)a(q^*) + (\beta^*-\beta)b(q^*) + (\delta^*-\delta)c(q^*)dt, \quad (72)$$

where

$$a = -\frac{\partial J}{\partial\alpha} = -2\int_0^T (u_t(t,x;q^*),p(t,x;q^*))dt, \quad (73)$$

$$c = -\frac{\partial J}{\partial \delta} = -2\int_0^T (\sin(u(t,x;q^*)), p(t,x;q^*))dt, \tag{74}$$

and

$$b = -\frac{\partial J}{\partial \beta} = -2\int_0^T (\nabla u(t,x), \nabla p(t,x))dt, \tag{75}$$

The optimality condition $DJ(q^*; q - q^*) \geq 0$ for any $q \in \mathcal{P}$ is

$$(\alpha^* - \alpha)a(q^*) + (\beta^* - \beta)b(q^*) + (\delta^* - \delta)c(q^*) \geq 0 \tag{76}$$

for any $(\alpha, \beta, \delta) \in P$.

In addition, the optimal coefficient $q^* \in \mathcal{P}$ for nonzero (a, b, c) can be compactly written as

$$\alpha^* = \frac{1}{2}\{\text{sign}(a) + 1\}\alpha_{max} - \frac{1}{2}\{\text{sign}(a) - 1\}\alpha_{min}, \tag{77}$$

$$\beta^* = \frac{1}{2}\{\text{sign}(b) + 1\}\beta_{max} - \frac{1}{2}\{\text{sign}(b) - 1\}\beta_{min}, \tag{78}$$

and

$$\delta^* = \frac{1}{2}\{\text{sign}(c) + 1\}\delta_{max} - \frac{1}{2}\{\text{sign}(c) - 1\}\delta_{min}, \tag{79}$$

for more detail see [5].

Now we have the following Theorem

Theorem 21. *If the optimal coefficient q^* is located in the interior $int\,\mathcal{P}$ of the admissible set $\mathcal{P}$, then*

$$a = 0, \quad b = 0, \quad \text{and} \quad c = 0 \quad \text{in} \quad \Omega.$$

Proof. In the interior of $\mathcal{P}$, $\frac{\partial J}{\partial \alpha} = \frac{\partial J}{\partial \beta} = \frac{\partial J}{\partial \delta} = 0$. Thus $a = b = c = 0$. □

Theorem 22. *Consider the sine-Gordon equation (1) with a constant diffusion coefficient β. Let the admissible set be*

$$P = [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]$$

with $\beta_{min} > 0$.

Let the objective function be defined by

$$J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|^2_{L^2(0,T;H)}.$$

Then the mapping $q \to J(q)$ from $int\,\mathcal{P} \subset \mathbb{R}^3$ into $\mathbb{R}$ is differentiable. Its gradient $\nabla J(q) = (a, b, c)$, where a, b, c are defined in (80),(82), and (81). If the parameter $q^* \in int\,\mathcal{P}$ is optimal, then $\nabla J(q^*) = 0$.

Proof. To show that the mapping $q \to J(q)$ from $int\,\mathcal{P} \subset \mathbb{R}^3$ into $\mathbb{R}$ is differentiable it suffices to show that $\nabla J(q) = (a, b, c)$ is continuous in $\mathcal{P}$ where

$$a = -\frac{\partial J}{\partial \alpha} = -2\int_0^T (u_t(t, x; q^*), p(t, x; q^*))dt, \tag{80}$$

$$c = -\frac{\partial J}{\partial \delta} = -2\int_0^T (\sin(u(t, x; q^*)), p(t, x; q^*))dt, \tag{81}$$

and

$$b = -\frac{\partial J}{\partial \beta} = -2\int_0^T (\nabla u(t, x), \nabla p(t, x))dt, \tag{82}$$

Arguing as in section 2, we can conclude that (69) has a unique weak solution $p \in W(0, T)$. Suppose $h(q^*) = \delta^* \cos(u(q^*)) - 1$ and $g(q^*) = k_2(u(q^*; t) - z_d^2)$. From Theorem 10 the mappings $q^* \to u(q^*)$, $q^* \to h(q^*)$, and $q^* \to g(q^*)$ from $\mathcal{P}$ into $C([0, T]); V)$ are continuous. Similarly the mapping $q^* \to u'(q^*)$ from $\mathcal{P}$ into $C([0, T]); H)$ is continuous. Continuity of $q^* \to p(q^*)$ $\mathcal{P}$ into $C([0, T]); V)$ and $q^* \to p'(q^*)$ $\mathcal{P}$ into $C([0, T]); H)$ can be proved similarly as in Theorem 10. Thus partial derivatives a, b, c defined in (80),(82), and (81) are continuous. Hence by [17] the mapping $q \to J(q)$ from $int\,\mathcal{P} \subset \mathbb{R}^3$ into $\mathbb{R}$ is differentiable. $\square$

5. Computational Algorithm

As mentioned in section 2, let $\left\{w_j\right\}_{j=1}^{\infty}$ be eigenfunctions of $-\beta\Delta + I$ that form an orthonormal basis in H. Then $\{\frac{w_j}{\sqrt{\lambda_j}}\}_{j=1}^{\infty}$ is an orthonormal basis on V as in section 2 . Fix $N \in \mathbb{N}$. Let $V_N = span\{w_1, w_2, ..., w_N\}$. Let $P_N : H \to V_N$ be the projection operator defined by $P_N v = \sum_{j=1}^{N}(v, w_j)w_j$ for any $v \in H$. As defined in section 2, the approximate solution of (10) is

$$u_N(t, x) = \sum_{j=1}^{N} g_{jN}(t)w_j(x) \tag{83}$$

that satisfies

$$\frac{d^2}{dt^2}(u_N, w_j) + \alpha\frac{d}{dt}(u_N, w_j) + a_\beta(u_N, w_j) + \delta(\sin(u_N), w_j) = (f, w_j) + (u, w_j)$$
$$u_N(0) = P_N u_0 \quad \text{and} \quad \frac{d}{dt}u_N(0) = P_N u_1 \quad \text{for} \quad \text{any} \quad j \in \mathbb{N}. \tag{84}$$

Let $\bar{g}_N = \{g_{jN}\}_{j=1}^N \in \mathbb{R}^N$. We can rewrite (84) as the following vector differential equation

$$\bar{g}_N''(t) + \alpha\bar{g}_N'(t) + \beta\Lambda\bar{g}_N(t) = \bar{F}(t, \bar{g}_N) \tag{85}$$

with the initial data $\vec{g}_N(0)$ and $\vec{g}_N'(0)$.

Let $\bar{Z}_1(t) = \bar{g}_N(t)$ and $\bar{Z}_2(t) = \bar{g}_N'(t)$. Then the initial value problem (85) can be reduced into the following system of first order ODEs

$$\begin{aligned} &\bar{Z}_1'(t) = \bar{Z}_2(t) \\ &\bar{Z}_2'(t) = -\alpha\bar{Z}_2(t) - \beta\Lambda\bar{Z}_1(t) + \bar{F}(t, \bar{u}_N) \\ &\bar{Z}_1(0) = \bar{g}_N(0), \quad \bar{Z}_2(0) = \bar{g}_N'(0). \end{aligned} \tag{86}$$

The approximate solution of (12) is

$$u_N(t, x) = \sum_{j=1}^{N} g_{jN}(t)\sqrt{2}\cos((j-1)\pi x). \tag{87}$$

Now we compute the approximate solution of the adjoint system

$$\begin{aligned} &p'' - \alpha^* p' + A_\beta^* p + (\delta^* \cos(u(q^*) - 1))p = k_2(u(q^*; t) - z_d^2) \\ &p(T) = 0, \quad p'(T) = k_1(u(q^*; T) - z_d^1). \end{aligned} \tag{88}$$

Let $p(T-s, x) = w(s, x)$ for any $x \in (0, 1)$, then we have $p_t(T-s, x) = -w_s(s, x)$ and $p_{tt}(T-s, x) = w_{ss}(s, x)$. The adjoint system (88) can be written as

$$\begin{aligned} &w'' + \alpha w' + A_\beta w + (\delta\cos(u(q) - 1))w = k_2(u(q; t) - z_d^2) \\ &w(0, x) = 0 \quad w'(0, x) = k_1(u(q^*; T) - z_d^1). \end{aligned} \tag{89}$$

The approximate solution of the adjoint system (89) is given by

$$\begin{aligned} &\langle y_N'', w_k\rangle + \alpha(y_N', w_k) + (A_\beta y_N, w_k) + \delta(P_N\cos(u_N(q))y_N, w_k) \\ &= (k_2 P_N(u_N(q; t) - z_d^2), w_k) + (y_N, w_k) \\ &y_N(0) = Q_N 0, \quad y_N'(0) = P_N k_1(u(q^*; T) - z_d^1) \end{aligned} \tag{90}$$

where $y_N = \sum_{j=1}^N h_j(t)w_j(x)$.

Equation (90) is equivalent to the following vector differential equation

$$\bar{h}_N''(s) + \alpha\bar{h}_N'(s) + \beta\Lambda\bar{h}_N(s) = \bar{H}(s, \bar{h}_N) \tag{91}$$

with the initial data $\vec{h}_N(0)$ and $\vec{h}_N'(0)$.

6. Numerical Results

For our numerical experiments we choose to use a Fourier series method for the solution of the sine-Gordon equation (1), and MATLAB function *fminicon* for the minimization of the cost functional. As described in section 2 eigenfunctions of the operator A_β, $w_j = \cos(\pi(j-1)x), j = 1, 2, ...,$ are chosen as an orthonormal basis in H. As described in section 5, let $P_N : H \to V_N$ be the projection operator defined from H onto $V_N = \text{span}\{w_1, w_2, ..., w_N\}$. Expanding the functions in (21) into the Fourier cosine series we have

$$\begin{aligned} &g_k'' + \alpha g_k' + \beta_k g_k + \delta S_k = F_k \\ &g_k(0) = P_N u_0, \quad g_k'(0) = P_N u_1, \end{aligned} \tag{92}$$

where $\beta_k = \beta[1 + (\pi(k-1))^2]$, $g_k(t), F_k(t), P_N u_0$ and $P_k u_1$ are the Fourier coefficients of the solution $u_N(t)$ in (21). Similarly $S_k(t)$ is the Fourier cosine coefficient of $P_N \sin(u_N)(t)$. The cost functional $J_N(q)$ can be written as

$$J_N(q) = k_2 \sum_{i=1}^{M} \sum_{k=1}^{N} [Y_k(q; t_i) - Z(t_i)]^2 + k_1 \sum_{k=1}^{N} [Y_k(q; T) - Z(T)]^2, \tag{93}$$

where $k_1 + k_2 > 0$ and $Z(t_i)$ for $i = 1, 2, ..., T$ are observations for the parameter set $\bar{q} = (\bar{\alpha}, \bar{\beta}, \bar{\delta})$.

In all the numerical experiments we used observation times $t_j = T.j/K$ where $j = 0, 1, 2, ..., K$ and $T = 4$. The model values are specified in the following table

Time and spatial intervals	$[0, T] \times [0, 1] = [0, 4] \times [0, 1]$
Admissible set	$\mathcal{P}_{ad} = [0.1, 1] \times [0.1, 1] \times [0, 2]$
Initial conditions	$u_0(x) = \sin(\pi x), \quad u_1(x) = x$
Forcing function	$f(t, x) = 1$
Dimension of system of ODE = N	64
Number of Partitions in [0,4] = M	64
Number of Partitions in [0,1] = K	128

Table 1: Parameter values for numerical simulations

To simulate the data $z_d^1(T, x)$ and $z_d^2(t, x)$, let $\bar{q} = (.2, .2, .3) \in \mathcal{P}_{ad}$ be the set of test parameters. Numerical solution of (1) is computed by using 4th order Runge-Kutta method. Since real data always contain some noise, we set

$$z_d(t, x) = u(\bar{q}; t, x) + \epsilon\gamma(x), \tag{94}$$

where ϵ is noise level and $\gamma(x)$ is a random variable uniformly distributed on interval [-.5,.5].

Let $q_0 \in \mathcal{P}_{ad}$ be an arbitrary chosen set of parameters. A MATLAB function called *fminicon* is used for minimization of the cost functional J_N. The minimizers q_N^*, minimum values of functional $J_N(q_N^*)$, and error

$$E = \frac{\|q^* - \bar{q}\|_{\mathbb{R}^3}}{\|\bar{q}\|_{\mathbb{R}^3}}$$

at different noise levels ϵ are given in the following tables. The first row of each table shows that the identification algorithm is successful for data z_d without noise, whereas the precision of the identification decreases with the increasing noise level. Without loss of generalities we can assume that $k_2 = 1$ in all the examples. Our experiments revealed that for $\epsilon = 0$, identification algorithm is successful for any k_1. For $\epsilon = 0.001$, the best identification is achieved for $k_1 = 1$, and for $\epsilon = 0.01$, the best identification is achieved for $k_1 = 2$.

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.1998, 0.1996, 0.3017)	9.7130e-008	0.0041
0.001	(0.1945, 0.1991, 0.2726)	0.0029	0.0679
0.01	(0.2737, 0.2751, 0.1910)	0.3458	0.3674

Table 2: Identification results for $k_1 = 0$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.2001, 0.2001, 0.3000)	1.7996e-007	2.1820e-004
0.001	(0.2056, 0.2040, 0.3031)	0.0155	0.0182
0.01	(0.1218, 0.1470, 0.2870)	1.6254	0.2312

Table 3: Identification results for $k_1 = 1$ and $k_2 = 1$

ϵ	q_N^*	$J_N(q_N^*)$	E
0	(0.2000, 0.2000, 0.3000)	2.2272e-007	7.4062e-005
0.001	(0.2013, 0.2026, 0.2905)	0.1534	0.0242
0.01	(0.1901, 0.1887, 0.3541)	14.0577	0.1362

Table 4: Identification results for $k_1 = 25$ and $k_2 = 1$

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