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DECOMPOSITIONS OF MATRICES OVER A FINITE CHAIN

Yizhi Chen[1 §], Haiqing Wang[2], Hui Luo[3]

Department of Mathematics
Huizhou University
Huizhou, Guangdong, 516007, P.R. CHINA

Abstract: Let $L = \{0, 1, ..., l\}$ denote a finite chain, $M_{m,n}(L)$ be the additive semigroup of all the $m \times n$ matrices over L. In this paper, we firstly give some subdirect decompositions of a finite chain L, and then show that if there is a subdirect embedding from L to the direct product $\prod_{i=1}^{h} L_i$ of subchains $L_1, L_2, \cdots, L_h$, then there will be a corresponding subdirect embedding from the semigroup $M_{m,n}(L)$ to semigroup $\prod_{i=1}^{h} \mathrm{M}_{m,n}(L_i)$. Based on the above results, it is also proved that a matrix $A \in M_{m,n}(L)$ can be decomposed into the sum of matrices over some special subchains of L which generalizes and extends the corresponding results obtained by [1].

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Key Words: decomposition, matrix, finite chain, subdirect product, semigroup, semiring

1. Introduction and Preliminaries

A semiring R is an algebraic structure $(R, +, \cdot)$ consisting of a non-empty set R together with two binary operations $+$ and $\cdot$ on R such that $(R, +)$ and $(R, \cdot)$ are semigroups connected by distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in R$. A semiring R is called a partially ordered semiring if it admits a compatible ordering $\leq$, i.e., $\leq$ is a partial order on R

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§Correspondence author

satisfying the following condition: for any $a, b, c, d \in R$, if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$ and $ac \leq bd$. A partially ordered semiring R is called a totally ordered semiring if the imposed partial order is a total order.

Hereafter, $L = \{0, 1, ..., l\}$ $(l \geq 1)$ denotes a finite chain with usual ordering. For any $a, b \in L$, define the addition and the multiplication on L as follows:

$$a + b = max\{a, b\} \text{ and } ab = min\{a, b\}.$$

Then it is easy to see that $(L, \leq, +, \cdot)$ is a totally ordered semiring.

Let $M_{m,n}(L)$ denote the set of all the $m \times n$ matrices over L. For any $A = (a_{ij}), B = (b_{ij}) \in M_{m,n}(L)$, define $+$ in $M_{m,n}(L)$ by

$$A + B = (a_{ij} + b_{ij}).$$

Then, clearly, $(M_{m,n}(L), +)$ is a semigroup, and we will simply denote it by $M_{m,n}(L)$.

Fuzzy matrices are one class of important matrices, and there are a series of papers in the literature considering fuzzy matrices (for example, see [1]-[13]). From [2], it is known that the decomposition of a fuzzy matrix is an important technique for the study of fuzzy matrices. In [1], Zhao etc. translated the problem of the decomposition of a fuzzy matrix into the corresponding one of the matrix over a finite chain L. It was shown that a matrix over a finite chain L can be decomposed into the sum of some matrices over its special subchains by using some order-preserving semiring homomorphisms.

In this paper, we will study some concrete decompositions of the matrices over a finite chain. We will firstly give some ways to decompose L into the subdirect product of some of its subchains, and then show that if there is a subdirect embedding from L to the direct product $\prod_{i=1}^{h} L_i$ of subchains $L_1, L_2, \cdots, L_h$, then there will be a corresponding subdirect embedding from semigroup $M_{m,n}(L)$ to semigroup $\prod_{i=1}^{h} \mathrm{M}_{m,n}(L_i)$. Based on the above results, it is proved that a matrix $A \in M_{m,n}(L)$ can be decomposed into the sum of matrices over some special subchains of L which generalizes and extends the corresponding results obtained by [1].

For convenience, we will continue to introduce some concepts.

An algebra A is said to be a subdirect product of an indexed family $A_i (i \in I)$of algebras if it satisfies $A \leq \Pi_{i \in I} A_i$ and $A\pi_i = A_i$ for each $i \in I$, where π_i means the projective mapping from A to A_i [13].

If a homomorphism ϕ from A to $\Pi_{i \in I} A_i$ is injective, then it is called a embedding homomorphism. For an embedding homomorphism $\phi : A \to \Pi_{i \in I} A_i$, if $A\phi$ is a subdirect product of the A_i, then ϕ is called subdirect. Also, we will say that A is isomorphic to the subdirect product of $\{A_i\}_{i \in I}$ (or A has a subdirect decomposition of $\{A_i\}_{i\in I}$) . An algebra A is called subdirectly irreducible if for every subdirect embedding $\phi : A \to \Pi_{i\in I}A_i$, there is an $i \in I$ such that $\phi \circ \pi_i : A \to A_i$ is an isomorphism. By this definition, it is not hard to see that any two-element algebra is subdirectly irreducible.

For notations and terminologies occurred but not mentioned in this paper, the readers are referred to [12],[13].

2. Subdirect Decompositions of a Finite Chain

In this section, we will give some subdirect decompositions for a finite chain L.

Theorem 1. *Let L be a finite chain and $L = \bigcup_{i=1}^{h} L_i$, where $L_i \bigcap L_j = \{0\}$ if $i \neq j$. Then L is isomorphic to a subdirect product of $L_i (1 \le i \le h)$.*

Proof. Let L be a finite chain and $L = \bigcup_{i=1}^{h} L_i$, where $L_i \bigcap L_j = \{0\}$ if $i \neq j$. Assume that $L_i = \{0, p_{i_1}, ..., p_{it_i}\}$ $(t_i \ge 1)$ is a subchain of L satisfying $0 = p_{i_0} < p_{i_1} < ... < p_{it_i}$, $t_1 + t_2 + ... + t_h = l$.

For any $a \in L$, define a mapping

$$\phi : L \to \prod_{i=1}^{h} L_i$$

by

$$a \mapsto (a_1, a_2, ..., a_h),$$

where

$$a_i = \begin{cases} p_{it_i}, & a \ge p_{it_i}, \\ p_{it_k}, & a \ge p_{it_k}, a \not\ge p_{i(t_k+1)} (0 \le t_k \le t_i - 1). \end{cases}$$

Then, it is clear that $L\phi \le \prod_{i=1}^{h} L_i$ and $(L\phi)\pi_i = L_i$ for $i = 1, 2..., h$, where π_i is the projective mapping from $L\phi$ to L_i. Thus, $L\phi$ is a subdirect product of $L_i (1 \le i \le h)$.

Note that ϕ is an isomorphism from L to $L\phi$, and then L is isomorphic to a subdirect product of $L_i (i = 1, 2, ..., h)$. □

Example 1. Let $L = \{0, 1, 2, 3, 4, 5\}$ be a finite chain. Take $L_1 = \{0, 1\}$, $L_2 = \{0, 2, 4\}$, $L_3 = \{0, 3, 5\}$, and define

$$\phi : L \to \prod_{i=1}^{3} L_i,$$

$$0 \mapsto (0,0,0),$$
$$1 \mapsto (1,0,0),$$
$$2 \mapsto (1,2,0),$$
$$3 \mapsto (1,2,3),$$
$$4 \mapsto (1,4,3),$$
$$5 \mapsto (1,4,5).$$

Then it is easy to check that L is isomorphic to a subdirect product of L_1, L_2 and L_3.

In Theorem 1, put $L_i = \{0, i\}(i = 1, 2, ..., l)$. Notice that any two-element chain is subdirectly irreducible, then we have

Corollary 1. *Let $L = \{0, 1, ..., l\}$ be a finite chain. Then L is isomorphic to a subdirect product of subdirectly irreducible members $L_i = \{0, i\}(i = 1, 2, ..., l)$.*

Also, in Theorem 1, if we take $h = \lceil \frac{l}{2} \rceil$, which is the least integer greater than or equal to $\frac{l}{2}$, then we also have

Corollary 2. *Let $L = \{0, 1, ..., l\}$ be a finite chain. Then L is isomorphic to a subdirect product of L_i, where $L_i = \{0, 2i-1, 2i\}(i = 1, 2, ..., h-1)$, $L_h = \{0, 2h-1, 2h\}$ when $l = 2h$, $L_h = \{0, 2h-1\}$ when $l = 2h-1$.*

Further, denote $h = \lceil \frac{l}{k-1} \rceil$, which is the least integer greater than or equal to $\frac{l}{k-1}$, we will similarly have

Corollary 3. *Let $L = \{0, 1, ..., l\}$ be a chain. Then L is isomorphic to a subdirect product of L_i, where $L_i = \{0, ..., (k-1)i-1, (k-1)i\}(i = 1, 2, ..., h-1)$, $L_h = \{0, ..., (k-1)h-1, (k-1)h\}$when $h = \lceil \frac{l}{k-1} \rceil = \frac{l}{k-1}$, $L_h = \{0, (k-1)(h-1), ..., l\}$ when $\lceil \frac{l}{k-1} \rceil > \frac{l}{k-1}$.*

Example 2. Let $L = \{0, 1, 2, 3, 4, 5\}$ be a finite chain. Take $L_1 = \{0, 1, 2\}$, $L_2 = \{0, 3, 4\}$, $L_3 = \{0, 5\}$, and define

$$\phi : L \to \prod_{i=1}^{3} L_i,$$

$$0 \mapsto (0,0,0),$$
$$1 \mapsto (1,0,0),$$
$$2 \mapsto (2,0,0),$$

$$3 \mapsto (2,3,0),$$
$$4 \mapsto (2,4,0),$$
$$5 \mapsto (2,4,5).$$

Then we can also check that L is isomorphic to a subdirect product of L_1, L_2 and L_3.

3. The Matrices over a Finite Chain

In this section, by the subdirect decompositions of L obtained in Section 2, we will firstly get some concrete subdirect decompositions of $M_{m,n}(L)$, and then give a more general decomposition of the matrices in $M_{m,n}(L)$ which will generalize and extend the decomposition theorem of a matrix over a finite chain L in [1].

Theorem 2. *If L is a subdirect product of $L_i (i = 1, 2, ..., h)$, then $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)$ $(i = 1, 2, ..., h)$.*

Proof. Assume that L is a subdirect product of L_i $(i = 1, 2, ..., h)$, then we have $L \leq \prod_{i=1}^{h} L_i$, $L\pi_i = L_i (i = 1, 2, ..., h)$. Define a mapping

$$\varphi : M_{m,n}(L) \to \prod_{i=1}^{h} M_{m,n}(L_i)$$

by

$$A = (a_{ij}) \mapsto ((a_{ij}\pi_1), ..., (a_{ij}\pi_h)).$$

In the following, we can show that $M_{m,n}(L) \cong (M_{m,n}(L))\varphi$. To verify this, we only need to prove that φ a monomorphism.

Actually, for any $A = (a_{ij}), B = (b_{ij}) \in M_{m,n}(L)$, we have

$$\begin{aligned}
A\varphi = B\varphi \quad &\Longleftrightarrow \quad (a_{ij})\varphi = (b_{ij})\varphi \\
&\Longleftrightarrow \quad ((a_{ij}\pi_1), ..., (a_{ij}\pi_h)) = ((b_{ij}\pi_1), ..., (b_{ij}\pi_h)) \\
&\Longleftrightarrow \quad (a_{ij}\pi_1) = (b_{ij}\pi_1), ..., (a_{ij}\pi_h) = (b_{ij}\pi_h) \\
&\Longleftrightarrow \quad a_{ij}\pi_1 = b_{ij}\pi_1, ..., a_{ij}\pi_h = b_{ij}\pi_h \\
&\Longleftrightarrow \quad l_1 = l'_1, ..., l_h = l'_h (\text{put} \quad a_{ij} = (l_1, ..., l_h), \\
&\qquad\quad b_{ij} = (l'_1, ..., l'_h) \in \prod_{i=1}^{h} L_i)
\end{aligned}$$

$$\begin{array}{rl} \Longleftrightarrow & a_{ij} = b_{ij} \\ \Longleftrightarrow & A = B. \end{array}$$

Hence, φ is injective.

Secondly, for any $A = (a_{ij}), B = (b_{ij}) \in M_{m,n}(L)$,

$$\begin{array}{rcl} (A+B)\varphi & = & ((a_{ij}) + (b_{ij}))\varphi \\ & = & ((a_{ij} + b_{ij}))\varphi \\ & = & (((a_{ij} + b_{ij})\pi_1), ..., ((a_{ij} + b_{ij})\pi_h)) \\ & = & ((a_{ij}\pi_1 + b_{ij}\pi_1), ..., (a_{ij}\pi_h + b_{ij}\pi_h)) \\ & & (\text{since} \quad \pi_1, ..., \pi_h \quad \text{are homomorphisms}) \\ & = & ((a_{ij}\pi_1) + (b_{ij}\pi_1), ..., (a_{ij}\pi_h) + (b_{ij}\pi_h)) \\ & = & ((a_{ij}\pi_1), ..., (a_{ij}\pi_h)) + ((b_{ij}\pi_1), ..., (b_{ij}\pi_h)) \\ & = & A\varphi + B\varphi. \end{array}$$

Thus, φ is a homomorphism.

Now, we will show that $(M_{m,n}(L))\varphi$ is a subdirect product of $M_{m,n}(L_i)(i = 1, 2, ..., h)$. $(M_{m,n}(L))\varphi \leq \prod_{i=1}^{h} M_{m,n}(L_i)$ is clear. We only need to verify that $[(M_{m,n}(L))\varphi]\Pi_1 = M_{m,n}(L_1), ..., [(M_{m,n}(L))\varphi]\Pi_h = M_{m,n}(L_h)$, where Π_i $(i = 1, 2, ..., h)$ are the projective mappings from $M_{m,n}(L)$ to $M_{m,n}(L_i)$.

In fact, for any $A^{(k)} = (a_{ij}^{(k)}) \in M_{m,n}(L_k)(k = 1, 2, ..., h)$, since $L\pi_k = L_k$, there exists $b_{ij}^{(k)} \in L$ such that $b_{ij}^{(k)}\pi_k = a_{ij}^{(k)}$. Now, put $B^{(k)} = (b_{ij}^{(k)})$, then $(B^{(k)}\varphi)\Pi_k = [(b_{ij}^{(k)})\varphi]\Pi_k = (b_{ij}^{(k)}\pi_k) = A^{(k)}$. And then we have $[(M_{m,n}(L))\varphi]\Pi_k = M_{m,n}(L_k)$.

Hence, $(M_{m,n}(L))\varphi$ is a subdirect product of $M_{m,n}(L_i)(i = 1, 2, ..., h)$. And then $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)$ $(i = 1, 2, ..., h)$. □

Note that the composition of two isomorphisms is also an isomorphism, then the following theorem is clear.

Theorem 3. *If L is isomorphic to a subdirect product of $L_i(i = 1, 2, ..., h)$, then $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)(i = 1, 2, ..., h)$.*

Remark 1. The above two theorems show that some subdirect decompositions of the semigroup $(M_{m,n}(L), +)$ can be decided by the corresponding subdirect decompositions of L.

Corollary 4. *Let L be a finite chain. Then $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)$, where $L_i = \{0, i\}(i = 1, 2, ..., l)$.*

Corollary 5. *Let L be a finite chain. Then $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)$, where L_i and h are constructed as Corollary 2.*

Corollary 6. *Let L be a finite chain. Then $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)$, where L_i and h are constructed as Corollary 3.*

Remark 2. Corollary 4, Corollary 5 and Corollary 6 give some concrete subdirect decompositions of $M_{m,n}(L)$.

Further, we can also obtain the following theorem.

Theorem 4. Assume that L is isomorphic to a subdirect product of $L_i (i = 1, 2, ..., h)$ and φ is the corresponding isomorphism from $M_{m,n}(L)$ to the subdirect product of $M_{m,n}(L_i)$. Then for any $A \in M_{m,n}(L)$ satisfying $A\varphi = (A_1, A_2, ..., A_h)$, we have $A = \sum_{i=1}^{h} A_i$.

Proof. Assume that ϕ is the isomorphism from L to the subdirect product of $L_i (i = 1, 2, ..., h)$ and φ is the corresponding isomorphism from $M_{m,n}(L)$ to the subdirect product of $M_{m,n}(L_i)$. By the proof of Theorem 1, for any $a \in L$, we have $a = \sum_{i=1}^{h} (a\phi)\pi_i$, where $a\phi = ((a\phi)\pi_1, (a\phi)\pi_2, ..., (a\phi)\pi_h)$. Thus, for any $A = (a_{ij}) \in M_{m,n}(L)$, together with the proof of Theorem 2, we have $A = (a_{ij}) = ((a_{ij}\phi)\pi_1 + (a_{ij}\phi)\pi_2 + ... + (a_{ij}\phi)\pi_h) = ((a_{ij}\phi)\pi_1) + ((a_{ij}\phi)\pi_2) + ... + ((a_{ij}\phi)\pi_h)) = A_1 + A_2 + ... + A_h$, where $A_i \in M_{m,n}(L_i)(i = 1, 2, ..., h)$ and $A\varphi = (A_1, A_2, ..., A_h)$. □

Now, together with the results obtained in Section 2, we immediately have the following corollaries.

Corollary 7. *Let L be a finite chain. Assume that φ is the corresponding isomorphism from $M_{m,n}(L)$ to the subdirect product of $M_{m,n}(L_i)$, where $L_i = \{0, i\}(i = 1, 2, ..., l)$. Then for any $A \in M_{m,n}(L)$ satisfying $A\varphi = (A_1, A_2, ..., A_l)$, we have $A = \sum_{i=1}^{l} A_i$.*

Corollary 8. *Let L be a finite chain. Let φ be the corresponding isomorphism from $M_{m,n}(L)$ to the subdirect product of $M_{m,n}(L_i)(i = 1, 2, ..., h)$, where L_i and h are constructed as Corollary 2. Then for any $A \in M_{m,n}(L)$ satisfying $A\varphi = (A_1, A_2, ..., A_h)$, we have $A = \sum_{i=1}^{h} A_i$.*

Corollary 9. *Let L be a finite chain. Let φ be the corresponding isomorphism from $M_{m,n}(L)$ to the subdirect product of $M_{m,n}(L_i)(i = 1, 2, ..., h)$, where L_i and h are constructed as Corollary 3. Then for any $A \in M_{m,n}(L)$ satisfying $A\varphi = (A_1, A_2, ..., A_h)$, we have $A = \sum_{i=1}^{h} A_i$.*

Remark 3. Corollary 7 shows that a matrix over a finite chain can be decomposed into the sum of some matrices over the two elements subchains of the finite chain, which is just the cut matrices decomposition theorem of the matrices over a finite chain; Corollary 8 and Corollary 9 show that a matrix over a finite chain can be decomposed into the sum of some matrices over some of its special subchains, which is just the Theorem 3 in [1]. Thus, our Theorem 4 generalizes and extends the corresponding results in [1].

Example 3. Let $L = \{0, 1, 2, 3, 4, 5\}$ be a finite chain. Take $L_1 = \{0, 1\}$, $L_2 = \{0, 2\}$, $L_3 = \{0, 3\}$ $L_4 = \{0, 4\}$ and $L_5 = \{0, 5\}$, then by Corollary 4, we have that $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)(i = 1, 2, 3, 4, 5)$. Also, if we take

$$A = \begin{pmatrix} 4 & 1 & 2 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 5 & 3 & 4 \end{pmatrix} \in M_{3,4}(L),$$

then we have $A = \sum_{i=1}^{5} A_i$, where

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in M_{3,4}(L_1), A_2 = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix} \in M_{3,4}(L_2)$$

$$A_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 3 & 3 \end{pmatrix} \in M_{3,4}(L_3), A_4 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \end{pmatrix} \in M_{3,4}(L_4),$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix} \in M_{3,4}(L_5).$$

Hence, the matrix A over a finite chain L can be decomposed into the sum of some cut matrices.

Example 4. Let $L = \{0, 1, 2, 3, 4, 5\}$ be a finite chain. Take $L_1 = \{0, 1\}$, $L_2 = \{0, 2, 4\})$ and $L_3 = \{0, 3, 5\}$, then by Theorem 2, $M_{m,n}(L)$ is isomorphic to a subdirect product of $M_{m,n}(L_i)(i = 1, 2, 3)$. Now, if we take

$$A = \begin{pmatrix} 4 & 1 & 2 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 5 & 3 & 4 \end{pmatrix} \in M_{3,4}(L),$$

then we have $A = A_1 + A_2 + A_3$, where

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in M_{3,4}(L_1), A_2 = \begin{pmatrix} 4 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 0 & 4 & 2 & 4 \end{pmatrix} \in M_{3,4}(L_2),$$

$$A_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & 3 & 3 \end{pmatrix} \in M_{3,4}(L_3).$$

Thus, the matrix A over a finite chain L can be decomposed into the sum of some matrices over some of its special subchains L_1, L_2 and L_3.

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