SOME REMARKS ON THE QUASI F-MOMENT PROBLEM

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Abstract: This paper is devoted to give some properties of the so called quasi $F$-moment problem generated by orthogonal polynomials. Different relations between different types of quasi $F$-moment problem are explained. Some transformations between different types of quasi $F$-moment problems are given.

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1. Introduction

In his fundamental papers In [2] authors introduced the definition of $F$-moment problem, where $F$ be a closed subset of $\mathbb{R}$, i.e. characterizing the real sequences $s = (s_n)_{n \geq 0}$ of the form

$$s_n = \int_F x^n d\mu(x), \quad n \geq 0,$$

where $\mu \in E_+(\mathbb{R})$ is supported by the closed set $F$. In this paper, we will...
consider the following two quasi $F$-moment problems: first the quasi Hausdorff moment sequence (QH):

$$h_n = \int_0^1 p_n(x) d\mu(x), \quad n \geq 0,$$

where $p_n(x)$ represents orthogonal polynomial on $[0,1]$ and the quasi Stieltjes moment sequence (QS):

$$h_n = \int_0^\infty q_n(x) d\nu(x), \quad n \geq 0,$$

where $q_n(x)$ represents orthogonal polynomial on $[0,\infty]$. Suppose that the polynomials $p_n(x) = \sum_{n=0}^\infty a_n x^n$, $q_n(x) = \sum_{n=0}^\infty b_n x^n$ with positive coefficients i.e. $a_n, b_n \geq 0$. So the set QS can be considered as a subset of $[0,\infty[^N_0$ with the product topology.

2. Some Properties of the Sets QH and QS

The following lemma are in fact, an adaption of whatever done for semigroups in [2, Berg et al]. We will not repeat the proof, whenever the proof for semigroups can be applied to the polynomials with necessary modification.

**Lemma 1.** A sequence $g = (g_n)_{n \geq 0}$ of orthogonal polynomial is the sequence of moments of a measure $\nu \in E_+ (\mathbb{R})$ where

$$E_+ (\mathbb{R}) = \{ \mu \in M_+ (\mathbb{R}); \sum_{n=0}^\infty b_n \int |x^n| d\nu(x) = \int |q_n(x)| d\nu(x) < \infty, \quad \text{for all} \quad n \geq 0 \}$$

i.e., is of the form

$$g_n = \int q_n(x) d\nu(x), \quad n = 0, 1, 2, \ldots$$

if and only if $g \in \mathbb{P}(\mathbb{N}_0)$.

**Proposition 2.** For a sequence of orthogonal polynomial $g = (g_n)_{n \geq 0}$ the following conditions are equivalent:

(i) $g, E_1 g \in \mathbb{P}(\mathbb{N}_0)$;

(ii) $t = (g_0, 0, g_1, 0, g_2, 0, \ldots) \in \mathbb{P}(\mathbb{N}_0)$;
(iii) There exists $\nu \in M_+([0, \infty])$ such that

$$g_n = \int_0^\infty q_n(x)d\nu(x), \quad n \geq 0.$$  

Proof. The proof of the two directions ”(i)⇒(ii)” and ”(iii)⇒(i)” are similar to Theorem 6.2.5 [2]. Now we sufficiently concern our efforts to prove the direction ”(ii)⇒(iii)”. By the above lemma there is $\sigma \in E_+(\mathbb{R})$ such that

$$t_{2n} = g_n = \int q_{2n}(x)d\sigma(x),$$

$$t_{2n+1} = \int q_{2n+1}(x)d\sigma(x) \quad n \geq 0.$$  

hence

$$g_n = \int_0^\infty q_n(x)d\nu(x), \quad n \geq 0.$$  

where $\nu$ is the image measure of $\sigma$ under the continuous mapping $x \mapsto x^2$ of $\mathbb{R}$ into $[0, \infty[$.

**Theorem 3.** The set $QS$ is a closed set stable under point-wise sums, products and multiplication by non-negative scalars.

Proof. The above proposition tells us that $(g_n) \in QS$ if and only if $(g_n)\text{and}(g_{n+1})$ are positive definite. This shows that $QS$ is a closed set. Its stable under point-wise sum and multiplication by nonnegative scalars, but it also stable under point-wise products this is a real consequence of the following remark; let $(g_n), (f_n) \in QS$ with measures $\nu, \eta$ respectively i.e.,

$$g_n = \int_0^\infty q_n(x)d\nu(x),$$

$$f_n = \int_0^\infty q_n(x)d\eta(x).$$

Then $(g_n f_n)$ is the quasi moment sequence of the product convolution measure $\nu \diamond \eta$.  

3. Main Result

Lemma 4. Let $p \geq 1, d_j > 0, 0 < q_j < 1, j = 1, ..., p$ be given. Then $g_0 = 1,$
$$g_n = \prod_{k=0}^{n-1} (1 + d_1 q_k(q_1) + ... + d_p q_k(q_p))^{-1}, \quad n \geq 1$$
is a QS moment sequence.

Proof. Consider the entire function of $p$ complex variables:
$$f(z_1, ..., z_p) = \prod_{k=0}^{\infty} (1 + z_1 q_k(q_1) + ... + z_p q_k(q_p))$$
The power series expansion of $f$ can be written
$$f(z) = f(z_1, ..., z_p) = \sum_{\alpha} b_{\alpha} z^{\alpha}$$
where we use the multi-index notation
$$z = (z_1, ..., z_p), \alpha = (\alpha_1, \alpha_2, ..., \alpha_p), z^{\alpha} = z_1^{\alpha_1} ... z_p^{\alpha_p}$$
where the sum is over all integers $\alpha_1 \geq 0, ..., \alpha_p \geq 0$. The coefficients $b_{\alpha} = b_{\alpha}(q)$
of the power series are positive as sums of products of powers of $q_k(q_1), ..., q_k(q_p)$.
Let
$$\nu = \frac{1}{f(d_1, ..., d_p)} \sum_{\alpha} b_{\alpha} d^{\alpha_1} q_k^{\alpha_1}$$
Then $\nu$ is a probability measure with compact support. The $n^{th}$ quasi moment of $\nu$ is
$$g_n = \frac{1}{f(d)} \sum_{\alpha} b_{\alpha} d^{\alpha} q_k^{\alpha}$$
$$= f(d_1 q_n(q_1), ..., d_p q_n(q_p))$$
$$= \prod_{k=0}^{n-1} (1 + d_1 q_k(q_1) + ... + d_p q_k(q_p))^{-1}.$$

Theorem 5. (Main Result) Let $(h_n)$ be a non vanishing quasi Hausdorff moment sequence. Then $(g_n)$ defined by $g_0 = 1$ and $g_n = \frac{1}{h_0...h_n}$ for $n \geq 1$ is a normalized quasi Stieltjes moment sequence.
Proof. Any non-negative measure $\mu$ on $[0,1]$ is a weak limit of a sequence of discrete measures of the form $a_1\delta_{x_1} + ... + a_p\delta_{x_p}$ where $a_j > 0, j = 1, ..., p$ and $0 < x_1 < x_2 < ... < x_p < 1[1]$. By the closedness of QS (see, Prop 3), it is enough to prove Theorem 5 for discrete measures of this type i.e., to prove that

$$g_n = \prod_{k=1}^{n} (a_1q_k(x_1) + ... + a_pq_k(x_p))^{-1}$$

with ($g_0=1$) belongs to QS. We have

$$g_n = \left(\frac{1}{a_p}\right)^n [q_k(x_p)^{-\frac{n+1}{2}}] \prod_{k=1}^{n} \left(1 + \frac{a_1q_k(x_1)}{a_pq_k(x_p)} + ... + \frac{a_{p-1}q_k(x_{p-1})}{a_pq_k(x_p)}\right)^{-1}$$

which is the point-wise product of three QS moment sequences, namely $(\frac{1}{a_p})^n$ and the moment sequence $[q_k(x_p)^{-\frac{n+1}{2}}]$ and Lemma 4. A representing measure is the product convolution of 3 corresponding representing measures.

**Theorem 6.** Let $(g_n)$ be a normalized quasi Stieltjes moment sequence generated by a quasi Hausdorff moment sequence $(h_n)$ i.e., $g_0 = 1$ and $g_n = \frac{1}{h_0...h_n}$ for $n \geq 1$. If $h_\infty = c > 0$ then $(g_n)$ is determinate and the support $S$ of the uniquely determined representing measure satisfies $\frac{1}{c} \in S \subseteq [0, \frac{1}{c}]$. The sequence $(g_n)$ is a quasi Hausdorff moment sequence if and only if $h_\infty \geq 1$.

Proof. Suppose that $h_\infty = c > 0$. Then clearly $g_n \leq \frac{1}{c^n}$, which shows that the support $S$ of $\mu$ is contained in $[0, \frac{1}{c}]$, and then $\mu$ is determinate. On the other hand, since $h_n \rightarrow c$ there exists to any $\epsilon > 0$ an $N \in \mathbb{N}$ such that

$$g_n \geq \frac{(c+\epsilon)^N}{h_1...h_n} \left(\frac{1}{c+\epsilon}\right)^n$$

So, $\frac{1}{c} \in S$. Finally if $h_\infty \geq 1$, then $S$ is a subset of the unit interval, so $(g_n)$ is a HS. Conversely, if $(g_n)$ is a HS and in particular decreasing, we get from $s_n \leq s_{n-1}$ that $h_n \geq 1$ and hence $h_\infty \geq 1$.

As an application of the above two Theorems we get:

**Corollary 7.** For an arbitrary HS $(h_n)$ the sequence $(g_n)$ defined by $g_0 = 1$ and $g_n = 1/(1 + h_1)...(1 + h_n)$ for $n \geq 1$ is a HS.
4. General Transformations

In his fundamental paper [3] Berg introduced a non-linear injective transformation $T$ from the set of non-vanishing normalized Hausdorff moment sequences $(a_n)$

$$a_n = \int_{0}^{1} x^n d\mu(x), \quad n \geq 0, \quad a_0 = 1$$

to the set of normalized Stieltjes moment sequences $(s_n)$ by the formula

$$T[(a_n)_n] = \frac{1}{a_1...a_n}$$

This section is devoted to present some general results about the multi-dimensional cases. Let

$$\tilde{a}_n = \int_{[0,1] \times [0,1]} x^n d\mu(x)$$

where $n = (n_1, n_2), x = (x_1, x_2)$ and $\mu$ is a non-negative Borel measure on $\mathbb{R}^2$. Using the notation $x^n = x_1^{n_1}x_2^{n_2}$ for $n \in \mathbb{N}_0^2$ the n-th moment of $\mu \in M$ is defined by

$$\tilde{s}_n = \int_{\mathbb{R}^2} x^n d\mu(x), \quad n \geq 0, \quad a_0 = 1 \quad (1)$$

where $M$ is the set of Borel measures on $\mathbb{R}^2$. We will denote by $\tilde{S}$ the set of all sequences of the form (1).

**Theorem 8.** The set $\tilde{S}$ is a closed set stable under point-wise sums, products and multiplication by non-negative scalars.

**Theorem 9.** Let $p \geq 1, c_j > 0, 0 < q_j < 1, j = 1, ..., p$ be given. Then

$$s(n) = \prod_{k=\{0\}}^{n \backslash \{1\}} (1 + c_1 q_1^k + ... + c_p q_p^k)^{-1}$$

$$= \prod_{k_1=0}^{n_1-1} \prod_{k_2=0}^{n_2-1} (1 + c_1 q_1^{k_1} q_2^{k_2} + ... + c_p q_p^{k_1} q_p^{k_2})^{-1}$$

belongs to is $\tilde{S}$, where $n = (n_1, n_2); k = (k_1, k_2); q_i = (q_{i,1}, q_{i,2})$.

**Proof.** Consider the entire function of $p$ complex variables:

$$f(z_1, ..., z_p) = \prod_{k=\{0\}}^{\{\infty\}} (1 + z_1 q_1^k + ... + z_p q_p^k)$$
The power series expansion of \( f \) can be written

\[
f(z) = f(z_1, ..., z_p) = \sum_{\alpha} b_{\alpha} z^\alpha
\]

where we use the multi-index notation

\[
z = (z_1, ..., z_p), \quad \alpha = (\alpha_1, \alpha_2, ..., \alpha_p), \quad z^\alpha = z_1^{\alpha_1} ... z_p^{\alpha_p}
\]

where the sum is over all integers \( \alpha_1 \geq 0, ..., \alpha_p \geq 0 \). The coefficients \( b_{\alpha} = b_{\alpha}(q) \) of the power series are positive as sums of products of powers of \( q_1, ..., q_p \). Let

\[
\mu = \frac{1}{f(c_1, ..., c_p)} \sum_{\alpha} b_{\alpha} c_\alpha \delta_{q^\alpha}
\]

Then \( \mu \) is a probability measure with compact support. The \( n^{th} \) quasi moment of \( \mu \) is

\[
s(n) = \frac{1}{f(c)} \sum_{\alpha} b_{\alpha} c_\alpha (q^\alpha)^n = \frac{f(c_1 q_1^n, ..., c_p q_p^n)}{f(c_1, ..., c_p)} = \prod_{k=0}^{n-1} \left( 1 + c_1 q_1^k + ... + c_p q_p^k \right)^{-1}.
\]

**Theorem 10.** Let \( \tilde{s}(n) \) belongs to the set \( \tilde{S} \). Then the sequence \( (T(n)) \) defined by

\[
T(n) = \frac{1}{\tilde{s}(1) ... \tilde{s}(n)}
\]

belongs to the set \( \tilde{S} \).

**References**


