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A NOTE ON MULTIPLIERS OF BANACH ALGEBRAS

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Abstract: In this paper, we give some new results for multipliers on Banach algebras and we give new proof for the second transpose of a multiplier on a Banach algebra is a multiplier on second dual of it.

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1. Introduction

The notion of multipliers initialed by definition of centralizers in abstract algebra. Wendel [9] first introduced centralizer mapping for Banach algebras. Wang [8] has studied operators similar to centralizers in the context of a commutative Banach algebra and some useful result about Banach algebras and $C^{\star}-$algebras considered in [5]. Brešer in [2] studied centralizer of prime rings, and gave many useful results. At first time Helgason, used multiplier instead of centralizer in [4]. The general theory of multipliers on a Banach algebra has been developed by Birtal in [1]. Ülger in [7], considered closeness of range of

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multipliers on Banach algebras. He proved that when Banach algebra A has a bounded approximate identity such that every proper closed ideal of A is contained in a proper closed ideal with a bounded approximate identity. Then a multiplier on A has a closed range if and only if it factors as a product of an idempotent multiplier and an invertible multiplier.

A mapping T from a Banach algebra A into itself is said to be a multiplier (centralizer) of A if

$$x(Ty) = (Tx)y$$

for all $x, y \in A$. The linear operator, T is said to be a right or left multiplier (centralizer) if

$$T(xy) = (Tx)y \quad \text{or} \quad T(xy) = x(Ty),$$

respectively, for all $x, y \in A$. The first dual and second dual of Banach algebra A, denoted by $A^{\prime}$ and $A^{\prime\prime}$, respectively.

Let $\mathcal{M}(A)$ be the collection of the all bounded linear operators on A. Then the collection of the multipliers of A is a closed commutative subalgebra of $\mathcal{M}(A)$ which contains the identity operator (see [5]). Recently Daws studied on $\mathcal{M}(A)$ as Banach A-bimodule (see [3]).

2. Main Results

In this section, our aim is to give some new algebraic results for multipliers of algebras. An algebra A is called without order if for all $x \in A$, $xA = \{0\}$ implies that $x = 0$, or, for all $x \in A$, $Ax = \{0\}$ implies that $x = 0$. At first, we recall the following Lemma, from [6] (see [6, Lemma 1.3.2]).

Lemma 2.1. *Let A be a without order algebra. Let $x, z \in A$ and $x \neq 0$, such that $xyz = zyx$ for all $y \in A$. Then there exists $\lambda \in \mathbb{C}$, such that $x = \lambda z$.*

Theorem 2.2. *Let A be a without order Banach algebra, and let T and $T^{\prime}$ be left $(right)$ multiplier of A, such that*

$$T(x)T^{\prime}(y) = T^{\prime}(x)T(y), \tag{2.1}$$

for all $x, y \in A$. If $T \neq 0$, then there exists $\lambda \in \mathbb{C}$ such that $T^{\prime}(x) = \lambda T(x)$.

Proof. By replacing y by yz in 2.1, we have

$$T(x)yT^{\prime}(z) = T^{\prime}(x)yT(z) \tag{2.2}$$

for all $x, y, z \in A$ and in particular $T(x)yT^{\square}(x) = T^{\square}(x)yT(x)$. If $T(x) \neq 0$, then there exists $\lambda(x) \in \mathbb{C}$ such that $T^{\square}(x) = \lambda T(x)$ (Lemma 2.1). Therefore if $T(x) \neq 0$ and $T(z) \neq 0$, we have

$$(\lambda(x) - \lambda(z))T(x)yT(z) = 0,$$

for all $y \in A$. Hence, this implies that $\lambda(x) = \lambda(z)$. Thus, we show that, there exists $\lambda \in \mathbb{C}$, such that $T^{\square}(x) = \lambda T(x)$ holds for all $x \in A$. Now if we suppose that $T(x) = 0$, according to 2.2 and $T \neq 0$, we conclude that $T^{\square}(x) = 0$. Thus the proof is complete. □

Now we extend the result of Theorem 2.2, for many multipliers of algebra A and give a similar result.

Theorem 2.3. *Let A be a without order algebra, and let T, F, G and H be left (right) multipliers of A. Suppose that*

$$T(x)G(y) = H(x)F(y), \tag{2.3}$$

for all $x, y \in A$. If $T \neq 0$ and $F \neq 0$, then there exists $\lambda \in \mathbb{C}$ such that $G(x) = \lambda F(x)$ and $H(x) = \lambda T(x)$, for all $x \in A$.

Proof. Similarly to proof of previous Theorem, we replace y by yz in $T(x)G(y) = H(x)F(y)$. So we have

$$T(x)yG(z) = H(x)yF(z), \tag{2.4}$$

for all $x, y, z \in A$. Now let $y = yF(w)$, $w \in A$. Then

$$\begin{aligned} T(x)yF(w)G(z) &= H(x)yF(w)F(z) \\ &= T(x)yG(w)F(z), \end{aligned}$$

and so we have

$$T(x)y(F(w)G(z) - G(w)F(z)) = 0.$$

Since $T \neq 0$ and A is a prime algebra, these imply that $F(w)G(z) - G(w)F(z) = 0$ or $F(w)G(z) = G(w)F(z)$, for all $y, w \in A$. Since $F \neq 0$, therefore there exists $\lambda \in \mathbb{C}$ such that $G(y) = \lambda F(y)$ (Theorem 2.2), for all $y \in A$. Thus

$$T(x)y\lambda F(z) = H(x)yF(z),$$

and so

$$(\lambda T(x) - H(x))yF(z) = 0,$$

for all $x, y, z \in A$. Therefore $F \neq 0$, $\lambda T(x) = H(x)$, for all $x \in A$. □

Let A be a Banach algebra. We denote the center A, by $Z(A)$ such that

$$Z(A) = \{a \in A : \quad a.x = x.a \quad for \quad all \quad x \in A\}.$$

Let T be a nonzero left multiplier of Banach algebra A. Then T is a nonzero left multiplier on every nonzero ideal of A. Because, let I be a nonzero ideal of Banach algebra A. Let T be a nonzero left multiplier of Banach algebra A, such that T on I is zero. Let $y \in A$ and $x \in I$ then we have

$$0 = T(xy) = xT(y) \neq 0.$$

This is a contradiction. Therefore T is a nonzero left multiplier on every nonzero ideal of A.

Theorem 2.4. *Let A be a noncommutative without order algebra, and I be a nonzero left ideal of A. Let T be a multiplier map of A such that $T(I) \subset Z(A)$, then $T = 0$.*

Proof. Let $u, v \in I$, Thus $T(u), T(v) \in Z(A)$. Then

$$\begin{aligned} 0 &= T(uv).v - v.T(uv) = u.T(v).v - vu.T(v) \\ &= uv.T(v) - vuT(v) = (uv - vu)T(v). \end{aligned}$$

Therefore, either $T(v) = 0$ or v is in the center of I. This implies I is the union of subsets $K := \{u \in I : T(u) = 0\}$ and $H := \{a \in I : u \in Z(I)\}$. It is clear that K and H are additive subgroups of I. But this is imposable, because union of two subsets of a group, when is as a subgroup or group which one of them be subset of other. Therefore $K = I$ or $H = I$. If $H = I$, then I is commutative and therefore A is commutative, which is contradiction. Hence $K = I$, therefore by argumentation of before theorem, $T = 0$. □

In some cases, results for without order Banach algebras remained true, when we replace it by prime Banach algebras by semisimple Banach algebras. Nevertheless, obtained result from Theorem 2.4 is not true when we replace by prime Banach algebras by semi simple Banach algebras. Because assume that A is a following Banach algebra:

$$A = \left\{ \begin{bmatrix} a & b & c \\ d & e & 0 \\ 0 & 0 & f \end{bmatrix} : a, b, c, e, d, f \in \mathbb{C} \right\}.$$

The algebra A is a noncommutative and semisimple Banach algebra with the following center

$$Z(A) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} : a, b \in \mathbb{C} \right\}.$$

Now, define

$$T : A \longrightarrow A \quad \text{by} \quad T \begin{bmatrix} a & b & c \\ d & e & 0 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Then T is a multiplier and $T(A) \subset Z(A)$. This shows that Theorem 2.4 for semisimple Banach algebras not true.

Let A be a Banach algebra and let $\pi : A \times A \longrightarrow A$ denote the product of A, such that $\pi(a, b) = ab$ for each $a, b \in A$. Then the transpose of π is defined to be the continuous bilinear mapping $\pi^* : A^* \times A \longrightarrow A^*$ given by formula

$$\langle \pi^*(a^*, b), c \rangle := \langle a^*, \pi(b, c) \rangle \quad (a \in A^*, b, c \in A).$$

We continue to define

$$\pi^{**} : A^{**} \times A^* \longrightarrow A^* \quad \text{and} \quad \pi^{***} : A^{**} \times A^{**} \longrightarrow A^{**},$$

the latter map extends π in the sense that $\pi^{***}|A \times A = \pi$. In fact π^{***} is the unique extension of π such that $\pi^{***}(., b^{**})$ is $\sigma(A^{**}, A^*)$ to $\sigma(A^{**}, A^*)$ continuous for every b^{**} in A^{**}, and $\pi^{***}(a, .)$ is $\sigma(A^{**}, A^*)$ to $\sigma(A^{**}, A^*)$ continuous for every a in A. Let $T : A \longrightarrow A$ be a right multiplier then we can write

$$(T \circ \pi)(a, b) = \pi(a, T(b)) \quad (a, b \in A). \tag{2.5}$$

Let $T : A \longrightarrow A$ be a multiplier, then readily from the double-limit definition of the product of elements in A^{**}, we can conclude that T^{**} is a multiplier on A^{**}. Now, here we give another proof of this statement.

Theorem 2.5. *Let $T : A \longrightarrow A$ be a multiplier, then the second transpose of T is the multiplier of A^{**}.*

Proof. Let T be a left multiplier of A. Then

$$T \circ \pi(a, b) = \pi(a, T(b)) \quad (a, b \in A),$$

also for all $a, b, c \in A$ we have

$$\begin{aligned}\langle (T\circ\pi)'(a', b), c\rangle &= \langle a', \pi(b, T(c))\rangle = \langle \pi'(a', b), T(c)\rangle \\ &= \langle T'\circ\pi'(a', b), c\rangle,\end{aligned}$$

since c is arbitrary therefore

$$(T\circ\pi)'(a', b) = T'\circ\pi'(a', b). \tag{2.6}$$

By 2.6 we have

$$\begin{aligned}\langle (T\circ\pi)''(a'', b'), c\rangle &= \langle a'', (T\circ\pi)'(b', c)\rangle = \langle a'', T'\circ\pi'(b', c)\rangle \\ &= \langle T''a'', \pi'(b', c)\rangle = \langle \pi''(T''a'', b'), c\rangle,\end{aligned}$$

therefore

$$(T\circ\pi)''(a'', b') = \pi''(T''a'', b'), \tag{2.7}$$

and

$$\begin{aligned}\langle (T\circ\pi)'''(a'', b''), c'\rangle &= \langle a'', (T\circ\pi)''(b'', c')\rangle = \langle a'', \pi''(T''b'', c')\rangle \\ &= \langle \pi'''(a'', T''b''), c'\rangle.\end{aligned}$$

Thus,

$$(T\circ\pi)'''(a'', b'') = \pi'''(a'', T''b''). \tag{2.8}$$

On the other hands we have

$$\begin{aligned}\langle (T\circ\pi)'(a', b), c\rangle &= \langle a', T\circ\pi(b, c)\rangle = \langle T'a', \pi(b, c)\rangle \\ &= \langle \pi'(T'a', b), c\rangle. \end{aligned}\tag{2.9}$$

From 2.9 we have

$$\begin{aligned}\langle (T\circ\pi)''(a'', b'), c\rangle &= \langle a'', (T\circ\pi)'(b', c)\rangle = \langle a'', \pi'(T'b', c)\rangle \\ &= \langle \pi''(a'', T'b'), c\rangle,\end{aligned}\tag{2.10}$$

and from 2.10, we conclude that

$$\begin{aligned}\langle (T\circ\pi)'''(a'', b''), c'\rangle &= \langle a'', (T\circ\pi)''(b'', c')\rangle = \langle a'', \pi''(b'', T'c')\rangle \\ &= \langle \pi'''(a'', b''), T'c'\rangle \\ &= \langle T''\circ\pi'''(a'', b''), c'\rangle.\end{aligned}\tag{2.11}$$

Then from 2.8 and 2.11, we have

$$T''\circ\pi'''(a'', b'') = \pi'''(a'', T''b'')$$

for all $a'', b'' \in A''$. This shows that T'', is a left multiplier. For the right case, proof is similar. □

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