FRAME BOUNDS FOR SUBSPACES OF $L_2(\mathbb{R})$

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Abstract: Let $H$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We say a set $\{f_k : k \in \mathbb{Z}\}$ is a (fundamental) frame for $H$ if there exist $A, B > 0$ such that for each $f \in H$,

$$A \|f\|^2_H \leq \sum_{n \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \|f\|^2_H.$$  \hspace{1cm} (1)

In case $\{f_k : k \in \mathbb{Z}\}$ is a frame for the subspace $\text{span}\{f_k : k \in \mathbb{Z}\}$, we say that $\{f_k : k \in \mathbb{Z}\}$ is a frame sequence.

A Weyl-Heisenberg frame sequence is a frame sequence which is generated by translated and modulated versions of $L_2$-functions.

In this paper, we characterize Weyl-Heisenberg frame sequences using infinite Hermitian matrices and obtain the optimal frame bounds in terms of the operator norms of these matrices. This work is inspired by a paper by Casazza and Christensen, where sufficient conditions for a Weyl-Heisenberg system to

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be a frame sequence are studied.

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## 1. Introduction

Let $H$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We say a set

$$\{ f_k : k \in \mathbb{Z} \}$$

is a (fundamental) frame for $H$ if there exist $A, B > 0$ such that for each $f \in H$,

$$A \| f \|^2_H \leq \sum_{n \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \| f \|^2_H.$$  \hspace{1cm} (2)

The sharpest possible choices of $A$ and $B$ are called the lower frame bound and the upper frame bound respectively. A frame is said to be tight if the upper frame bound is equal to the lower frame bound. Parseval’s identity shows that every orthonormal basis is a tight frame with frame bounds equal to one. If a set $\{ f_k : k \in \mathbb{Z} \}$ satisfies the frame condition (2) for a subspace $\text{span} \{ f_k : k \in \mathbb{Z} \}$, then we say that $\{ f_k : k \in \mathbb{Z} \}$ is a frame sequence. The notion of frames was developed by Duffin and Schaeffer in their work on non-harmonic Fourier analysis. Since then, frames have enjoyed widespread study, with applications to harmonic analysis, operator theory, and applied mathematics.

In this paper, we study Weyl-Heisenberg frame sequences in the separable Hilbert space $L_2(\mathbb{R})$. A Weyl-Heisenberg system is obtained by applying discrete translations and modulations to a subset $F \subset L_2(\mathbb{R})$ of window functions defined on $\mathbb{R}$. Explicitly, for given two positive numbers $a$ and $p$, we define the translation and modulation of a function $f \in L_2(\mathbb{R})$ by:

Translation by $n$ : \hspace{1cm} $(E_n f)(t) := f(t + na), \ t \in \mathbb{R}, \ n \in \mathbb{Z}$

Modulation by $m$ : \hspace{1cm} $(M_m f)(t) := e^{i\frac{2\pi}{p}mt} f(t), \ t \in \mathbb{R}, \ m \in \mathbb{Z}$.

We then define the Weyl-Heisenberg system generated by $F$ to be the set

$$\{ M_m E_n f : f \in F, \ m, n \in \mathbb{Z} \}.$$  

One of the difficult problems concerning Weyl-Heisenberg systems is how to check that a given system is a frame for $L_2(\mathbb{R})$. In general this is a still open
problem. Sufficient conditions were established by Daubechies [4], and later, Ron and Shen [8] characterized Weyl-Heisenberg frames for $L_2(\mathbb{R})$ using dual Gramian matrices when $a/p = 1$. In their characterization they applied the techniques of dual Gramian matrices to shift-invariant frames for $L_2(\mathbb{R})$, which can be viewed as a Fourier transform of Weyl-Heisenberg frames for $L_2(\mathbb{R})$.

Frame sequences were studied by Benedetto and Li in [1], in which they analyzed the shift-invariant system $\{ g(\cdot - n) : n \in \mathbb{Z} \}$, $g \in L_2(\mathbb{R})$, and gave necessary and sufficient conditions that this is a frame sequence. Their characterization is in terms of the zero-set of the function $\sum_{n \in \mathbb{Z}} |g(t-n)|^2$, $t \in \mathbb{R}$. To state their result, we first define the Fourier transform on $L_1(\mathbb{R})$ by

$$\hat{f}(\lambda) := \int_{\mathbb{R}} f(t) e^{-2\pi i \lambda t} dt,$$

and extend it in the usual way to an isometry from $L_2(\mathbb{R})$ onto itself.

**Result 1.** (Benedetto and Li [1]) Let $g \in L_2(\mathbb{R})$. Then $\{ g(\cdot - n) : n \in \mathbb{Z} \}$ is a frame sequence with bounds $A, B$ if and only if

$$0 < A \leq \sum_{n \in \mathbb{Z}} |\hat{g}(t+n)|^2 \leq B,$$

for almost every $t$ for which $\sum_{n \in \mathbb{Z}} |\hat{g}(t+n)|^2 \neq 0$.

Casazza and Christensen in [2] found a sufficient condition for a Weyl-Heisenberg system to be a frame sequence in which the duality condition $a = p$ is not necessary. Their result can be considered a subspace version of a result by Ron and Shen [9].

**Result 2.** (Casazza and Christensen [2]) Let $g \in L_2(\mathbb{R})$, $a, p > 0$,

$$N_g := \left\{ t \in \mathbb{R} : \sum_{n \in \mathbb{Z}} |g(t-na)|^2 = 0 \right\}, \quad (3)$$

and suppose that

$$\tilde{A} := \inf_{t \in [0,a] - N_g} \left[ \sum_{n \in \mathbb{Z}} |g(t-na)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g(t-na)\bar{g}(t-na-kp) \right| \right] > 0,$$

$$\tilde{B} := \sup_{t \in [0,a]} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |g(t-na)\bar{g}(t-na-kp)| < \infty.$$
Then \( \left\{ e^{i \frac{2\pi}{p} m(t)} g(t - na) : m, n \in \mathbb{Z} \right\} \) is a frame for \( L_2(\mathbb{R} - N_g) \) with bounds \( pA, pB \), where \( L_2(\mathbb{R} - N_g) \) is the set of functions in \( L_2(\mathbb{R}) \) that vanishes at \( N_g \).

The proof of Result 2 is based on the following identity, which is valid for any bounded and compactly supported function \( f \in L_2(\mathbb{R} - N_g) \), provided \( g \) satisfies the condition \( \tilde{B} < 0 \):

\[
\sum_{m,n \in \mathbb{Z}} \left| \langle f, e^{i \frac{2\pi}{p} m(t)} g(t - na) \rangle \right|^2 = p \int |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - na)|^2 \, dt \\
+ p \sum_{k \neq 0} \int \bar{f}(t) f(t - kp) \sum_{n \in \mathbb{Z}} g(t - na) \bar{g}(t - na - kp) \, dt.
\]

One finds that the conditions \( \tilde{A} > 0 \) and \( \tilde{B} < \infty \) in Result 2 are equivalent to the hypotheses used by Daubechies [5] in the construction of \( L_2 \)-frames. These two conditions are sufficient conditions for a system to be an \( L_2 \)-frame. However, they are not necessary. Moreover, the constants \( \tilde{A}, \tilde{B} \) are known to be valid frame bounds; however they are generally (and generically) different from the sharpest bounds, i.e., the frame bounds.

It is the purpose of the present paper not only to give a characterization of Weyl-Heisenberg frame sequences but also to provide the sharpest frame sequence bounds. In order to do that, we adopt the idea of dual Gramian analysis from Ron and Shen [8]. To state our result regarding the characterization of Weyl-Heisenberg frame sequences, we first define an infinite matrix \( Q_g(t) \), for a given \( g \in L_2(\mathbb{R}) \) and almost every \( t \in \mathbb{R} \), with entries given by:

\[
Q(t)(j,k) := Q_g(t)(j,k) := p \sum_{n \in \mathbb{Z}} g(t + an + pj) \bar{g}(t + an + kp).
\]

We note that each entry of \( Q_g(t) \) is an \( a \)-periodic function. We also define \( N := N_g \) as

\[
N_g := \left\{ t \in [0,a] : \sum_{n \in \mathbb{Z}} |g(t + na + kp)|^2 = 0, \text{ for some } k \in \mathbb{Z} \right\}.
\]

In Section 3, we present some of the basic properties of this infinite matrix \( Q(t) \) in detail.

**Corollary 3.** Let \( g \in L_2(\mathbb{R}) \), \( a, p > 0 \), and suppose that

\[
\sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |g(t + an + pj)|^2 < \infty,
\]
for almost every $t \in [0, a]$. Define

$$A := \text{ess sup}_{t \in [0, a] - N} \| Q(t)^{-1} \|,$$

$$B := \text{ess sup}_{t \in [0, a] - N} \| Q(t) \|.$$

If $A < \infty$, then the Weyl-Heisenberg system

$$\left\{ e^{i \frac{2 \pi}{p} \cdot j} g(\cdot - na) : m, n \in \mathbb{Z} \right\}$$

is a frame for $L^2(\mathbb{R} - N)$ with the lower frame bound $A$ and the upper frame bound $B$, where $L^2(\mathbb{R} - N)$ is the set of functions in $L^2(\mathbb{R})$ that vanish at $N$.

We note that for almost every $t \in [0, a]$, the diagonal entries and the off-diagonal entries of $Q(t)$ are

$$Q(t)(j, j) = p \sum_{n \in \mathbb{Z}} |g(t + an + jp)|^2$$

and

$$Q(t)(j, k) = p \sum_{n \in \mathbb{Z}} g(t + an + jp) \overline{g}(t + an + kp).$$

It is well known that if $Q(t)$ is diagonally dominant, then it is invertible and furthermore, that for any $j \in \mathbb{Z}$, and for almost every $t \in [0, a]$,

$$\frac{1}{\| Q(t)^{-1} \|} \geq |Q(t)(0, 0)| - \sum_{k \neq 0} |Q(t)(0, k)|.$$

Equivalently, we have

$$\frac{1}{\| Q(t)^{-1} \|} \geq p \sum_{n \in \mathbb{Z}} |g(t + ap)|^2 - p \sum_{k \neq 0} \sum_{n \in \mathbb{Z}} g(t + ap) \overline{g}(t + ap + kp).$$

Thus the assumption that $Q(t)$ is diagonally dominant for almost every $t \in [0, a] - N$, i.e.,

$$A := \inf_{t \in [0, a] - N} \left[ \sum_{n \in \mathbb{Z}} |g(t + ap)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g(t + ap) \overline{g}(t + ap + kp) \right| \right] > 0,$$

leads immediately to Result 2. Corollary 3 does not assume this diagonally dominate condition, and in fact we provide an example of a Weyl-Heisenberg frame sequence for which $Q(t)$ is not diagonally dominant for almost every
In order to state the corresponding necessary condition, we define an infinite positive matrix $Q$ to be strongly positive if there exists a constant $c > 0$ such that for any $x \in \ell_2(\mathbb{Z})$,

$$x^* Q x \geq c \|x\|_{\ell_2(\mathbb{Z})}^2.$$ 

**Corollary 4.** Let $g \in L_2(\mathbb{R})$, $a, p > 0$, and suppose that

$$\sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |g(t + an + pj)|^2 < \infty,$$

for almost every $t \in [0, a]$. If the Weyl-Heisenberg system

$$\{ e^{i \frac{2\pi}{p} m} g(\cdot - na) : m, n \in \mathbb{Z} \}$$

is a frame for $L_2(\mathbb{R} - N)$ with the lower frame bound $A$ and the upper frame bound $B$, then for any $x \in \ell_2(\mathbb{Z})$, and for almost every $t \in [0, a] - N$,

$$A \|x\|_{\ell_2(\mathbb{Z})}^2 \leq x^* Q(t) x \leq B \|x\|_{\ell_2(\mathbb{Z})}^2.$$

This paper is laid out as follows: In §2, we review $L_1$-periodization and the bracket product in $L_2(\mathbb{R})$. In §3, we introduce modulation systems and associated bounded linear operators, and we define the corresponding infinite Hermitian matrices. In §4, we present our main theorems, which characterize modulation frame sequences using infinite Hermitian matrices. Finally, we discuss Weyl-Heisenberg frame sequences and the relationship between Weyl-Heisenberg frames and infinite quadratic forms.

### 2. Notation and Preliminaries

Throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L_2(\mathbb{R})$. For a subset $\mathcal{U} \subset \mathbb{R}$, we define $L_2(\mathcal{U})$ to be the set of functions in $L_2(\mathbb{R})$ that vanish on $\mathbb{R} \setminus \mathcal{U}$, and define $L_2^0(\mathcal{U})$ to be the set of functions in $L_2(\mathcal{U})$ that have compact support. Similarly, we define $\ell_2^0(\mathbb{Z})$ to be the set of sequences in $\ell_2(\mathbb{Z})$ that have compact support. From here on, we let $p$ be a fixed positive real number, and let $T$ denote the interval $[0, p]$.

Given a system $\{f_k : k \in \mathbb{Z}\}$ in a separable Hilbert space $H$, the frame operator $S : H \to H$ is defined by

$$S f := \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle f_k.$$
It is well known that if the system \( \{ f_k : k \in \mathbb{Z} \} \) is a frame for \( H \), then the operator \( S \) is bounded, self-adjoint, positive, and invertible. If the system \( \{ f_k : k \in \mathbb{Z} \} \) is a frame for \( H \), then so is the system \( \{ S^{-1} f_k : k \in \mathbb{Z} \} \) and we have the following perfect reconstruction property for all \( f \) in \( H \):

\[
f = \sum_{k \in \mathbb{Z}} \langle f, S^{-1} f_k \rangle f_k = \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle S^{-1} f_k.
\]

This series is the \( L_2 \)-convergent expansion and is minimal in the sense that for all \( \{ x_k \}_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z}) \) with

\[
f = \sum_{k \in \mathbb{Z}} x_k f_k
\]

there holds

\[
\sum_{k \in \mathbb{Z}} |\langle f, S^{-1} f_k \rangle|^2 \leq \sum_{k \in \mathbb{Z}} |x_k|^2.
\]

We note that if the frame operator \( S : H \to H \) is bounded for a given system \( \{ f_k : k \in \mathbb{Z} \} \), then for any \( f \in H \), we have

\[
\langle f, S f \rangle = \sum_{n \in \mathbb{Z}} |\langle f, f_k \rangle|^2.
\]

Thus a set \( \{ f_k : k \in \mathbb{Z} \} \) satisfies the frame condition (2) if and only if

\[
A \|f\|_H^2 \leq \langle f, S f \rangle \leq B \|f\|_H^2, \quad \forall f \in H.
\]

For further references on frames, we refer the reader to [3], [7], [8] and [11].

In this paper, we use the notion of a modulation system, which consists of modulated versions of functions \( g_n \in L_2(\mathbb{R}) \), \( n \in \mathbb{Z} \). In particular, Weyl-Heisenberg systems are a special case of modulation systems. One can consider modulation frames for \( L_2(\mathbb{R}) \) as a Fourier transform of shift-invariant frames for \( L_2(\mathbb{R}) \), which are characterized using dual Gramian matrices by Ron and Shen in [8]. However, in working directly with modulation frames, we are able to characterize Weyl-Heisenberg frame sequences and obtain the optimal frame bounds. We adopt the dual Gramian matrix technique from [8] in the time domain to obtain the best bounds for modulation frame sequences.

First, we recall the following \( L_1 \)-periodization result:

**Proposition 5.** (\( L_1 \)-Periodization, [10]) Let \( f \in L_1(\mathbb{R}) \). Then the series

\[
\sum_{n \in \mathbb{Z}} f(\cdot + pn)
\]

converges in \( L_1(T) \).
For $f, g \in L_2(\mathbb{R})$, we define the bracket product of $f$ and $g$:

$$[f, g](t) := \sum_{k \in \mathbb{Z}} f(t + kp)\overline{g}(t + kp),$$

as the $L_1$-periodization of $f \overline{g}$. This bracket product will be useful later in discussing the frame operator with regard to modulation systems.

Let $f, g \in L_2(\mathbb{R})$ such that $[g, g] \in L_\infty(T)$. Then, by the Cauchy-Schwarz inequality,

$$\int_T |[f, g](t)|^2 \, dt \leq \| [g, g] \|_{L_\infty(T)} \| f \|_{L_2(\mathbb{R})}^2.$$

This inequality can be extended to countable systems of functions:

**Proposition 6.** Let $G := \{ \ g_n \in L_2(\mathbb{R}) : n \in \mathbb{Z} \}$ such that

$$\sum_{n \in \mathbb{Z}} [g_n, g_n] \in L_\infty(T).$$

Then for any $f \in L_2(\mathbb{R})$, \( \sum_{n \in \mathbb{Z}} |[f, g_n]|^2 \) is in $L_1(T)$.

**Proof.** Let $f \in L_2(\mathbb{R})$. Then, by the Cauchy-Schwarz inequality, we have

$$\int_T \left| \sum_{n \in \mathbb{Z}} [f, g_n](t) \right|^2 \, dt \leq \int_T \sum_{n \in \mathbb{Z}} [f, f](t)[g_n, g_n](t) \, dt \leq M \| f \|_{L_2(\mathbb{R})}^2. \quad \square$$

### 3. Modulation Frames

Let $g \in L_2(\mathbb{R})$. From now on, we will denote modulation by an integer $m$ in the following way:

$$M_m g(t) := e^{i \frac{2\pi}{\tau} mt} g(t), \quad t \in \mathbb{R}.$$

If $G := \{ g_n : n \in \mathbb{Z} \}$ is a system of functions in $L_2(\mathbb{R})$, then the modulation system generated by $G$ is the system

$$\{ M_m g_n : m, n \in \mathbb{Z} \}.$$

A modulation system that is also a frame for $L_2(\mathcal{V}), \mathcal{V} \subset \mathbb{R}$, will be called a modulation frame sequence for $L_2(\mathcal{V})$. We define

$$N_G := \left\{ t \in \mathbb{R} : \sum_{n} |g_n(t + kp)|^2 = 0 \text{ for some } k \in \mathbb{Z} \right\},$$
Then we (formally) define $S_G$ on $L_2(\mathcal{U})$, the subspace of functions in $L_2(\mathbb{R})$ that vanish outside of $\mathcal{U}$, by

$$S_G f := \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle f, M_m g_n \rangle M_m g_n.$$ 

It follows that for any $f \in L_2(\mathcal{U})$,

$$S_G f = p \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left[ \sum_{j \in \mathbb{Z}} \frac{1}{p} \int_{T+jp} R f(t) g_n(t) e^{-i \frac{2\pi}{p} mt} dt e^{i \frac{2\pi}{p} mt} g_n(t) \right].$$

Substitution yields that

$$S_G f = p \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left[ \int_T [f, g_n](t) e^{-i \frac{2\pi}{p} mt} dt e^{i \frac{2\pi}{p} mt} g_n(t) \right].$$

Since $[f, g_n] \in L_2(T)$ and $\left\{ e^{i \frac{2\pi}{p} m \cdot} \right\}_{m \in \mathbb{Z}}$ is an orthogonal basis of $L_2(T)$,

$$\sum_{m \in \mathbb{Z}} \frac{1}{p} \int_T [f, g_n](t) e^{-i \frac{2\pi}{p} mt} dt e^{i \frac{2\pi}{p} mt} = [f, g_n](t).$$

Consequently, for any $f \in L_2(\mathcal{U})$, we have

$$S_G f = p \sum_{n \in \mathbb{Z}} [f, g_n] g_n. \quad (5)$$

Using this equality, we can now show that $S_G$ is a bounded linear operator on $L_2(\mathcal{U})$.

**Proposition 7.** Let $G := \{ g_n \in L_2(\mathbb{R}) : n \in \mathbb{Z} \}$ such that

$$\left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L_\infty(T)} < \infty.$$
Then $S_G : L_2(\mathcal{U}) \to L_2(\mathcal{U})$ defined by

$$S_G f := \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle f, M_m g_n \rangle M_m g_n$$

is a bounded linear operator.

Proof. Let $M = \| \sum_{n \in \mathbb{Z}} [g_n, g_n] \|_{L_\infty(T)}$. We have

$$\int_{\mathcal{U}} |S_G f(t)|^2 \, dt = p^2 \int_{\mathcal{U}} \left| \sum_{n \in \mathbb{Z}} [f, g_n](t) g_n(t) \right|^2 \, dt \quad (6)$$

$$\leq p^2 \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |[f, g_n](t)|^2 \sum_{n \in \mathbb{Z}} |g_n(t)|^2 \, dt \quad (7)$$

$$= p^2 \sum_{j \in \mathbb{Z}} \int_{T+jp} \sum_{n \in \mathbb{Z}} |[f, g_n](t)|^2 \sum_{n \in \mathbb{Z}} |g_n(t)|^2 \, dt \quad (8)$$

$$\leq p^2 \int_{T} \sum_{n \in \mathbb{Z}} |[f, g_n](t)|^2 \sum_{n \in \mathbb{Z}} |g_n(t+jp)|^2 \, dt, \quad (9)$$

where (7) follows from Cauchy-Schwarz and (9) from periodicity of $\sum_{n \in \mathbb{Z}} |[f, g_n]|^2$.

Since $\sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |g_n(\cdot + jp)|^2 \in L_\infty(T)$, together with the monotone convergence theorem and Cauchy-Schwarz, we have:

$$p^2 \int_{T} \sum_{n \in \mathbb{Z}} |[f, g_n](t)|^2 \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |g_n(t+jp)|^2 \, dt$$

$$\leq Mp^2 \int_{T} \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |f(t+kp)g_n(t+kp)|^2 \, dt$$

$$\leq Mp^2 \int_{T} \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} |f(t+kp)|^2 \right) \left( \sum_{k \in \mathbb{Z}} |g_n(t+kp)|^2 \right) \, dt$$

$$= Mp^2 \int_{T} \sum_{k \in \mathbb{Z}} |f(t+kp)|^2 \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |g_n(t+kp)|^2 \, dt$$

$$\leq M^2 p^2 \int_{T} \sum_{k \in \mathbb{Z}} |f(t+kp)|^2 \, dt.$$ 

Since $|f|^2 \in L_1(\mathbb{R})$, by Proposition 5, we conclude that

$$\|S_G f\|^2_{L_2(\mathcal{U})} \leq M^2 p^2 \int_{\mathcal{U}} |f(t)|^2 \, dt = M^2 p^2 \|f\|^2_{L_2(\mathcal{U})},$$
which shows that $S_G$ is a bounded operator with an upper bound $Mp$. It is clear that $S_G$ is linear.

We use the boundedness of the linear operator $S_G$ to find an upper frame sequence bound for a modulation system as follows:

**Proposition 8.** Let $G := \{ g_n : n \in \mathbb{Z} \} \subset L^2(\mathbb{R})$ such that

$$M := \left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L^\infty(T)} < \infty.$$

Then

$$\langle f, S_G f \rangle \leq Mp \| f \|_{L^2(\mathcal{U})}^2.$$

**Proof.** By the Cauchy-Schwarz inequality and Proposition 7, we have

$$\langle f, S_G f \rangle \leq \| f \|_{L^2(\mathcal{U})} \| S_G f \|_{L^2(\mathcal{U})} \leq Mp \| f \|_{L^2(\mathcal{U})}^2. \quad \square$$

We note that this upper bound is not optimal, and in Theorem 11 we will give the upper frame sequence bound for $\langle f, S_G f \rangle$.

Consider the equality (5) for $f \in L^2_0(\mathcal{U})$:

$$S_G f(t) = p \sum_{n \in \mathbb{Z}} [f, g_n](t) g_n(t) \quad = \quad \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f(t + kp) \overline{g_n}(t + kp) g_n(t), \quad = \quad \sum_{k \in \mathbb{Z}} \left[ \sum_{n \in \mathbb{Z}} g_n(t) \overline{g_n}(t + kp) \right] f(t + kp),$$

where swapping the order of summation is valid because $f$ has compact support. Since $[f, g_n]$ is $p$-periodic, we have that

$$S_G f(t + jp) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} g_n(t + jp) \overline{g_n}(t + kp) f(t + kp).$$

An immediate consequence of the above is

$$\begin{bmatrix}
  \vdots \\
  S_G f(t - p) \\
  S_G f(t) \\
  S_G f(t + p) \\
  \vdots 
\end{bmatrix} = Q(t) \cdot \begin{bmatrix}
  \vdots \\
  f(t - p) \\
  f(t) \\
  f(t + p) \\
  \vdots 
\end{bmatrix}, \quad (10)$$
where
\[
Q(t) := \left[ p \sum_{n \in \mathbb{Z}} g_n(t + jp) \overline{g_n}(t + kp) \right]_{(j,k) \in \mathbb{Z}^2}, \quad t \in \mathbb{R}.
\]

We note that \(Q(t)\) is an infinite Hermitian matrix. We will show that \(Q(t)\) is a bounded operator from \(\ell_2(\mathbb{Z})\) to \(\ell_2(\mathbb{Z})\) for almost every \(t \in \mathbb{R}\). Furthermore, \(Q(t)\) is uniformly bounded for almost every \(t \in \mathbb{R}\) so that we can establish frame sequence bounds for the modulation system generated by \(G\) using \(\|Q(t)\|\). If any of the entries of \(Q(t)\) is not well-defined or is not finite, then we define \(\|Q(t)\| := \infty\). In a similar way, we define \(\|Q(t)^{-1}\| := \infty\) if \(Q(t)\) is not well-defined, or is not invertible.

**Proposition 9.** Let \(G := \{g_n : n \in \mathbb{Z}\} \subset L_2(\mathbb{R})\) such that
\[
M := \left\| \sum_{n \in \mathbb{Z}} \left[ g_n, g_n \right] \right\|_{L_\infty(T)} < \infty,
\]
and let, for almost every \(t \in \mathbb{R}\), \(Q(t)\) be the infinite matrix:
\[
Q(t) := \left[ p \sum_{n \in \mathbb{Z}} g_n(t + jp) \overline{g_n}(t + kp) \right]_{(j,k) \in \mathbb{Z}^2}.
\]

Then
\[
\text{ess sup}_{t \in \mathbb{R}} \|Q(t)\| \leq M.
\]

**Proof.** Let \(x \in \ell_2^C(\mathbb{Z})\). Then for almost every \(t \in \mathbb{R}\),
\[
\|Q(t) \cdot x\|_{\ell_2(\mathbb{Z})}^2 \quad (11)
\]
\[
= \sum_{k \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z}} x_m \sum_{n \in \mathbb{Z}} g_n(t + mp) \overline{g_n}(t + kp) \right|^2 \quad (12)
\]
\[
\leq \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |x_m|^2 \sum_{n \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} g_n(t + mp) \overline{g_n}(t + kp) \right|^2 \quad (13)
\]
\[
= \|x\|_{\ell_2(\mathbb{Z})}^2 \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} g_n(t + mp) \overline{g_n}(t + kp) \right|^2 \quad (14)
\]
\[
\leq \|x\|_{\ell_2(\mathbb{Z})}^2 \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |g_n(t + mp)|^2 \sum_{n \in \mathbb{Z}} |g_n(t + kp)|^2 \quad (15)
\]
\[
\|x\|_{\ell_2(\mathbb{Z})}^2 \leq \left( \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |g_n(t + kp)|^2 \right) \leq M^2 \|x\|_{\ell_2(\mathbb{Z})}^2,
\]
where inequalities (13) and (15) are obtained by the Cauchy-Schwarz inequality over \( m \) and \( n \), respectively. The reordering from (15) to (16) is valid because the sequences are absolutely convergent, and (17) follows from the hypothesis. Since \( \ell_2'(\mathbb{Z}) \) is a dense subset of \( \ell_2(\mathbb{Z}) \), we conclude that \( Q(t) \) is a bounded linear operator on \( \ell_2(\mathbb{Z}) \) with bound \( M \).

Also note that for almost every \( t \in \mathbb{R} \), \( Q(t) \) is a positive operator since for any \( f \in L_2^C(\mathcal{U}) \),

\[
[f(t + kp)]_{k \in \mathbb{Z}}^\ast Q(t)[f(t + kp)]_{k \in \mathbb{Z}} = \sum_n |[f, g_n]|^2(t) \geq 0,
\]
which is a \( p \)-periodic function.

### 4. Main Results

In our next lemma, we establish an identity which is the key to our main theorems.

**Lemma 10.** Let \( G := \{g_n : n \in \mathbb{Z}\} \subset L_2(\mathbb{R}) \) such that

\[
\left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L_\infty(T)} < \infty,
\]
and let \( S_G : L_2(\mathcal{U}) \to L_2(\mathcal{U}) \) be the operator defined by

\[
S_G f := \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle f, M_m g_n \rangle M_m g_n.
\]

Then for any \( f \in L_2(\mathcal{U}) \),

\[
\langle f, S_G f \rangle = \int_T x_f^\ast(t)Q(t)x_f(t) dt,
\]
where \( x_f(t) \) is the vector \( [f(t + kp)]_{k \in \mathbb{Z}} \).
Proof. Let $M = \left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L^\infty(T)}$ and let $f \in L^2_2(\mathcal{U})$. It follows that

$$\langle f, S_G f \rangle = \int_{\mathbb{R}} f(t) \cdot \overline{S_G f(t)} \, dt = \sum_{k \in \mathbb{Z}} \int_{T + pk} f(t) \cdot \overline{S_G f(t)} \, dt = \sum_{k \in \mathbb{Z}} \int_{T} f(t + kp) \cdot \overline{S_G f(t + kp)} \, dt.$$

Since $f$ has compact support, we can interchange the sum and the integral in the last term. This, together with equality (10), yields that for any $f \in L^2_2(\mathcal{U})$,

$$\langle f, S_G f \rangle = \int_{T} x^*_f(t)Q(t)x_f(t) \, dt.$$

Since $S_G$ and $Q(t)$ are bounded operators for almost every $t \in T$, this equality holds for any $f \in L^2_2(\mathcal{U})$. □

Theorem 11. Let $G := \{g_n : n \in \mathbb{Z} \} \subset L^2(\mathbb{R})$ such that

$$\left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L^\infty(T)} < \infty.$$

Define

$$A := \text{ess sup}_{t \in T} \left\| Q(t)^{-1} \right\|;$$

$$B := \text{ess sup}_{t \in T} \left\| Q(t) \right\|.$$

If $A < \infty$, then the modulation system $\{ M_m g_n \}_{m,n \in \mathbb{Z}}$ generated by $G$ is a frame for $L^2(\mathcal{U})$ with the lower frame bound $A$ and the upper frame bound $B$.

Proof. Let $f \in L^2_2(\mathcal{U})$ and $x_f(t) := [f(t + kp)]_{k \in \mathbb{Z}}$. Then, since $Q(t)$ is a positive operator, we have

$$\frac{x^*_f(t)x_f(t)}{\left\| Q(t)^{-1} \right\|} \leq x^*_f(t)Q(t)x_f(t) \leq \left\| Q(t) \right\| x^*_f(t)x_f(t),$$

for almost every $t \in T$. Thus we have

$$\frac{1}{\text{ess sup}_{t \in T} \left\| Q(t)^{-1} \right\|} \int_{T} x^*_f(t)x_f(t) \, dt.$$
\[ \leq \int_T \langle x_f^*(t)Q(t)x_f(t) \rangle dt \]
\[ \leq \sup_{t \in T} \|Q(t)\| \int_T x_f^*(t)x_f(t) dt. \]

The result follows from Lemma 10 and
\[ \int_T x_f^*(t)x_f(t) dt = \|f\|_{L^2(U)}^2. \]

\[ \blacksquare \]

Theorem 11 shows that the strong positivity of \(Q(t)\) for almost every \(t \in T\) implies the frame property of \(G\). In the next theorem, we show the opposite direction.

**Theorem 12.** Let \(G := \{g_n : n \in \mathbb{Z}\} \subset L_2(\mathbb{R})\) such that
\[ \left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L^\infty(T)} < \infty, \]
and \(\{M_mg_n\}_{m,n \in \mathbb{Z}}\) be a frame for \(L^2(U)\) with the lower frame bound \(A\) and the upper frame bound \(B\). Then for any \(x \in \ell_2(\mathbb{Z})\), and for almost every \(t \in T\),
\[ A \|x\|_{\ell_2(\mathbb{Z})}^2 \leq x^*Q(t)x \leq B \|x\|_{\ell_2(\mathbb{Z})}^2. \]

**Proof.** We consider the function
\[ f := \sum_{k \in \mathbb{Z}} x(k) \chi_{U \cap [kp,k(p+1))}, \]
where \(x = [x(k)]_{k \in \mathbb{Z}}\). Since \(\|f\|_{L^2(U)} = \|x\|_{\ell_2(\mathbb{Z})}\), \(f \in L^2(U)\) and
\[ \langle f, S_Gf \rangle = \int_T x_f^*(t)Q(t)x_f(t) dt, \quad x_f(t) := [f(t+kp)]_{k \in \mathbb{Z}}. \]

Let \(E\) be any measurable subset of \(T \cap \text{supp } f\). Then for \(\tilde{E} := \bigcup_{n \in \mathbb{Z}} E + np\), we have
\[ \int_{\tilde{E}} f(t)\overline{S_Gf(t)} dt = \int_E x_f^*(t)Q(t)x_f(t) dt. \]

The assumption that \(\{M_mg_n\}_{m,n \in \mathbb{Z}}\) is a frame for \(L^2(U)\) with the lower frame bound \(A\) and the upper frame bound \(B\) implies that
\[ A \int_E x_f^*(t)x_f(t) dt \leq \int_E x_f^*(t)Q(t)x_f(t) dt \leq B \int_E x_f^*(t)x_f(t) dt. \]
Since $E$ is an arbitrary measurable subset of $T \cap \text{supp}(f)$, we obtain the desired bounds for almost every $t \in T$:

$$A \| x \|_{l^2(\mathbb{Z})}^2 \leq x^* Q(t) x \leq B \| x \|_{l^2(\mathbb{Z})}^2.$$  \qed

As a special case, take the generating set $G$ to be the set of translations of a function $g \in L^2(\mathbb{R})$ by a positive constant $a$:

$$G := \{ g_n := g(\cdot + na) : n \in \mathbb{Z} \},$$

then the modulation system $\{ M_m g_n \}_{m,n \in \mathbb{Z}}$ is a Weyl-Heisenberg system, and Theorem 11 and Theorem 12 yield Corollary 3 and Corollary 4.

The following corollary is also an immediate consequence of Theorem 11 and Theorem 12, in the case when all entries of $Q(t)$ are constants independent of $t$.

**Corollary 13.** Let $G := \{ g_n : n \in \mathbb{Z} \} \subset L^2(\mathbb{R})$ such that

$$\left\| \sum_{n \in \mathbb{Z}} [g_n, g_n] \right\|_{L^\infty(I)} < \infty.$$

Assume that each entry of the corresponding infinite matrix $Q := Q(t)$ of $G$ is constant. Then the following are equivalent:

1. The infinite matrix $Q$ is strongly positive.
2. The modulation system $\{ M_m g_n \}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(U)$.

Corollary 13 generalizes the relationship between Weyl-Heisenberg frames and infinite quadratic forms, as originally appeared in [6]. We note that the infinite matrix $Q$ corresponding to the Weyl-Heisenberg frame

$$\{ e^{im} g_n : g_n = g(\cdot + 2n\pi), m, n \in \mathbb{Z} \}, \quad g = \chi_{[0,2\pi]} + \chi_{[2\pi,4\pi]} + \chi_{[6\pi,8\pi]}$$

is exactly the same as the infinite matrix of Example 5.1 in [6]:

$$Q(t) = \begin{bmatrix}
0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},$$

for any $t \in [0,2\pi)$. It is interesting to note that $Q$ is strongly positive, but is not diagonally dominant.
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