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FIXED POINTS, MAXIMAL ELEMENTS AND EQUILIBRIA OF GENERALIZED GAMES

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Abstract: In this paper we prove some existence theorem for pair of maximal elements for Ψ-condensing correspondences which are either L_C-majorized or $\nu u-$ majorized and whose domain are non-compact sets in locally convex topological vector spaces. We also generalize prefrence correspondences with respect to socio-economic game theory, N-Person game theory, etc.

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1. Introduction

It is well known that the famous Browder Fixed Point Theorem (see [27]) in equivalent to a maximal element theorem (see [9]). As an application of maximal element theorem, the general equilibrium existence theorem can be proved in generalized abstract economy with preference correspondences. In [28], Border established that the existence of equilibria theorem are equivalent to some

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classical fixed point theorem.Thus it is evident that the fixed point theorems have applications in other disciplines (e.g. game theory, optimization theory and economics). At the same time starting from the contexts of other disciplines (e.g. economics) we can restate and reobtain classical results in mathematics.

Thus in this paper, we propose the map to be related to abstract economy where as the map is related Social/Organizational/Government system etc.This new concept will be very helpful in analyzing socio economic concept related to abstract economy, corporate sector economy, public sector economy, other organizational economy as well.

Let E be a vector space and A⊂E. We shall denote C_0A the convex hull of A. If A is a subset of a topological space X, the interior of A in X is denoted by int_XA and the closure of A is denoted by Cl_XA or simply intA and ClA if there is no ambiguity respectively.

Let X be a set, we shall denote by 2^X the family of all subsets of X. Let X and Y besets$and\ FG : X \to 2^Y$. Then

1. The graph of F, denoted by Graph F, is the set $\{(x, y) \in XY : y \in F(x)\}$;
2. The map $F \cap G : X \to 2^Y$ is defined by $(F \cap G)(x) = F(x) \cap G(x)$ for each $x \in X$

Suppose X and Y are topological spaces and $F : X \to 2^Y$, the (1) F is said to be lower semi-continuous (respectively, upper semi-continuous) on X if or any closed (respectively, open) subset U of Y, the set $\{x \in X : F(x) \subset U\}$ is closed (respectively; open) in X;(2) F has open lower sections if $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for each $y \in Y$ and (3) F has a maximal elements if there exists a point $x \in X$such that $F(x) = \emptyset$

If X is a set, Y is subset of a vector space and $F : X \to 2^Y$such that for each $x \in X$, $C_0F(x) \subset Y$ then the map $C_0F : X \to 2^Y$ is defined by$(C_0F)(x) = C_0F(x)$ for each $x \in X$. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are collections of sets and $F_i : \prod_{j \in I} X_j \to 2_i^Y$ for each $i \in I$ then the map $\prod_{i \in I} F_i : \prod_{i \in I} X_i \to 2^{\prod_{j \in I} Y_i}$ is defined by $\left(\prod_{i \in I} F_i\right)(x) = \prod_{i \in I} F_i(x)$ for each $x \in \prod_{i \in I} X_i$

We note that if X is a topological space, Y is a topological vector space and $F : X \to 2^Y$ in lower semi-continuous, it is easy to see that C_0F is lower semi-continuous e.g see{1}.

Let I be a infinite countable set of agents (players). An abstract and socio-economy $G = (X_i A_i P_{2i+1})$ and $J = (X_i, A_i Q_{2i+2})$, respectively are defined as a family of $(X_i, A_i P_{2i+1} Q_{2i+2})$, where X_i is a topological space, $A_{ii} \prod_{j \in X_j} \to 2_i^X$ is a constraint correspondence and $P_{2i+1} Q_{2i+2} : \prod_{j \in X_j} \to 2_i^X$ are two abstract and socio preference maps respectively.

Let X be a Hausdorff topological vector space. Then a mapping $\Psi : 2^X \to C$ is called a measure of non-compactness provided that the following conditions hold for any $AB \in 2^X$;

1. $\Psi(A) = 0$ if and only if A is precompact,
2. $\Psi\left(\bar{C}_0 A\right) = \Psi(A)$, where $\bar{C}_0 A$ denotes the closed convex hull of A,
3. $\Psi(A \bigcup B) = \max?\Psi(A)\Psi(B)$.

It follows from (3) above that if $A \subset B$, then $\Psi(A) \leq \Psi(B)$ The above notion is a generalization of the set-measure of non-compactness and ball-measure of non-compactness [14] defined in terms of a family of seminorms when X is a locally convex topological vector space or a single norm when X is a banach space. For more detail refer, see [7].

Let $\Psi : 2^X \to C$ be a measure of non-compactness of X and $D \subset X$. A mapping $T : D \to 2^X$ is called Ψ-condensing provided that if $z \subset D$ and $\Psi(T(z)) \geq \Psi(z)$, then z is relatively compact. If $T : D \to 2^X$ is compact mapping (i.e. T(D)) is pre-compact. Then T is $\Psi-$ condensing for any measure of non-compactness Ψ. Various $\Psi-$ condensing mappings which are not compact have been considered in [6], [7], [8], [17], etc. Moreover, where the measure of non-compactness Ψ is either the set measure of non-compactness or ball-measure of non-compactness Ψ-condensing mappings are called condensing mappings e.g. [6], [8], [17] etc.

2. Preliminaries

Common Maximal Element. Recall that a frechet space is a locally convex Hausdorff topological vector space whose topology is induced by a complete translation invariant metric.

Lemma 1. *Let D be a non empty closed and convex subset of a frechet space X and $\Psi : 2^X \to C$ be a measure of non-compactness. Suppose a multivalued correspondence $F : D \to 2^D$ is upper semi-continuous and $\Psi-$ condensing with non-compact and convex values. Then F has a fixed point.*

3. Equilibria in Locally Convex Topological Vector Spaces

We give the following theorem;

Theorem 1. *Let$X$ be a non empty closed and convex subset of a locally convex topological vector space E. Suppose $ABP : X \to 2^X$ and $ABQ : X \to 2^X$ are such that:*

1. *For each $x \in X A(x)$ is non empty and $C_0 A(x) \subset B(x)$;*
2. *For each $y \in X A^{-1}(y)$ is compactly open in X;*
3. *$A \bigcap P$ and $A \bigcap Q$ are of class L_C;*
4. *The mapping$A$ is$\Psi-$ condensing.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in \bar{B}(\hat{x})$ and $A(\hat{x}) \bigcap P(\hat{x}) = \emptyset$ and $A(\hat{x}) \bigcap Q(\hat{x}) = \emptyset$.

Proof. Let $M = \left\{x \in X : x \notin \bar{B}(x)\right\}$. Then M is open in X
Define $\varphi : X \to 2^X$ by

$$\varphi(x) = \begin{cases} A(x) \bigcap P(x), & if\, x \notin M; \\ A(x) & if\, x \in M. \end{cases}$$

Since $A \bigcap P$ is of classL_C, for each $x \in X x \notin (A(x) \bigcap P(x))$ and there exists a correspondence $\beta : X \to 2^X$ such that (a) for each $x \in X \beta(x) \subset A(x) \bigcap P(x)$;

(b) for each $y \in X \beta^{-1}(y)$,is compactly open in X; and

(c) $\{x \in X : \beta(x) \neq \emptyset\} = \{x \in X : A(x) \bigcap P(x) \neq \emptyset\}$. Now we also define $\psi : X \to 2^X$ by

$$\psi(x) = \begin{cases} \beta(x), & if\, x \notin M; \\ A(x) & if\, x \in M. \end{cases}$$

Then clearly for each $x \in X$, $\psi(x) \subset \varphi(x)$ and

$$\{x \in X : \psi(x) \neq \emptyset\} = \{x \in X : \varphi(x) \neq \emptyset\}.$$

If $y \in X$, then it is easy to see that $\psi^{-1}(y) = \left(M \bigcup \beta^{-1}(y)\right) \bigcap A^{-1}(y)$and is compactly open in X by the assumption (2) and (b). Moreover, if $x \in M$, then$x \notin \bar{B}(x)$, it follows from (1) that $x \notin C_0 A(x) = C_0 \varphi(x)$; and if $x \notin M$, then $x \notin C_0 (A(x) \bigcap P(x)) = C_0 \varphi(x)$ by (1). This shows that that φ is of class L_C. By the condition (4), φ is also Ψ- condensing since for each $x \in X\, \varphi(x)) \subset A(x)$which is Ψ- condensing. Hence φ satisfies all hypotheses of theorem 3.4 of Ghanshyam Mehta[1, P 693]By theorem 3.4 there exists a point $\hat{x} \in X$ such that $\varphi(\hat{x}) = \emptyset$, As $A(x) \neq \emptyset$, for all $x \in X$, we must have $\hat{x} \in \bar{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$ and the proof is completed.

Similarly it can be proved for $A(\hat{x}) \cap Q(\hat{x}) = \emptyset$.

Now we can prove the following existence theorem.

Theorem 2. *Let* $\Gamma_1 = (\mathrm{X_i};\mathrm{A_i};\mathrm{P_{2i+1}})_{\mathrm{i \in I}}$ *and* $\Gamma_2 = (\mathrm{X_i};\mathrm{A_i};\mathrm{Q_{2i+2}})_{\mathrm{i \in I}}$ *be a pair of generalized games, where* I *is a (Countable or uncountable) set of players. Suppose that the following conditions are satisfied for each* $i \in I$*:*

1. $\mathrm{X_i}$ *is a non empty closed convex, subset of a locally topological vector space* $\mathrm{E_i}$*;*
2. $\mathrm{A_i}$ *is upper semi-continuous with non empty compact convex values;*
3. *The mapping* $A : X \to 2^X$ *defined by* $A(x) = \prod_{i \in I} A_i(x)$ *for each* $x \in X$ *is* $\Psi-$*condensing, where* $\Psi : 2^{\prod_{j \in I} E_j} \to C$ *is a measure of non compactness;*
4. *For each* $x \in X \pi_i(x) \notin \mathrm{A_i}(x) \cap \mathrm{P_{2i+1}}(x)$ *and* $\pi_i(x) \notin \mathrm{A_i}(x) \cap \mathrm{Q_{2i+2}}(x)$*;*
5. *The set* $U_i = \{x \in X : \mathrm{A_i}(x) \cap \mathrm{P_{2i+1}}(x) \neq \emptyset \text{ and } \mathrm{A_i}(x) \cap \mathrm{Q_{2i+2}}(x) \neq \emptyset\}$ *is open in* X*;*
6. $\mathrm{A_i} \cap \mathrm{P_{2i+1}}$ *and* $\mathrm{A_i} \cap \mathrm{Q_{2i+2}}$ *are upper semi-continuous on* $\cup_i$ *such that, for each* $x \in U_i$*,* $(A_i \cap P_{2i+1})(x)$ *and* $(A_i \cap Q_{2i+2})(x)$ *are closed and convex.*

Then there exists $x^* \in X$ *such that, for each* $i \in I \pi_i(x^*) \in A_i(x^*)$ *and* $A_i(x^*) \cap P_{2i+1}(x^*) = \emptyset$ *and* $A_i(x^*) \cap Q_{2i+2}(x^*) = \emptyset$.

Proof. Foreach given i$\in I$, we define a correspondence $F_i : X \to 2_i^X$ by

$$F_i(x) = \begin{cases} A_i(x) \cap P_{2i+1}(x), & if\; x \in U_i \\ A_i(x) & if\; x \notin U_i. \end{cases}$$

Here F_i is upper semi-continuous with nonempty compact convex values. Now define $F : X \to 2^X$ by $F(x) := \prod_{i \in I} F_i(x)$ for each $x \in X$. Then F is upper semi-continuous with nonempty compact convex values by theorem 7.3.14 of Klein and Thompson [2,P-88]. As $F(x) \subset A(x)$ for each $x \in X$ and A *is* $\Psi-$condensing, F is also $\Psi-$ condensing. Note that X is closed and convex subset of the locally convex topoligcal vector space$\prod_{i \in I} E_i$,thus, F satisfies all hypotheses of theorem 2.3 of Ghanshyam Mehta[1, P 692] *By theorem* 2.3; there exists $x^* \in X$, such that $x^* \in F(x^*)$. From our hypothesis (4), it follows that $x^* \in U_i$ for all $i \in I$. Therefore, we have that, for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_{2i+1}(x^*) = \emptyset$Similarly we can prove for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$and $A_i(x^*) \cap Q_{2i+2}(x^*) = \emptyset$

4. Equilibria in Frechet Spaces

In this section; we shall prove the existence of equilibria for generalized games in Frechet spaces which have any (countable or uncountable) set of players. We first have the following existence of equilibria for pair of generalized games.

Theorem 3. *Let* $\Gamma_1 = (\mathrm{X_i};\mathrm{A_i};\mathrm{P_{2i+1}})_{\mathrm{i}\in\mathrm{I}}$ *and* $\Gamma_2 = (\mathrm{X_i};\mathrm{A_i};\mathrm{Q_{2i+2}})_{\mathrm{i}\in\mathrm{I}}$ *be a pair of generalized game and let* $X = \prod_{i\in I} X_i$ *be paracompact, where* I *is a (Countable or uncountable) set of players. Suppose that the following conditions are satisfied for each* $i \in I$*:*

1. X_{i} *is a non empty closed and convex subset of a frechet space* $\mathrm{E_i}$*;*
2. A_{i} *is lower semi-continuous with nonempty closed convex values;*
3. *The mapping* $A : X \to 2^X$ *defined by* $A(x) := \prod_{i\in I} A_i(x)$ *is* $\Psi-$ *condensing for each* $x \in X$*, where* $\Psi : 2^{\prod_{j\in I} E_j} \to C$ *is a measure of noncompactness;*
4. *For each* $x \in X$*,* $\pi_i(x) \notin A_i(x) \cap P_{2i+1}(x)$ *and* $\pi_i(x) \notin A_i(x) \cap Q_{2i+2}(x)$*;*
5. *The set* $U_i = \{x \in X : \mathrm{A_i}(x) \cap \mathrm{P_{2i+1}}(x) \neq \emptyset \text{ and } \mathrm{A_i}(x) \cap \mathrm{Q_{2i+2}}(x) \neq \emptyset\}$ *is closed in* X*;*
6. *The mapping* $\mathrm{A_i} \cap \mathrm{P_{2i+1}}$ *and* $\mathrm{A_i} \cap \mathrm{Q_{2i+2}}$ *are lower semicontinuous on* U_i *such that for each* $x \in U_i$*,* $\mathrm{A_i}(x) \cap P_{2\mathrm{i}+1}(x)$ *and* $\mathrm{A_i}(x) \cap \mathrm{Q_{2i+2}}(x)$ *are closed and convex. Then there exists* $x^* \in X$ *such that, for each* $i \in I$

$\pi_i(x^*) \in A_i(x^*)$ *and* $\mathrm{A_i}(x^*) \cap P_{2\mathrm{i}+1}(x^*) = \emptyset$*;* $\mathrm{A_i}(x^*) \cap Q_{2\mathrm{i}+2}(x^*) = \emptyset$.

Proof. Foreach given $i \in I$, we define a correspondence $F_i : X \to 2_i^X$ by

$$F_i(x) = \begin{cases} A_i(x) \cap P_{2i+1}(x), & if\ x \in U_i \\ A_i(x) & if\ x \notin U_i. \end{cases}$$

Here F_i is lower semi-continuous with nonempty closed and convex values. Then by Michael's selection theorem [3, theorem 3.2″] and remark of Aubin [5,P551] there exists a continuous (single – valued) mapping $f_i : X \to X_i$ such that $f_i(x) \in F_i(x)$ for each $x \in X$.Now define $f : X \to X$ by $f(x) := \{f_i(x)\}$, for each $x \in X$, of course f is continuous and $f(x) \in F(x) = \prod_{i\in I} F_i(x) \subset \prod_{i\in I} A_i(x)$. Note that A *is* $\Psi-$condensing, then f is $\Psi-$ condensing. Since X is a nonempty closed and convex subset of the locally convex topological vector space $\prod_{i\in I} E_i$, f satisfies all hypotheses of theorem 2.3 of Ghanshyam Mehta

[1,P692]. Now it is obvious that there exists $x^* \in X$ such that $f(x^*) = x^*$ Note that, for each $i \in I$, if $x_i^* \in U_i$, then $\pi_i(x^*) = f_i(x^*) \in A_i(x^*) \cap P_{2i+1}(x^*)$, which contradicts our assumption (4). Hence, for each $i \in I$, we must have $\pi_i(x^*) \notin U_i$ and thus, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_{2i+1}(x^*) = \emptyset$

Similarly it can be established that $\pi_i(x^*) \in A_i(x^*)$and $A_i(x^*) \cap Q_{2i+2}(x^*) \neq \emptyset$.

Theorem 4. *Let* $\Gamma_1 = (X_i; A_i; P_{2i+1})_{i \in I}$ *and* $\Gamma_2 = (X_i; A_i; Q_{2i+2})_{i \in I}$ *be a pair of generalized game where* I *is any set of players. Suppose that the following conditions are satisfied for each* $i \in I$*:*

1. X_i *is a non empty closed and convex subset of a frechet space* E_i*;*
2. A_i *is upper semi-continuous with nonempty compact convex values;*
3. *The mapping* $A : X \to 2^X$ *defined by* $A(x) = \prod_{i \in I} A_i(x)$ *for each* $x \in X$ *is* $\Psi-$ *condensing, where* $\Psi : 2^{\prod_{j \in I} E_j} \to C$ *is a measure of non-compactness;*
4. *The set* $U_i := \{x \in X : A_i(x) \cap P_{2i+1}(x) \neq \emptyset$ *and* $A_i(x) \cap Q_{2i+2}(x) \neq \emptyset \}$ *is para-compact and open in* X
5. *The mapping* $A_i \cap P_{2i+1}$ *and* $A_i \cap Q_{2i+2}$ *are lower semi-continuous on* U_i *such that for each* $x \in U_i$, $A_i(x) \cap P_{2i+1}(x)$ *and* $A_i(x) \cap Q_{2i+2}(x)$ *are closed and convex.*

Then there exists $x^* \in X$ *such that, for each* $i \in I$ *either*

$$\pi_i(x^*) \in A_i(x^*) \cap P_{2i+1}(x^*)$$

and

$$\pi_i(x^*) \in A_i(x^*) \cap Q_{2i+2}(x^*)$$

or

$$\pi_i(x^*) \in A_i(x^*)$$

and

$$A_i(x^*) \cap P_{2i+1}(x^*) \neq \emptyset;$$

$$\pi_i(x^*) \in A_i(x^*)$$

and

$$A_i(x^*) \cap Q_{2i+2}(x^*) \neq \emptyset.$$

Proof. For each i$\in I$, by our assumption (5), theorem 3.2" of Michael [3, p. 367] and remark of Aubin [5, P551], it follows that there exists a (Single-Valued) mapping $f_i : U_i \to X_i$ such that $f_i(x) \in A_i(x) \cap P_{2i+1}(x)$ for each $x \in U_i$

$$F_i(x) = \begin{cases} \{f_i(x)\}, & if\, x \in U_i; \\ A_i(x), & if\ x \notin U_i. \end{cases}$$

Then the conditions (2), (4) and lemma 4.2 of Ghanshyam Mehta [1,P694] imply that F_i is upper semicontinuous with nonempty compact convex values. Let $F : X \to 2^X$ be a mapping defined by $F(x) := \prod_{i \in I} F_i(x)$ for each $x \in X$ Then F is upper semi-continuous with nonempty compact convex values by theorem 7.3.14 of [2; P88] and moreover, $F(x) \subset A(x)$ for each $x \in X$. Since A is $\Psi-$ condensing, it is clear that F is alsoΨcondensing. Therefore, by theorem 2.3 of Ghanshyam Mehta [1,P692] there exists $x^* \in X$ such that $x^* \in F(x^*)$. It follows that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_{2i+1}(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_{2i+1}(x^*) = \emptyset$

Similarly, it can be established that for each $i \in I$, either

$$\pi_i(x^*) \in A_i(x^*) \cap Q_{2i+2}(x^*)$$

or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap Q_{2i+2}(x^*) = \emptyset$.

Theorem 3.1. *Let* $\Gamma_1 = (X_i; A_i; P_{2i+1})_{i \in I}$ *and* $\Gamma_2 = (X_i; A_i; Q_{2i+2})_{i \in I}$ *be a pair of generalized game and let* $X = \prod_{i \in I} X_i$*be para-compact, where* I *is a (Countable or Uncountable) set of players. Suppose that the following conditions are satisfied for each* $i \in I$*;*

1. X_i *is a non empty closed and convex subset of a finite, dimensional space*E_i*;*

2. A_i *is lower semi-continuous with nonempty convex values (not necessarily closed);*

3. *The mapping* $A : X \to 2^X$ *defined by* $A(x) := \prod_{i \in I} A_i(x)$ *for each* $x \in X$ *is*$\Psi-$ *condensing, where*$\Psi : 2^{\prod_{j \in I} E_j} \to C$ *is a measure of non-compactness;*

4. *For each* $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_{2i+1}(x)$ *and* $\pi_i(x) \notin A_i(x) \cap Q_{2i+2}(x)$*;*

5. *The set* $U_i := \{x \in X : A_i(x) \cap P_{2i+1}(x) \neq \emptyset \text{ and } A_i(x) \cap Q_{2i+2}(x) \neq \emptyset\}$ *is closed in* X*;*

6. *The mapping* $A_i \cap P_{2i+1}$ *and* $A_i \cap Q_{2i+2}$ *are lower semi-continuous on* U_i *such that for each* $x \in U_i$, $A_i(x) \cap P_{2i+1}(x)$ *and* $A_i(x) \cap Q_{2i+2}(x)$ *are convex (not necessarily closed).*

Then there exists $x^* \in X$ *such that, for each* $i \in I$

$\pi_i(x^*) \in A_i(x^*)$ *and* $A_i(x^*) \cap P_{2i+1}(x^*) = \emptyset$ *and also* $A_i(x^*) \cap Q_{2i+2}(x^*) = \emptyset$.

Theorem 4.1. *Let* $\Gamma_1 = (X_i; A_i; P_{2i+1})_{i \in I}$ *and* $\Gamma_2 = (X_i; A_i; Q_{2i+2})_{i \in I}$ *be a pair of generalized game where* I *is any set of players. Suppose that the following conditions are satisfied for each* $i \in I$*:*

1. X_i *is a non empty closed and convex subset of a finite dimensional space* E_i*;*
2. A_i *is upper semi-continuous with nonempty compact convex values;*
3. *The mapping* $A : X \to 2^X$ *defined by* $A(x) = \prod_{i \in I} A_i(x)$ *for each* $x \in X$ *is* $\Psi-$ *condensing; where* $\Psi : 2^{\prod_{j \in I} E_j} \to C$ *is a measure of non-compactness;*
4. *The set* $U_i := \{x \in X : A_i(x) \cap P_{2i+1}(x) \neq \emptyset \text{ and } A_i(x) \cap Q_{2i+2}(x) \neq \emptyset\}$ *is para-compact and open in* X
5. *The mapping* $A_i \cap P_{2i+1}$ *and* $A_i \cap Q_{2i+2}$ *are lower semi-continuous on* U_i *such that for each* $x \in U_i$, $A_i(x) \cap P_{2i+1}(x)$ *and* $A_i(x) \cap Q_{2i+2}(x)$ *are convex (not necessarily closed).*

Then there exists $x^* \in X$ *such that, for each* $i \in I$ *either*

$$\pi_i(x^*) \in A_i(x^*) \cap P_{2i+1}(x^*)$$

or $\pi_i(x^*) \in A_i(x^*)$ *and* $A_i(x^*) \cap P_{2i+1}(x^*) = \emptyset$.

Similarly, there exists $x^* \in X$ *such that for each* $i \in I$, *either*

$$\pi_i(x^*) \in A_i(x^*) \cap Q_{2i+2}(x^*)$$

or $\pi_i(x^*) \in A_i(x^*)$ *and* $A_i(x^*) \cap Q_{2i+2}(x^*) = \emptyset$.

Finally, we have the following;

Theorem 5. *Let* I *be any set of players. For each* $i \in I$, *suppose that* X_i *is a nonempty compact convex subset of a finite-dimensional space* E_i *and* $P_{2i+1} : X = \prod_{j \in I} X_j \to 2_i^X$ *and* $Q_{2i+2} : X = \prod_{j \in I} X_j \to 2_i^X$ *are lower semi-continuous* *with convex values on the set* $U_i := \{x \in X : P_{2i+1}(x) \neq \emptyset$ and $Q_{2i+2}(x) \neq \emptyset\}$. *If* U_i *is paracompact and either open or closed in* X *for each* $i \in I$, *then there exists* $x^* \in X$ *such that for each* $i \in I$ *either* $\pi_i(x^*) \in P_{2i+1}(x^*)$ *or* $P_{2i+1}(x^*) = \emptyset$ *and* $\pi_i(x^*) \in Q_{2i+2}(x^*)$ *or* $Q_{2i+2}(x^*) = \emptyset$.

Proof. Fix an arbitrary $i \in I$We shall first show that there exists an upper semicontinuous map $F_i : X \to 2_i^X$ with nonempty compact convex values such that $F_i(x) \subset P_{2i+1}(x)$for all $x \in U_i$. Indeed, since $P_{2i+1}:U_i \to 2_i^X$ is lower semicontinuous by theorem 3.2" of Michael [3,P.367] and remark of Aubin[5,P.551], let $f_i : U_i \to X_i$ be a single valued continuous map such that $f_i(x) \in P_{2i+1}(x)$ for all $x \in U_i$

Case 1. Suppose that U_i is open in X. Define $F_i : X \to 2_i^X$by;

$$F_i(x) = \begin{cases} \{f_i(x)\}, & if\, x \in U_i \\ X_i, & if\ x \notin U_i. \end{cases}$$

Then by lemma 4.2 of Ghanshyam Mehta [1;P694],F_i is upper semicontinuous and has non-empty compact convex values such that $F_i(x) = \{f_i(x)\} \subset P_{2i+1}(x)$for all $x \in U_i$.

Case 2. Suppose that U_i is closed in X. Define $G_i : X \to 2_i^X$by;

$$G_i(x) = \begin{cases} \{f_i(x)\}, & if\, x \in U_i \\ X_i, & if\ x \notin U_i. \end{cases}$$

Then by lemma 4.2 of GhanshyamMehta[1,P694] G_i is lower semicontinuous with nonempty compact convex values. By Theorem 3.2" of Michael [3,P.367] again, there exists a single-valued continuous map $g_i : X \to X_i$ such that $g_i(x) \in G(x)$ for all $x \in X$. Let $F_i : X \to 2_i^X$ be defined by $F_i(x) = \{g_i(x)\}$ for each $x \in X$. Then $F_i$is an upper semicontinuous (in fact, continuous) with nonempty compact convex values such that $F_i(x) = \{g_i(x)\} = \{f_i(x)\} \subset P_{2i+1}(x)$ for all $x \in U_i$

Now define $F : X \to 2^X$ by $F(x) = \prod_{i \in I} F_i(x)$ for each $x \in X$. Then F is upper semicontinuous (by theorem 7.3.14 of [2,P.88]) with nonempty compact and convex values. Since X is compact, by Fan[10] or Glicksberg[11] fixed point theorem, there exists $x^* \in X$ such that $x^* \in F(x^*)$. It follows that for each $i \in I$, either $\pi_i(x^*) \in P_{2i+1}(x^*)$ or $P_{2i+1}(x^*) = \emptyset$

Similarly it can be established that for each $i \in I$, either $\pi_i(x^*) \in Q_{2i+2}(x^*)$ or $Q_{2i+2}(x^*) = \emptyset$

5. Remarks

Our results generalize and improve corresponding results given by Tulcea [12], [15], Yannelis and Prabhakar [9], D. Gale and A. Mas-Colell [16], [19], Ghanshyam Mehta [1], [17], S.Y. Chang [18], M. Florenzano [20], G. Meha, K.K. Tan and X.Z. Yuan [21], and S. Toussain [22].

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