BOUNDARY CHARACTERISTIC ORTHOGONAL POLYNOMIALS FOR SINGULARLY PERTURBED PROBLEMS

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Abstract: In this paper we generate fitted mesh using the boundary characteristic orthogonal polynomials for numerically solving singularly perturbed boundary value problems. The method is based upon the orthogonal collocation method. Boundary characteristic orthogonal polynomials (BCOPs) are generated using the Gram - Schmit process from a set of linearly independent functions which also satisfy the given boundary conditions. The procedure is illustrated by taking several examples. After obtaining the fitted mesh, the problems have been solved using upwind finite difference method.

AMS Subject Classification: 65L11, 65L10, 65L50, 65M50
Key Words: boundary characteristic orthogonal polynomials (BCOPs), singularly perturbed boundary values problems (SPPs), fitted mesh methods, upwind finite difference

1. Introduction

Orthogonal polynomials have been extensively used in numerical approximations, for example, the famous Legendre, Chebyshev polynomials and many more. The importance of orthogonal projection and orthogonal decomposition,
particularly in the solution of systems of linear equations and in the least square
data fitting is also well known. Now a large number of books and research pa-
pers are available on orthogonal polynomials and their applications and some
good references can be found in [1, 2, 3].

Problems in which a small perturbation parameter, say $\epsilon$ is multiplied to
the highest derivative arise in various fields of science and engineering, for
instance fluid mechanics, elasticity, hydrodynamics, etc. The main concern
with such problems is the rapid growth or decay of the solution in one or more
narrow "layer region(s)". These kinds of problems are known in the literature
as singularly perturbed problems (SPPs).

Singular perturbation problems in consideration have shocks as boundary
layers or interior layers. For such kinds of problems the solution can be smooth
in most of the solution domain with small area where the solution changes very
quickly. To approximate their solution it is well known (see [4, 5]) that the
classical numerical methods cannot be used on uniform meshes; the reason is
that the error is unbounded [6] for arbitrary values of the singularly perturbed
parameter, $\epsilon$. So when solving such problems numerically, one would like to
adjust the discretization to the solution. In terms of mesh generation, we want
to have many points in the area where the solution has strong variations and a
few points in the area where the solution has weak variations and such method
is known as fitted mesh methods.

In orthogonal collocation, we first generate the orthogonal polynomials and
then we find the roots of those orthogonal polynomials (Each polynomial in
an orthogonal sequence has all $n$ of its roots real, distinct, and strictly inside
the interval of orthogonality) and treat them as collocation points. The choice
of collocation points is also critical and should not be arbitrary in realistic
problems. The dependence of the roots according to the mesh requirement lies
on the choice of the weight functions used in defining the inner product. In
Section 2 we discuss how to find the BCOPs and the corresponding roots and
numerical results have been presented in Section 3.

2. Generation of Mesh using BCOPs

Let us first define the inner product in the functional space for two functions
$f(x)$ and $g(x)$ defined over the domain $D \in R^n$ by

$$< f, g > = \int_D w(x)f(x)g(x)dD \tag{1}$$
where \( w(x) \) is the suitable chosen weight function according to the mesh defined over \( D \). The induced norm of a function using above inner product is, therefore, given as

\[
\|f\|^2 = \int_D w(x)f^2(x)dD. \tag{2}
\]

To generate an orthogonal sequence, we can start with the set

\[ \{h(x)f_i(x)\}, i = 0, 1, 2, 3,... \tag{3} \]

where \( h(x) \) is the chosen function which satisfy the given boundary conditions of a differential equation and \( f_i(x) \) are the linearly independent functions over the domain \( D \). Note that each \( h(x)f_i(x) \) will, therefore, also satisfy the same boundary conditions (if we have zero boundary conditions). Otherwise for non-zero boundary conditions, we have to choose \( f_i(x) \) which also satisfy the boundary conditions and independency.

To generate an orthogonal sequence \( \phi_i \), we apply the well known Gram-Schmidt process, which is given as

\[
\phi_1 = hf_1 \tag{4}
\]

\[
\phi_i = hf_i - \sum_{j=1}^{i-1} c_{ij}\phi_j, \quad i = 2, 3, 4,... \tag{5}
\]

where

\[
c_{ij} = \frac{\langle hf_i, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}. \tag{6}
\]

The orthogonal sequence can also be normalized by dividing each \( \phi_i \) by its norm.

**2.1. BCOPs Approximations**

First we try to approximate a given function, using the method of least square, which has boundary layers on both the boundaries by generating corresponding BCOPs. Suppose we take original function \( F(x) \) and write it as a linear combination of generated BCOPs \( \phi_i \) as

\[
F(x) = \sum_{j=0}^{N} d_j \phi_j(x). \tag{7}
\]

Then the \( d_j \)s can be calculated using the concept of orthogonality (Fourier-Legendre type).

\[
d_j = \frac{\langle F, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}. \tag{8}
\]
Suppose we start with the function
\[ F(x) = \frac{\exp((x-1)/\sqrt{\epsilon}) + \exp(-x/\sqrt{\epsilon})}{(1 + \exp(-1/\sqrt{\epsilon}))} - \cos(\pi x)^2; \quad x \in (0, 1), \]  
(9)
which has boundary layers on both the sides of the interval for small values of parameter \( \epsilon \) with zero boundary conditions. Let us we start with \( h(x) = x(1-x) \), as it satisfies the zero boundary conditions. The inner product for this function can be defined as
\[ < f, g > = \int_0^1 \frac{f(x)g(x)}{\sqrt{x(1-x)}} \, dx, \]  
(10)
with weight function \( w(x) = 1/\sqrt{x(1-x)} \), as it will be very helpful for generating finer mesh at the boundary points 0 and 1. As we know that the computation of the integral involved will become simpler if we deal with the polynomial functions \( f_i \)'s. So we start with one of the obvious choice of \( f_i \) as
\[ f_i = \{1, x, x^2, x^3, \ldots\} \]  
(11)
It is clear that the function \( hf_i \) also satisfies the zero boundary conditions. Using the above procedure to generate BCOPs, Some \( \phi_i \)'s can be given as
\[ \phi_0 = \frac{8x(1-x)\sqrt{6}}{3\sqrt{\pi}}; \]  
(12)
\[ \phi_1 = \frac{32(x^2(1-x) - (1/2)x(1-x))}{\sqrt{\pi}}; \]  
(13)
and so on. Similarly \( \phi_9 \) can also be given as
\[ \phi_9 = -2.330769986x(x-1)(2x-1)(3.27680 \times 10^5x^8 - 1.310720 \times 10^6x^7 \\
+ 2.146304 \times 10^6x^6 - 1.851392 \times 10^6x^5 + 9.02144 \times 10^5x^4 \\
- 2.47808 \times 10^5x^3 + 36256x^2 - 2464x + 55). \]  
(14)
For the above \( F(x) \), equation (7) can be written as
\[ F(x) = \sum_{j=0}^{9} d_j \phi_j(x) \]  
(15)
where $\phi_i$s as generated above. The corresponding $d_i$s are found using orthogonal property as

$$d_0 = -0.556648, \quad d_1 = -2.246921 \times 10^{-10},$$
$$d_2 = -0.756039, \ldots, d_8 = -0.025290, \quad d_9 = -1.16538 \times 10^{-13}.$$  \hfill \text{(16)}$$

All the $\phi_i$s as used in equation (15) has been plotted in the Figure 1 and the corresponding approximated $F(x)$ is plotted in the Figure 2. As we can see in the Figure 2 that both the functions are coinciding completely with each other.

### 3. Numerical Results

Now we generate the fitted mesh for a given problem using the BCOPs method and solve the singularly perturbed problems using upwind finite difference methods on non-uniform meshes.

**Test Problem 1.** (see [7]) We consider 1D linear reaction-diffusion problem as

$$-\epsilon u''(x) + u(x) = -\cos^2(\pi x) - 2\epsilon \pi^2 \cos(2\pi x), \quad x \in (0, 1), \quad 0 < \epsilon << 1,$$  \hfill \text{(17)}
with boundary conditions

\[ u(0) = 0, \ u(1) = 0 \]  \hspace{1cm} (18)

whose exact solution (as also discussed in equation (9)) is

\[ u(x) = \frac{\exp(-1(1-x)/\sqrt{\epsilon}) + \exp(-x/\sqrt{\epsilon})}{1 + \exp(-1/\sqrt{\epsilon})} - \cos^2(\pi x). \]

This problem has regular boundary layers of width \( O(\sqrt{\epsilon}) \) at \( x = 0 \) and \( x = 1 \).

Using the same inner product as defined above in equation (10) we get the same orthonormal polynomials as given above equations (12 -14), we find the roots of such polynomials and apply orthogonal collocation method to compute the numerical solution. The roots of few BCOPs other than 0,1 are given in Table 1.

In Table 2, we have shown the maximum error for various values of perturbation parameter \( \epsilon \) by using different BCOPs. Results looks better even for smaller values of \( \epsilon \). It is also clear from the Table that for each \( \phi_i \)s (say \( \phi_5 \)) maximum error is reducing as for higher degree BCOPs, the more roots are falling in the boundary layer region.

Similarly we can solve another type of problem i.e convection diffusion equation which has only one boundary layer.
Table 1: Roots for the corresponding $\phi_i$s for test problem 1

<table>
<thead>
<tr>
<th>$\epsilon \downarrow \phi_i \rightarrow$</th>
<th>$\phi_5$</th>
<th>$\phi_6$</th>
<th>$\phi_7$</th>
<th>$\phi_8$</th>
<th>$\phi_9$</th>
<th>$\phi_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0706</td>
<td>0.0555</td>
<td>0.0457</td>
<td>0.0384</td>
<td>0.0330</td>
<td>0.0287</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0501</td>
<td>0.0449</td>
<td>0.0392</td>
<td>0.0341</td>
<td>0.0299</td>
<td>0.0265</td>
</tr>
<tr>
<td>0.001</td>
<td>0.0288</td>
<td>0.0179</td>
<td>0.0049</td>
<td>0.0144</td>
<td>0.0244</td>
<td>0.0297</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.0095</td>
<td>0.0156</td>
<td>0.0224</td>
<td>0.0283</td>
<td>0.0314</td>
<td>0.0308</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.000964</td>
<td>0.0016</td>
<td>0.0026</td>
<td>0.0039</td>
<td>0.0057</td>
<td>0.0080</td>
</tr>
</tbody>
</table>

Table 2: Maximum error for various values of $\epsilon$ and mesh points for test problem 1

Test Problem 2. (see [5]) Convection diffusion equation in 1D can be given as

$$\epsilon u''(x) + 2u'(x) = 0, x \in (0, 1), 0 < \epsilon << 1,$$

with boundary conditions

$$u(0) = 1, \ u(1) = 0$$

This equation has boundary layer of order $O(\epsilon)$ at $x = 0$ and its exact solution is given as

$$u(x) = \frac{e^{-2x/\epsilon} - e^{-2/\epsilon}}{1 - e^{-2/\epsilon}}.$$  

To find the fitted mesh (finer mesh at $x = 0$) we define the inner product as

$$< f, g > = \int_0^1 \frac{f(x)g(x)}{\sqrt{x}} dx,$$

with weight function as $w(x) = 1/\sqrt{x}$ to have finer mesh near $x = 0$. In this case we choose $f_i$s as

$$f_i = \{1 - x, (1 - x)^2, (1 - x)^3, (1 - x)^4, \ldots\}$$
and \( h(x) = 1 - x \). We can see that each \( h f_i \) satisfies the non-zero boundary conditions as per the requirement. Using the above procedure to generate BCOPs, we give some \( \phi \) as mentioned below

\[
\phi_0 = \frac{\sqrt{15}(1 - x)}{4}; \\
\phi_1 = -\frac{3\sqrt{5}(-7(1 - x)^2 + 6(1 - x))}{8}; \\
\phi_5 = (.26757(x - 1))(7429x^5 - 14535x^4 + 9690x^3 - 2550x^2 + 225x - 3); \\
\phi_{10} = -(0.00019037(x - 1))(1.14353210^{10}x^{10} - 5.05281610^{10}x^9 \\
+ 9.42781510^{10}x^8 - 9.66955410^{10}x^7 + 5.945469010^{10}x^6 - 2.24229110^{10}x^5 \\
+ 5.09611610^{9}x^4 - 6.5756310^8x^3 + 4.251410^7x^2 - 1.04975010^6x + 4199); \\
\]

(24)

and so on. Similarly we can generate much higher order BCOPs. For this example we have generated the functions upto \( \phi_{14} \). The roots of the corresponding BCOPs (since \( x = 0 \) (left boundary point) is not the root for any \( \phi \) because of our choice of BCOPs, we add \( x = 0 \) as an additional root for all the \( \phi \) other than 0,1 are given in Table 3.

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \phi )</th>
<th>roots ( \rightarrow )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_{10} )</td>
<td>.00490</td>
<td>.04354</td>
</tr>
<tr>
<td>( \phi_{11} )</td>
<td>.00413</td>
<td>.03677</td>
</tr>
<tr>
<td>( \phi_{14} )</td>
<td>.00266</td>
<td>.02377</td>
</tr>
<tr>
<td></td>
<td>.78150</td>
<td>.86273</td>
</tr>
</tbody>
</table>

Table 3: Roots for the corresponding \( \phi \) for test problem 2

Table 4 discuss the maximum error for the test problem 2 for various values of \( \epsilon \) verses various \( \phi \). In this case too maximum error reduces as the value of \( \epsilon \) becomes smaller.

**Test Problem 3.** (see [8]) We consider another interior layer SPP problem as

\[
\epsilon u''(x) + xu'(x) = -\epsilon \pi^2 \cos(\pi x) - \pi x \sin(\pi x), x \in (-1, 1), 0 < \epsilon << 1, \quad (25)
\]

with boundary conditions

\[
u(-1) = -2, \ u(1) = 0. \quad (26)
\]
Its exact solution is not known and for small $\epsilon$ it gives a turning point near $x = 0$. Therefore we need finer mesh near the turning point to resolve the interior layer for small $\epsilon$. To find the fitted mesh (finer mesh at $x = 0$) we define the inner product as

$$< f, g > = \int_{-1}^{1} \frac{f(x) g(x)}{\sqrt{1-x^2}} dx,$$

(27)

with weight function as $w(x) = 1/\sqrt{1-x^2}$ to have finer mesh near $x = 0$. In this case we choose $f_i$s as

$$f_i = \{x, x^2, x^3, x^4, \ldots\}$$

(28)

and $h(x) = (x - 1)$. We can see that in this case $h(x)f_i$s do not satisfy the boundary conditions (i.e non zero b.c.) but at the point $x = 0$, where the interior layer occur, all the $h(x)f_i$s satisfy the given zero conditions. The roots of the few corresponding BCOPs other than 1 are given in Table 5 below.

<table>
<thead>
<tr>
<th>$\epsilon \downarrow \phi_i \rightarrow$</th>
<th>$\phi_5$</th>
<th>$\phi_{10}$</th>
<th>$\phi_{11}$</th>
<th>$\phi_{12}$</th>
<th>$\phi_{13}$</th>
<th>$\phi_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2421</td>
<td>0.2333</td>
<td>0.2285</td>
<td>0.2238</td>
<td>0.2194</td>
<td>0.2152</td>
</tr>
<tr>
<td>0.01</td>
<td>0.3107</td>
<td>0.2730</td>
<td>0.2572</td>
<td>0.2463</td>
<td>0.2405</td>
<td>0.2391</td>
</tr>
<tr>
<td>0.001</td>
<td>0.0520</td>
<td>0.1525</td>
<td>0.1757</td>
<td>0.1990</td>
<td>0.2218</td>
<td>0.2435</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.0055</td>
<td>0.0178</td>
<td>0.0210</td>
<td>0.0245</td>
<td>0.0283</td>
<td>0.0323</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.00054</td>
<td>0.0018</td>
<td>0.0021</td>
<td>0.0025</td>
<td>0.0029</td>
<td>0.0033</td>
</tr>
</tbody>
</table>

Table 4: Maximum error for various values of $\epsilon$ and mesh points for test problem 2

Figures 3 and 4 show the results for different values of $\epsilon$ and for $\epsilon = .000001$, we can see a thin interior layer at $x = 0$. | $\phi_i \downarrow roots \rightarrow$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_7$</td>
<td>.20449</td>
<td>.56782</td>
<td>.84597</td>
<td>-.18876</td>
</tr>
<tr>
<td>$\phi_8$</td>
<td>.00620</td>
<td>.35035</td>
<td>.65300</td>
<td>.87768</td>
</tr>
<tr>
<td>$\phi_9$</td>
<td>.16224</td>
<td>.46129</td>
<td>.71569</td>
<td>.90057</td>
</tr>
<tr>
<td>$\phi_{10}$</td>
<td>.00414</td>
<td>.28704</td>
<td>.54703</td>
<td>.76304</td>
</tr>
</tbody>
</table>

Table 5: Roots for the corresponding $\phi_i$s for test problem 3
Test Problem 4. (see [9]) We consider 2D linear reaction-diffusion problem as
\[-\epsilon^2 \Delta u(x, y) + 2u(x, y) = f(x, y), \text{ in } \Omega = (0, 1) \times (0, 1). \tag{29}\]
f has been chosen such that the exact solution of Eq. (29) is given as
\[u(x, y) = \]
Table 6: Maximum error for various values of \( \epsilon \) and mesh points for test problem 4

\[
\begin{array}{|c|ccccccc|}
\hline
\phi_i \downarrow \epsilon \rightarrow & 2^{-5} & 2^{-6} & 2^{-7} & 2^{-8} & 2^{-9} & 2^{-10} & 2^{-15} \\
\hline
\phi_8 & 0.00700 & 0.02057 & 0.01503 & 0.00424 & 0.00106 & 0.00026 & 0.00000026 \\
\phi_{10} & 0.01570 & 0.00685 & 0.02144 & 0.00848 & 0.00217 & 0.00054 & 0.00000053 \\
\hline
\end{array}
\]

This \( u(x, y) \) has typical boundary layers of width \( O(\epsilon) \). Since the exact solution is known, we can accurately measure the maximum error as given in the Table 6. We have taken the same \( \phi_i \)'s as discussed in example 1 (with zero boundary conditions). To solve 2D problem we have used the same 1D derivative matrices to generate Laplacian using tensor products, also known as *Kronecker products* as discussed in [10]. Table 6 gives the maximum errors, and Figure 5 shows the exact and computed results.

4. Conclusion

In this paper we have introduced the BCOPs methods to study the singularly perturbed problems having boundary and interior layers in one and two dimensions. The one of the advantage of BCOPs is that all the corresponding polynomials also satisfy the boundary conditions and there is no need to satisfy them separately. BCOPs have been used to solve problem in the vibration by treating them as the bases functions for the approximation. But in this paper we have taken the advantage of orthogonal collocation to develop fitted meshes to deal with SPPs.

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Figure 5: Solutions for example 4 (29)

References


