DECOMPOSITIONS OF VARIOUS COMPLETE GRAPHs INTO ISOMORPHIC COPIES OF THE 4-CYCLE WITH A PENDANT EDGE

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Abstract: Necessary and sufficient conditions are given for the existence of isomorphic decompositions of the complete bipartite graph, the complete graph with a hole, and the λ-fold complete graph into copies of a 4-cycle with a pendant edge.

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1. Introduction

A g-decomposition of graph G is a set of subgraphs of G, γ = {g₁, g₂, ..., gₙ}, where gᵢ ≅ g for i ∈ {1, 2, ..., n}, E(gᵢ) ∩ E(gⱼ) = ∅ for i ≠ j, and ∪ⁿᵢ₌₁E(gᵢ) = E(G). The gᵢ are called blocks of the decomposition. When G is a complete...
graph, the $g$-decomposition is often called a graph design. The study of graph designs and graph decompositions is a vibrant area of research [3, 4, 8]. Several studies have centered on $g$-decompositions of complete graphs into copies of a given graph $g$ with a small number of vertices [1, 2, 5, 6, 7]. This study takes a slightly different approach and concentrates on $g$-decompositions of different types of complete graphs for a given $g$. The $g$ which is the topic of this study is the 4-cycle with a pendant edge. We denote this graph as $H$. That is, $V(H) = \{a, b, c, d, e\}$ and $E(H) = \{(a, b), (b, c), (c, d), (a, d), (a, e)\}$; we represent this $H$ as $[a, b, c, d; e]$. See Figure 1.1. An $H$-decomposition of $K_v$ exists if and only if $v \equiv 0$ or 1 (mod 5), $v \geq 10$ [1].

Figure 1.1: We denote this graph as $H = [a, b, c, d; e]$

2. $H$-Decompositions of $K_{m,n}$

We assume the partite sets of the complete bipartite graph, $K_{m,n}$, are $V_m = \{0_1, 1_1, \ldots, (m-1)_1\}$ and $V_n = \{0_2, 1_2, \ldots, (n-1)_2\}$.

**Theorem 2.1.** There is an $H$-decomposition of $K_{m,n}$ if and only if $mn \equiv 0$ (mod 5), $m \geq 5$, and $n \geq 2$.

**Proof.** Since $|E(K_{m,n})| = mn$, and $H$ has 5 edges, $mn \equiv 0$ (mod 5) is necessary. Since $H$ is bipartite with one partite vertex set consisting of 2 vertices, both $m$ and $n$ must be at least 2.

Graph $H$ is bipartite itself and each of its partite sets has a single vertex of odd degree. If $m = 3$ and $n = 5k$ then an $H$-decomposition of $K_{m,n}$ would require $3k$ copies of $H$. However, $3k$ copies of $H$ can only produce a bipartite graph with at most $3k$ odd degree vertices in each partite set. But if $m = 5k$ then one of the partite sets contains $5k$ vertices of odd degree. So no $H$-decomposition of $K_{m,n}$ exists when $m = 3$ and $n = 5k$. Therefore $m \geq 5$.

**Case 1.** Suppose $m \equiv 0$ (mod 2) and $n \equiv 0$ (mod 5). Then an $H$-decomposition of $K_{m,n}$ is given by

\$
[(1+2i)_1, (5j)_2, (2i)_1, (1+5j)_2; (2+5j)_2], [(2i)_1, (3+5j)_2, (1+2i)_1, (4+5j)_2; \ldots]
\$
Throughout, we reduce vertex labels by a modulus appropriate for the vertex set we use.

**Case 2.** Suppose \( m \equiv 1 \pmod{2} \), \( m \geq 5 \), and \( n \equiv 0 \pmod{5} \). Then and \( H \)-decomposition of \( K_{m,n} \) is given by

\[
\{(0, (5j)2, 1), (1 + 5j)2; (4 + 5j)2\}, [31, (1 + 5j)2, 41, (2 + 5j)2; (5j)2],
\]

\[
[21, (2 + 5j)2, 01, (3 + 5j)2, (1 + 5j)2], [11, (3 + 5j)2, 31, (4 + 5j)2; (2 + 5j)2],
\]

\[
[41, (5j)2, 21, (4 + 5j)2; (3 + 5j)2], [(6 + 2i)1, (5j)2, (5 + 2i)1, (1 + 5j)2; (2 + 5j)2],
\]

\[
[(5 + 2i)1, (3 + 5j)2, (6 + 2i)1, (4 + 5j)2; (2 + 5j)2] \mid i = 0, 1, \ldots, (m - 5)/2 - 1, j = 0, 1, \ldots, n/5 - 1\}.
\]

In both cases, the given set is a decomposition of \( K_{m,n} \).

\[\square\]

### 3. \( H \)-Decompositions of \( K(v, w) \)

The **complete graph of order** \( v \) **with a hole of size** \( w \), \( K(v, w) \), **is the graph with vertex set** \( V(K(v, w)) = V_{v-w} \cup V_w \), **where we assume these sets are** \( V_{v-w} = \{0_1, 1_1, \ldots, (v-w-1)_1\} \) and \( V_w = \{0_2, 1_2, \ldots, (w-1)_2\} \), **and edge set** \( E(K(v, w)) = \{(a, b) \mid a, b \in V(K(v, w)) \text{ and } \{a, b\} \not\in V_w\}\).

**Theorem 3.1.** There is an \( H \)-decomposition of \( K(v, w) \) if and only if \(|E(K(v, w))| \equiv 0 \pmod{5} \), \( v - w \geq 4 \), and \( (v, w) \not\in \{(5, 1), (6, 1)\}\).

**Proof.** Of course, \(|E(K(v, w))| \equiv 0 \pmod{5} \) is necessary. If \( v - w = 1 \) then \( K(v, w) \) is a star and there clearly is no \( H \)-decomposition. We cannot have \( v - w = 2 \), since there is then no possible \( H \) a subgraph of \( K(v, w) \) which can contain the edge \((0_1, 1_1)\).

**Case 1a.** Suppose \( (v \text{mod } 5), (w \text{mod } 5) \in \{(0, 0), (1, 0), (1, 1), (2, 2), (3, 3), (4, 4)\} \) and \( v - w \geq 10 \). Now \( K(v, w) = K_{v-w} \cup K_{v-w,w} \) where the vertex set of \( K_{v-w} \) is \( V_{v-w} \) and the partite sets of \( K_{v-w,w} \) are \( V_{v-w} \) and \( V_w \). In each case, \( K_{v-w} \) can be decomposed \([1]\) and \( K_{v-w,w} \) can be decomposed by Theorem 2.1. Therefore \( K(v, w) \) can be decomposed.

**Case 1b.** Suppose \( v \equiv w \equiv 0 \pmod{5} \) and \( v - w = 5 \). A decomposition of \( K(10, 5) \) is given by the set \( \{0_1, 2_1, 1_1, 4_1; 1_2, 2_2, 3_1, 1_2, 4_1; 1_1, 1_1, 2_1, 4_1; 1_2; 0_1, 3_2, 3_1, 0_2, 4_1; 1_1, 4_2, 0_1; 4_1; 0_1, 2_2, 2_1, 3_2; 0_2\} \). If \( v = \)
$5 + 5k$ and $w = 5k$, then $K(v, w) = K(10, 5) \cup (k - 1) \times K_{5,5}$ where the partite sets of the the $i$th copy of $K_{5,5}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{(5 + 5i)_2, (6 + 5i)_2, \ldots, (9 + 5i)_2\}$. $K(10, 5)$ is decomposed above and $K_{5,5}$ can be decomposed by Theorem 2.1.

**Case 1c.** Suppose $v \equiv 1 \pmod{5}$ and $w \equiv 0 \pmod{5}$ and $v - w = 6$. A decomposition of $K(11, 5)$ is given by the set $[0_1, 0_2, 1_1, 1_2; 3_1], [1_1, 2_2, 0_1, 3_2; 4_1], [2_1, 0_2, 3_1, 1_2; 5_1], [3_1, 3_2, 2_1, 2_2; 4_2], [4_1, 0_2, 5_1, 1_2; 4_2], [5_1, 3_2, 4_1, 2_2; 4_2], [0_1, 1_1, 3_1, 4_1; 4_2], [1_1, 2_1, 4_1, 5_1; 4_2], [2_1, 3_1, 5_1, 0_1; 4_2]$}. If $v = 6 + 5k$ and $w = 5k$, then $K(v, w) = K(11, 5) \cup (k - 1) \times K_{6,5}$ where the partite sets of the the $i$th copy of $K_{6,5}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1, 5_1\}$ and $\{(5 + 5i)_2, (6 + 5i)_2, \ldots, (9 + 5i)_2\}$. $K(11, 5)$ is decomposed above and $K_{6,5}$ can be decomposed by Theorem 2.1.

**Case 1d.** Suppose $v \equiv w \equiv 1 \pmod{5}$ and $v - w = 5$. We know that if $v = 6$ and $w = 1$, then $K(6, 1) = K_6$ and no decomposition of $K_6$ exists [1]. First, $K(11, 6) = K(7, 2) \cup K_{5,4}$ where the partite sets of $K_{5,4}$ are $\{0_1, 1_1, \ldots, 4_1\}$ and $\{2_2, 3_2, 4_2, 5_2\}$. A decomposition of $K(7, 2)$ is given by the set $\{[4_1, 1_1, 2_1, 3_1; 0_2], [0_1, 1_1, 3_1, 0_2; 1_2], [2_1, 0_2, 1_1, 1_2; 0_1], [4_1, 1_2, 3_1, 0_1; 2_1]\}$, and $K_{5,4}$ can be decomposed by Theorem 2.1. If $v = 6 + 5k$ and $w = 1 + 5k$, then $K(v, w) = K(11, 6) \cup (k - 1) \times K_{5,5}$ where the partite sets of the the $i$th copy of $K_{5,5}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{(6 + 5i)_2, (7 + 5i)_2, \ldots, (10 + 5i)_2\}$. A decomposition of $K(11, 6)$ is given above and $K_{5,5}$ can be decomposed by Theorem 2.1.

**Case 1e.** Suppose $v \equiv w \equiv 2 \pmod{5}$ and $v - w = 5$. First, $K(12, 7) = K(10, 5) \cup K_{5,2}$ where the partite sets of $K_{5,2}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2\}$. $K(10, 5)$ can be decomposed by Case 1b and $K_{5,2}$ can be decomposed by Theorem 2.1. If $v = 7 + 5k$ and $w = 2 + 5k$, then $K(v, w) = K(12, 7) \cup (k - 1) \times K_{5,5}$ where the partite sets of the the $i$th copy of $K_{5,5}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{(7 + 5i)_2, (8 + 5i)_2, \ldots, (11 + 5i)_2\}$. $K(12, 5)$ can be decomposed as described above and $K_{5,5}$ can be decomposed by Theorem 2.1.

**Case 1f.** Suppose $v \equiv w \equiv 3 \pmod{5}$ and $v - w = 5$. A decomposition of $K(8, 3)$ is given by the set $\{[0_1, 1_2, 4_1, 2_2; 3_1], [2_1, 0_2, 3_1, 1_2; 2_2], [0_1, 1_1, 3_1, 2_1; 0_2], [4_1, 1_1, 2_2, 3_1; 0_1], [1_1, 2_1, 4_1, 0_2; 1_2]\}$. If $v = 8 + 5k$ and $w = 3 + 5k$, then $K(v, w) = K(8, 3) \cup (k - 1) \times K_{5,5}$ where the partite sets of the the $i$th copy of $K_{5,5}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{(3 + 5i)_2, (4 + 5i)_2, \ldots, (7 + 5i)_2\}$. A decomposition of $K(8, 3)$ is given above and $K_{5,5}$ can be decomposed by Theorem 2.1.

**Case 1g.** Suppose $v \equiv w \equiv 4 \pmod{5}$ and $v - w = 5$. First, $K(9, 4) = K(7, 2) \cup K_{5,2}$ where the partite sets of $K_{5,2}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{2_2, 3_2\}$. $K(7, 2)$ can be decomposed by Case 1d and $K(5, 2)$ can be decomposed by Theorem 2.1. If $v = 9 + 5k$ and $w = 4 + 5k$, then $K(v, w) = K(9, 4) \cup (k - 1) \times K_{5,5}$
where the partite sets of the the $i$th copy of $K_{5,5}$ are \{0,1,2,3,4\} and \{(4+5i), (5+5i), \ldots, (8+5i)\}. $K(9,4)$ can be decomposed as described above and $K_{5,5}$ can be decomposed by Theorem 2.1.

**Case 2.** Suppose $v \equiv 0 \pmod{5}$ and $w \equiv 1 \pmod{5}$. First, $K(5,1) = K_5$ and no decomposition of $K_5$ exists. A decomposition of $K(10,6)$ is given by
\[
\{[0_1, 1_2, 1_1, 0_2; 3_1], [3_1, 0_2, 2_1, 1_2; 1_1], [1_2, 2_1, 1_2, 1_3; 3_1], [0_1, 3_2, 1_2; 2_1], [1_1, 5_2, 3_1, 4_2; 2_1], [0_1, 5_2, 2_1, 4_2; 1_1]\}.\]
If $v = w + 4$ and $w \equiv 1 \pmod{5}$, $w \geq 11$, then $K(v, w) = K(10,6) \cup K_{v-w,w-6}$, where $V(K(10,6)) = V_{v-w} \cup \{0_2, 1_2, \ldots, 5_2\}$ and the hole is on vertex set $\{0_2, 1_2, \ldots, 5_2\}$, and the partite sets of $K_{v-w,w-6}$ are $V_{v-w}$ and $\{6_2, 7_2, \ldots, (w-1)2\}$. $K(10,6)$ is decomposed above and $K_{v-w,w-6}$ can be decomposed by Theorem 2.1. For the other values of $v$ and $w$ in this case, $K(v, w) = K_{v-w+1} \cup K_{v,w-1}$ where the vertex set of $K_{v-w+1}$ is $V_{v-w} \cup \{0_2\}$ and the partite sets of $K_{v-w,w-1}$ are $V_{v-w}$ and $V_w \setminus \{0_2\}$. $K_{v-w+1}$ can be decomposed [1] and $K_{v,w-1}$ can be decomposed by Theorem 2.1.

**Case 3.** Suppose $v \equiv 2 \pmod{5}$ and $w \equiv 4 \pmod{5}$. First, if $v - w = 3$, say $w = 4 + 5k$ and $v = 7 + 5k$, then $K(v, w)$ has $15(k+1)$ edges and an $H$-decomposition of $K(v, w)$ would consist of $3(k+1)$ copies of $H$. Similar to the proof of the nonexistence of a decomposition of $K_{3,5k}$ in Theorem 2.1, such a decomposition would have at most $3(k+1)$ odd degree vertices in the hole, but each of the $4+5k$ vertices in the hole are of odd degree. So no such decomposition exists. A decomposition of $K(12,4)$ is given by
\[
\{[7_1, 0_1, 6_1, 1_1; 1_2], [3_1, 4_1, 2_1, 5_1; 1_1], [6_1, 2_1, 7_1, 3_1; 4_1], [0_1, 4_1, 1_1, 5_1; 2_1], [7_1, 4_1, 5_1, 6_1; 0_2], [0_1, 1_1, 2_1, 3_1; 2_2][1_2, 0_1, 0_2, 1_1; 7_1], [0_2, 2_1, 1_2, 3_1; 6_1], [1_2, 4_1, 0_2, 5_1; 6_1], [2_2, 7_1, 3_2, 6_1; 0_1], [3_2, 5_1, 2_2, 4_1; 1_1], [2_2, 3_1, 3_2, 2_1; 1_1]\}.\]
For the other values of $v$ and $w$ in this case, $K(v, w) = K(12,4) \cup K(v-w,8) \cup K_{w-4}$ where $V(K(12,4)) = \{0_1, 1_1, \ldots, 7_1, 0_2, 1_2, 2_2, 3_2\}$ and the hole is on the vertex set $\{0_2, 1_2, 2_2, 3_2\}$, $V(K(v-w,8)) = V_{v-w} \cup V_w \setminus \{0_1, 1_1, \ldots, 7_1\}$ and the hole is on the vertex set $V_w$, and the partite sets of $K_{w-4}$ are $\{0_1, 1_1, \ldots, 7_1\}$ and $V_w \setminus \{0_2, 1_2, 2_2, 3_2\}$. $K(12,4)$ is decomposed above, $K(v-8, w)$ can be decomposed by Case 1, and $K_{w-4}$ can be decomposed by Theorem 2.1.

**Case 4.** Suppose $v \equiv 4 \pmod{5}$ and $w \equiv 2 \pmod{5}$. A decomposition $K(9,2)$ is given by
\[
\{[0_2, 0_1, 1_2, 1_1; 2_1], [0_2, 3_1, 1_2, 4_1; 5_1], [5_1, 6_1, 0_1, 2_1; 1_1], [2_1, 1_1, 4_1, 3_1; 6_1], [1_1, 3_1, 5_1; 1_1], [6_1, 4_1, 0_1, 3_1; 2_1], [1_2, 2_1, 4_1, 5_1; 6_1]\}.\]
For the other values of $v$ and $w$ in this case, $K(v, w) = K(9,2) \cup K(v-7, w) \cup K_{7,w-2}$ where $V(K(9,2)) = \{0_1, 1_1, \ldots, 6_1, 0_2, 1_2\}$ and the hole is on the vertex set $\{0_2, 1_2\}$, $V(K(v-7, w)) = V_{v-w} \cup V_w \setminus \{0_1, 1_1, \ldots, 6_1\}$ and the hole is on the vertex set $V_w$, and the partite sets of $K_{7,w-2}$ are $\{0_1, 1_1, \ldots, 6_1\}$ and $V_w \setminus \{0_2, 1_2\}$. $K(9,2)$ is decomposed above, $K(v-7, w)$ can be decomposed by Case 1, and $K_{7,w-2}$
can be decomposed by Theorem 2.1.

\[ \square \]

4. Decompositions of $\lambda K_v$

The $\lambda$-fold complete graph, $\lambda K_v$, is the multigraph with edge multiset $E(\lambda K_v) = \{\lambda \times (a, b) \mid a \neq b \text{ and } \{a, b\} \subset V(\lambda K_v)\}$.

**Theorem 4.1.** There is an $H$-decomposition of $\lambda K_v$ if and only if

(a) $v \equiv 0$ or $1 \pmod{5}$ and $v \geq 10$ when $\lambda = 1$, or

(b) $\lambda \equiv 0 \pmod{5}$ and $v \geq 5$.

**Proof.** Since $|E(\lambda K_v)| = \lambda v(v - 1)/2$ and $|E(H)| = 5$, then a necessary condition for an $H$-decomposition of $\lambda K_v$ is that $\lambda v(v - 1)/2 \equiv 0 \pmod{5}$, and the necessary conditions follow. For $v = 5$ and $\lambda = 2$, the set $\{[0, 2, 3, 4; 1], [3, 1, 2, 4; 0], [2, 0, 1, 4; 3], [1, 3, 0, 4; 2]\}$ forms a decomposition where $V(2K_5) = \{0, 1, 2, 3, 4\}$. For $v = 5$ and $\lambda = 3$, the set $\{[0, 3, 1, 4; 2], [1, 2, 3, 4; 0], [4, 3, 0, 2; 1], [2, 4, 0, 1; 3], [2, 1, 3, 0; 4], [3, 4, 0, 1; 2]\}$ forms a decomposition. For $v = 5$ and $\lambda \geq 4$, a decomposition follows by taking repeated copies of the decompositions from the $\lambda = 2$ and $\lambda = 3$ cases. For $v = 6$ and $\lambda = 2$, the set $\{[i, 1+i, 2+i, 4+i; 3+i] \mid i = 0, 1, \ldots, 5\}$ forms a decomposition where $V(2K_6) = \{0, 1, 2, 3, 4, 5\}$. For $v = 6$ and $\lambda = 3$, the set $\{[5, 2, 4, 3; 1], [2, 0, 4, 1; 3], [0, 2, 5, 4; 3], [2, 1, 5, 3; 4], [5, 2, 3, 4; 1], [2, 0, 3, 1; 4], [1, 4, 5, 0; 3], [0, 1, 4, 3; 5], [0, 1, 3, 5; 4]\}$ forms a decomposition. For $v = 6$ and $\lambda \geq 4$, a decomposition follows similarly to the case of $v = 5$. For $v \equiv 0$ or $1 \pmod{5}$, $v \geq 10$, an $H$-decomposition of $K_v$ exists, and hence an $H$-decomposition of $\lambda K_v$ exists. For the remaining values of $v$, we have $\lambda \equiv 0 \pmod{5}$, so in these cases it is sufficient to present the constructions for $\lambda = 5$ only.

**Case 1.** Suppose $v \equiv 0 \pmod{4}$, $v \geq 8$, say $v = 4k$ and $\lambda = 5$. For $v = 8$, consider the set $B_1 = \{2 \times [0, 1, 3, 2; 4], [\infty, 0, 3, 6; 1], [0, 3, \infty, 5; 1]\}$. For $v \geq 12$, consider the set:

$$B_1 = \{[\infty, 0, 2k - 5, 4k - 9; 1], [0, 2k - 3, \infty, 2k - 2; 2k - 1]\}$$

$$\cup \{2 \times [0, 1, 3, 2; 2k - 1], 2 \times [0, 3, 7, 4; 2k - 1]\}$$

$$\cup \{[0, 5 + 2i, 11 + 4i, 6 + 2i; 1 + i] \mid i = 0, 1, \ldots, k - 4\}$$

$$\cup \{[0, 5 + 2i, 11 + 4i, 6 + 2i; k - 2 + i] \mid i = 0, 1, \ldots, k - 4\}.$$
Define the permutation $\pi$ on $\{0, 1, 2, \ldots, v-2, \infty\}$ as $\pi = (\infty)(0, 1, 2, \cdots, v-2)$. Then the set $\gamma = \{\pi^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_1 \text{ and } i = 0, 1, \ldots, v-2\}$ is an $H$-decomposition of $\lambda K_v$ where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-2, \infty\}$.

Case 2. Suppose $v \equiv 1 \pmod{4}$, say $v = 4k + 1$ and $\lambda = 5$. Consider the set:

$$B_2 = \{[0, 1 + 2i, 3 + 4i, 2 + 2i; 1 + i] \mid i = 0, 1, \ldots, k - 1\} \cup \{[0, 1 + 2i, 3 + 4i, 2 + 2i; k + 1 + i] \mid i = 0, 1, \ldots, k - 1\}.$$ 

Define the permutation $\rho$ on $\{0, 1, 2, \ldots, v-1\}$ as $\rho = (0, 1, 2, \cdots, v-1)$. Then the set $\gamma = \{\rho^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_2 \text{ and } i = 0, 1, \ldots, v-1\}$ is an $H$-decomposition of $\lambda K_v$ where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-1\}$.

Case 3. Suppose $v \equiv 2 \pmod{4}$, $v \geq 10$, say $v = 4k + 2$ and $\lambda = 5$. Consider the set:

$$B_3 = \{[\infty, 0, 2k, 4k; 1], [0, 2k, \infty, 2k - 1; 2k + 2]\} \cup \{[0, 1 + 2i, 3 + 4i, 2 + 2i; 1 + i] \mid i = 0, 1, \ldots, k - 1\} \cup \{[0, 1 + 2i, 3 + 4i, 2 + 2i; k + 1 + i] \mid i = 0, 1, \ldots, k - 2\}.$$ 

Then the set $\gamma = \{\pi^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_3 \text{ and } i = 0, 1, \ldots, v-2\}$ is an $H$-decomposition of $\lambda K_v$ where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-2, \infty\}$, where $\pi$ is defined in Case 1.

Case 4. Suppose $v \equiv 3 \pmod{4}$, say $v = 4k + 3$ and $\lambda = 5$. For $v = 7$, consider the set $B_4 = \{2 \times [0, 1, 3, 2; 4], [0, 3, 6, 2; 1]\}$. For $v \geq 11$, consider the set:

$$B_4 = \{2 \times [0, 1, 3, 2; 2k + 1], 2 \times [0, 3, 7, 4; 2k + 1], [0, 2k - 3, 4k - 3, 2k - 2; 2k + 1]\} \cup \{[0, 5 + 2i, 11 + 4i, 6 + 2i; 1 + i] \mid i = 0, 1, \ldots, k - 3\} \cup \{[0, 5 + 2i, 11 + 4i, 6 + 2i; k - 1 + i] \mid i = 0, 1, \ldots, k - 3\}.$$ 

Then the set $\gamma = \{\rho^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_4 \text{ and } i = 0, 1, \ldots, v-1\}$ is an $H$-decomposition of $\lambda K_v$ where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-1\}$, where $\rho$ is defined in Case 2. $\blacksquare$

References


