International Journal of Pure and Applied Mathematics
Volume 75 No. 4 2012, 493-510
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

ORTHOGONAL VECTOR VALUED WAVELETS ON $\mathbb{R}_+$

P. Manchanda[1 §], Vikram Sharma[2]

[1,2]Department of Mathematics
Guru Nanak Dev University
Amritsar, 143005, INDIA

Abstract: Xia and Suter [15] have introduced the notion of vector valued multiresolution analysis on real line $\mathbb{R}$. Chen and Chang [1] have given an algorithm for construction of vector valued wavelets. Farkov [3] has studied the notion of multiresolution analysis on locally abelian groups and constructed the compactly supported orthogonal p-wavelets on $L^2(\mathbb{R}_+)$. In this paper, we introduce vector valued multiresolution p-analysis on positive half line. We find necessary and sufficient condition for the existence of associated vector valued wavelets. We construct vector valued wavelets on $\mathbb{R}_+$. Our approach is connected with Walsh-Fourier theory.

AMS Subject Classification: 42A38, 42A55, 42C10, 42C15, 42C40
Key Words: multiresolution p-analysis, Walsh function, Walsh-Fourier transform, orthogonal vector valued wavelets

1. Introduction

Wavelet theory has been studied extensively in both theory and applications. The main advantage of wavelets is their time-frequency localization property. The wavelet transform is a simple mathematical tool that cuts up data or functions into different frequency components, studies each components with a resolution matched to its scale. Many signals in areas like music, speech, image and video images can be efficiently represented by wavelets that are translations and dilations of a single function called mother wavelet with bandpass property.

Received: December 6, 2011

§Correspondence author

Multiresolution Analysis is the heart of wavelet theory. A multiresolution analysis on the set of real numbers $\mathbb{R}$, introduced by Mallat [9] is an increasing sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \bigcup_{j\in\mathbb{Z}} V_j$ is dense in $\{V_j\}_{j\in\mathbb{Z}}$ and which satisfies f(t)$\in V_j$ if and only if f(2t)$\in V_{j+1}$. Also there exists an element $\phi \in V_0$ called scaling function such that the collection of integer translates of ϕ, i. e $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is a complete orthonormal basis for V_0. In the definition of multiresolution analysis the dilation factor of 2 can be replaced by an integer $n > 2$ and one can construct n - 1 wavelets to generate the whole space $L^2(\mathbb{R})$. A similar generalization of multiresolution analysis can be made in higher dimensions by considering matrix dilations.

Walsh analysis or Dyadic harmonic analysis has been extensively studied: both aspects theory as well as applications, see Golubov *et al.* [5], Schipp *et al.* [11]. In his papers, Lang [6-8] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the rules and procedures of Mallat and Daubechies via scaling filters. These wavelets turn out to be certain lacunary Walsh series on the $\mathbb{R}_+$. Later on, Farkov [4] extended the results of Lang[6-8] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Subsequently, Protasov and Farkov [10] constructed dyadic compactly supported wavelets in $L^2(\mathbb{R}_+)$, whereas Farkov [3] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}_+)$ and proved necessary and sufficient conditions for scaling filters with p^n many terms$(p, n \geq 2)$ to generate a p-MRA in $L^2(\mathbb{R}_+)$. The approach adopted by Farkov is connected with Walsh-Fourier transform and the elements of M-band wavelet theory.

The paper is organized as follows. In Section 2, we explain certain results of Walsh-Fourier analysis. We present brief review of generalized Walsh functions, Walsh-Fourier transforms and its various properties, multiresolution p-analysis in $L^2(\mathbb{R}_+)$ introduced by Farkov [3]. We introduce the concept of vector valued multiresolution p-analysis on $\mathbb{R}_+$ in Section 3. In Section 4, necessary and sufficient condition for the existence of associated vector valued wavelets is given. We construct a vector valued multiresolution analysis on the positive half line with a compactly supported vector valued scaling function Φ.

2. Walsh-Fourier Analysis

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Z}_+ = \{0, 1, ...\}$. Denote by [x] the integer part of x. For $x \in \mathbb{R}_+$ and for any positive integer j,

$$x_j = [p^j x](mod\, p), x_{-j} = [p^{1-j} x](mod\, p) \tag{2.1}$$

where $x_j, x_{-j} \in \{0, 1, ..., p-1\}$. It is clear that for each $x \in \mathbb{R}_+$, $\exists\, k = k(x) \in \mathbb{N}$ such that $x_{-j} = 0, \forall\, j > k$.

Consider on $\mathbb{R}_+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j} \tag{2.2}$$

with

$$\xi_j = x_j + y_j (mod\, p), j \in \mathbb{Z} \setminus \{0\}, \tag{2.3}$$

where $\xi_j \in \{0, 1, 2, ..., p-1\}$ and x_j, y_j are calculated by (2.1). As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}_+$.

For $x \in [0, 1)$, let $r_0(x)$ is given by

$$r_o(x) = \begin{cases} 1, & x \in [0, 1/p) \\ \varepsilon_p^j, & x \in [jp^{-1}, (j+1)p^{-1}), j = 1, 2, ..., p-1 \end{cases} \tag{2.4}$$

where $\varepsilon = \exp(\frac{2\pi i}{p})$.

The extension of the function r_0 to $\mathbb{R}_+$ is defined by the equality $r_0(x+1) = r_0(x), x \in \mathbb{R}_+$. Then the generalized Walsh functions $\{\omega_m(x)\}_{m \in \mathbb{Z}_+}$ are defined by

$$\omega_0(x) \equiv 1, \quad \omega_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^{\mu_j},$$

where $m = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, 2, \dots, p-1\}, \quad \mu_k \neq 0$.

For $x, w \in \mathbb{R}_+$, let

$$\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j) \right) \tag{2.5}$$

where $x_j$and w_j are given by (2.1). We note that

$$\chi\left(x, \frac{m}{p^{n-1}}\right) = \chi\left(\frac{m}{p^{n-1}}, x\right) = \omega_m\left(\frac{x}{p^{n-1}}\right), \; \forall\, x \in [0, p^{n-1}), m \in \mathbb{Z}_+.$$

The **Walsh-Fourier transform** of a function $f \in L^2(\mathbb{R}_+)$ is defined by

$$\tilde{f}(w) = \int_{\mathbb{R}_+} f(x) \overline{\chi(x, w)} dw \tag{2.6}$$

where $\chi(x,w)$ is given by (2.5). The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform [5, 11]. In particular, if $f \in L^2(\mathbb{R}_+)$, then $\tilde{f} \in L^2(\mathbb{R}_+)$ and

$$\|f\|_{L^2(\mathbb{R}_+)} = \|\tilde{f}\|_{L^2(\mathbb{R}_+)}. \tag{2.7}$$

If $x, y, w \in \mathbb{R}_+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, w) = \chi(x,w)\chi(y,w), \chi(x \ominus y, w) = \chi(x,w)\overline{\chi(y,w)}, \tag{2.8}$$

see Golubov *et al.* [5], Schipp *et al.* [11]. Thus for fixed x and w, the equality (2.8) holds for all $y \in \mathbb{R}_+$ except for countably many. It was shown by Golubov *et al.* [5] that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthomormal basis in $L^2(0,1)$.

According to Schipp *et al.* [11] for any $\phi \in L^2(\mathbb{R}_+)$, we have

$$\int_{\mathbb{R}_+} \phi(t)\overline{\phi(t \ominus k)}dt = \int_{\mathbb{R}_+} \tilde{\phi}(w)\overline{\tilde{\phi}(w)}\overline{\chi(k,w)}dw \tag{2.9}$$

Let $\{w\}$ be the fractional part of w. For any $\phi \in L^2(\mathbb{R}_+)$ and $k \in \mathbb{Z}_+$, we have $\chi(k,w) = \chi(k,\{w\})$. Therefore $\chi(k, w+l) = \chi(k,w), l \in \mathbb{Z}_+$. It follows from (2.9) that

$$\begin{aligned} \int_{\mathbb{R}_+} \phi(t)\overline{\phi(t \ominus k)}dt &= \sum_{l \in \mathbb{Z}_+} \int_0^1 |\tilde{\phi}(w)|^2 \overline{\chi(k,w)}dw \\ &= \int_0^1 (\sum_{l \in \mathbb{R}_+} |\tilde{\phi}(w+1)|^2)\overline{\chi(k,w)}dw \end{aligned}$$

Therefore, a necessary and sufficient condition for a system $\{\phi(t \ominus k) | k \in \mathbb{Z}_+\}$ to be orthonormal in $L^2(\mathbb{R}_+)$ is

$$\sum_{l \in \mathbb{R}_+} |\tilde{\phi}(w+1)|^2 = 1 \quad a.e. \tag{2.10}$$

Multiresolution p-analysis in $L^2(\mathbb{R}_+)$ defined by Farkov [3] is as follows:

Definition 2.1. A multiresolution p-analysis on $L^2(\mathbb{R}_+)$ is a nested sequence of closed subspaces $V_j, j \in \mathbb{Z}$ of $L^2(\mathbb{R}_+)$ such that following hold:

(a). $V_j \subset V_{j+1}, j \in \mathbb{Z}$.

(b). $\bigcup_j V_j$ is dense in $L^2(\mathbb{R}_+)$ and $\bigcap_j V_j = 0$.

(c). $\mathbf{f}(t) \in V_j$ if and only if $\mathbf{f}(pt) \in V_{j+1}$.

(d). $\mathbf{f}(t) \in V_0 \Rightarrow \mathbf{f}(t \oplus k) \in V_0$ for all $k \in \mathbb{Z}_+$.

(e). there exists a function called scaling function $\Phi \in V_0$ such that its translations $\Phi_k(t) = \Phi(t \ominus k), k \in \mathbb{Z}_+$, form an orthonormal basis for V_0.

The function ϕ is called the scaling function in $L^2(\mathbb{R}_+)$.

Farkov has given a general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}_+)$ arising from scaling filters with p^n many terms. For all integer $p \geq 2$ these wavelets are identified with certain lacunary Walsh series on $\mathbb{R}_+$. In this new setting Farkov [3] has proved the extension of classical results concerning necessary and sufficient condition of wavelets associated with the classical multiresolution analysis.

The following theorem by Farkov [3] generalizes A. Cohen′s result, see Daubechies [2]:

Theorem 2.1. *Let*

$$m_0(w) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\chi(k,w)}$$

be a polynomial satisfying the following conditions:

(i). $m_0(0) = 1$.

(ii). $\sum_{j=0}^{p^n-1} |m_0(sp^{-n} \oplus jp^{-1})|^2 = 1$ *for* $s = 0, 1, \dots p^{n-1} - 1$.

(iii). There exists a W-compact set E such that $0 \in int(E)$, $\mu(E) = 1$, $E \equiv [0,1)(mod\ \mathbb{Z}_+)$ and

$$\inf_{j\in\mathbb{N}} \inf_{w\in E} |m_0(p^{-j}w)| > 0$$

If the Walsh-Fourier transform if $\phi \in L^2(\mathbb{R}_+)$ can be written as

$$\tilde{\phi}(w) = \prod_{j=1}^{\infty} m_0(p^{-j}w),$$

then ϕ is scaling function in $L^2(\mathbb{R}_+)$.

3. Vector-Valued Multiresolution p-Analysis on $\mathbb{R}_+$

We use the following notations. Let $\mathbb{C}$ be the set of all complex numbers, $\mathbf{I_N}$ and $\mathbf{O}$ represent N×N identity matrix and the zero matrix respectively.

$L^2(\mathbb{R}_+, \mathbb{C}^N)$ represents the set of square integrable vector-valued functions $\mathbf{f}(t)$ on positive half line, $\mathbb{R}_+$ i. e. ,

$$L^2(\mathbb{R}_+, \mathbb{C}^N) = \{\mathbf{f}(t) = \left(\begin{array}{c} f_1(t), f_2(t), , \ldots \quad , f_N(t) \end{array} \right)^T : t \in \mathbb{R}_+, \\ f_v(t) \in L^2(\mathbb{R}_+), v = 1, 2, \ldots N\}$$

where T denotes Transpose.

For $\mathbf{f} \in L^2(\mathbb{R}_+, \mathbb{C}^N)$, $\|\mathbf{f}\|_{L^2(\mathbb{R}_+,\mathbb{C}^N)}$ is the norm of the function $\mathbf{f}$, i. e,

$$\|\mathbf{f}\|_{L^2(\mathbb{R}_+,\mathbb{C}^N)} = \sqrt{\sum_{v=1}^{N} \int_{\mathbb{R}_+} |f_v(t)|^2 dt}$$

and integration of $\mathbf{f}$ is given by

$$\int_{\mathbb{R}_+} \mathbf{f}(t) dt = \left(\begin{array}{c} \int_{\mathbb{R}_+} f_1(t)dt, \int_{\mathbb{R}_+} f_2(t)dt, , \ldots \quad , \int_{\mathbb{R}_+} f_N(t)dt \end{array} \right)^T .$$

The Walsh-Fourier transform of $\mathbf{f}(t)$ is defined by

$$\tilde{\mathbf{f}}(w) = \int_{\mathbb{R}_+} \mathbf{f}(t) \overline{\chi(k, w)} dt = \left(\begin{array}{c} \tilde{f}_1(w), \tilde{f}_2(w), \ldots, \tilde{f}_N(w) \end{array} \right)^T . \tag{3.1}$$

For two vector valued functions $\mathbf{f}, \mathbf{h} \in L^2(\mathbb{R}_+, \mathbb{C}^N)$, their symbol inner product is defined by

$$\langle \mathbf{f}, \mathbf{h} \rangle_{L^2(\mathbb{R}_+,\mathbb{C}^N)} = \int_{\mathbb{R}_+} \mathbf{f}(t) \mathbf{h}(t)^* dt,$$

where '*' means complex conjugate and transpose. The inner product defined above is matrix valued(usually it is a scaler valued).

A sequence $\{\mathbf{f}_k(t)\}_{k \in \mathbb{Z}_+} \subset \mathbf{U} \subseteq L^2(\mathbb{R}_+, \mathbb{C}^N)$ is called orthonormal set of $\mathbf{U}$, if it satisfies

$$\langle \mathbf{f}_k(t), \mathbf{f}_n(t) \rangle = \delta_{k,n} \mathbf{I}_N, \tag{3.2}$$

where $\delta_{k,n}$ is the Kronecker delta such that $\delta_{k,n} = 1$ when $k = n$ and $\delta_{k,n} = 0$ when $k \neq n$.

Definition 3.1. We say that $\mathbf{f}(t) \in \mathbf{U} \subseteq L^2(\mathbb{R}_+, \mathbb{C}^N)$ is an orthogonal vector-valued function in $\mathbf{U}$ if its translations $\{\mathbf{f}(t \ominus k)\}_{k\in\mathbb{Z}_+}$ is an orthonormal set in $\mathbf{U}$, i. e. ,

$$\langle \mathbf{f}(t \ominus k), \mathbf{f}(t \ominus n)\rangle = \delta_{k,n}\mathbf{I}_N, \quad k, n \in \mathbb{Z}_+. \tag{3.3}$$

Definition 3.2. A sequence $\{\mathbf{f}_k(t)\}_{k\in\mathbb{Z}_+} \subset \mathbf{U} \subseteq L^2(\mathbb{R}_+, \mathbb{C}^N)$ is called an orthonormal basis of $\mathbf{U}$ if it satisfies (3.2), and for any $\mathbf{h}(t) \in \mathbf{U}$, there exists a unique sequence of $N \times N$ constant matrices $\{A_k\}_{k\in\mathbb{Z}_+}$ such that

$$\mathbf{h}(t) = \sum_{k\in\mathbb{Z}_+} A_k \mathbf{f}_k(t).$$

The multiresolution analysis approach is one of the main approaches in the construction of wavelets. We introduce vector-valued multiresolution p-analysis on positive half line and give the definition for associated orthogonal vector valued wavelets.

Definition 3.3. A vector valued multiresolution p-analysis on $L^2(\mathbb{R}^+, \mathbb{C}^N)$ is a nested sequence of closed subspaces $V_j, \in \mathbb{Z}$ of $L^2(\mathbb{R}_+, \mathbb{C}^{\mathbb{N}})$ such that following hold:

(a). $V_j \subset V_{j+1}, j \in \mathbb{Z}$.

(b). $\bigcup_j V_j$ is dense in $L^2(\mathbb{R}_+, \mathbb{C}^N)$ and $\bigcap_j V_j = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector of $L^2(\mathbb{R}^+, \mathbb{C}^{\mathbb{N}})$.

(c). $\mathbf{f}(t) \in V_j$ if and only if $\mathbf{f}(pt) \in V_{j+1}$.

(d). $\mathbf{f}(t) \in V_0 \Rightarrow \mathbf{f}(t \oplus k) \in V_0$ for all $k \in \mathbb{Z}_+$.

(e). there exists a function called scaling function $\Phi \in V_0$ such that its translations $\Phi_k(t) = \Phi(t \ominus k), k \in \mathbb{Z}_+$, form an orthonormal basis for V_0.

Now $\Phi(t) \in V_0 \Rightarrow \Phi(pt) \in V_1$, by(e)

$$\begin{aligned} \delta_{0,k}\mathbf{I}_N &= \int_{\mathbb{R}_+} \Phi(t)\Phi(t \ominus k)dt \\ &= \int_{\mathbb{R}_+} \sqrt{p}\Phi(pt)\sqrt{p}\Phi(pt \ominus k)dt. \end{aligned}$$

So $\Phi_{1,k}(t) = \{\sqrt{p}\Phi(pt \ominus k)\}_{k\in\mathbb{Z}_+}$ form an orthonormal basis for V_1.

Therefore, The space V_j is defined by

$$V_j = \mathbf{clos}_{L^2(\mathbb{R}_+,\mathbb{C}^N)}(\mathbf{span}\{p^{\frac{j}{2}}\Phi(p^j t \ominus k)\}, k \in \mathbb{Z}_+), j \in \mathbb{Z}. \tag{3.4}$$

Now $\Phi \in V_1$, there exists a sequence of $N \times N$ constant matrices $\{R_k\}_{k\in\mathbb{Z}_+}$ such that

$$\Phi(t) = \sum_{k\in\mathbb{Z}_+} R_k \Phi(pt \ominus k) \quad , t \in \mathbb{R}_+. \tag{3.5}$$

If the sequence $\{R_k\}_{k\in\mathbb{Z}_+}$ is finite, we say that $\Phi(t)$ is a compactly supported vector-valued function. By Walsh Fourier Transform, we have

$$\tilde{\Phi}(w) = \mathcal{R}(w/p)\tilde{\Phi}(w/p), \quad w \in \mathbb{R}_+. \tag{3.6}$$

where

$$\mathcal{R}(w) = \frac{1}{p}\sum_{k\in\mathbb{Z}_+} R_k \overline{\chi(k, w)}. \tag{3.7}$$

Noting that $\chi(k, w + l) = \chi(k, w), k, l \in \mathbb{R}_+$, so $\mathcal{R}(w)$ is 1-periodic function of w. By (3.6), we have

$$\tilde{\Phi}(w) = \mathcal{R}(w/p)\mathcal{R}(w/p^2)\ldots\tilde{\Phi}(0) = \prod_{l=1}^{\infty}\mathcal{R}(w/p^l)\tilde{\Phi}(0). \tag{3.8}$$

Let $W_j, j \in \mathbb{Z}$ denote the orthocomplement subspace of V_j in V_{j+1} and there exists $p-1$ vector valued functions $\Psi_m(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N), m \in \mathbf{\Lambda}$, where $\mathbf{\Lambda} = \{1, 2, \ldots, p-1\}$, such that their translations and dilations form Riesz basis of W_j, i. e,

$$W_j = \mathbf{clos}_{L^2(\mathbb{R}_+,\mathbb{C}^N)}(\mathbf{span}\{p^{\frac{j}{2}}\Psi_m(p^j t \ominus k)\}, m \in \mathbf{\Lambda}, k \in \mathbb{Z}_+), j \in \mathbb{Z}. \tag{3.9}$$

We say that $\{\Phi(t), \Psi_1(t), \Psi_2(t), \ldots, \Psi_{p-1}(t)\}$ is a vector-valued wavelet system. For each $m \in \mathbf{\Lambda}$, $\Psi_m(t) \in W_0 \subset V_1$, there exist $p-1$ sequences of $N\times N$ constant matrices $\{S_k^{(m)}\}_{k\in\mathbb{Z}_+}$ such that

$$\Psi_m(t) = \sum_{k\in\mathbb{Z}_+} S_k^{(m)}\Phi(pt \ominus k), \quad , m \in \mathbf{\Lambda}, t \in \mathbb{R}_+. \tag{3.10}$$

By taking Walsh-Fourier Transform, the refinement equation (3.10) becomes

$$\tilde{\Psi}_m(w) = \mathcal{S}^{(m)}(w/p)\tilde{\Phi}(w/p), \quad w \in \mathbb{R}_+, m \in \mathbf{\Lambda} \tag{3.11}$$

where

$$\mathcal{S}^{(m)}(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}_+} S_k^{(m)} \overline{\chi(k, w)}. \tag{3.12}$$

If $\Phi(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N)$ is an orthogonal vector-valued scaling function, then it follows from (3.3) that

$$\langle \Phi(t), \Phi(t \ominus k) \rangle = \delta_{0,k} \mathbf{I}_N, \quad k \in \mathbb{Z}_+ \tag{3.13}$$

We say $p-1$ vector-valued functions $\Psi_1(t), \Psi_2(t), \ldots, \Psi_{p-1}(t)$ are orthogonal vector-valued wavelet functions associated with the orthogonal vector-valued scaling function $\Phi(t)$, if they satisfy

$$\langle \Psi_m(t), \Phi(t \ominus k) \rangle = \mathbf{O}, \quad m \in \mathbf{\Lambda}, k \in \mathbb{Z}_+, \tag{3.14}$$

and the family $\{\Psi_m(t \ominus k), m \in \mathbf{\Lambda}\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis of the subspace W_0. So we have

$$\langle \Psi_m(t), \Psi_n(t \ominus k) \rangle = \delta_{0,k} \delta_{m,n} \mathbf{I}_N, \quad m, n \in \mathbf{\Lambda}, k \in \mathbb{Z}_+. \tag{3.15}$$

The following lemma, which will be used in next section, gives a characterization in the frequency domain of an orthogonal vector-valued function $\mathbf{f}(t)$.

Lemma 3.1. *Let $\mathbf{f}(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N)$. Then $\mathbf{f}(t)$ is an orthogonal vector-valued function if and only if*

$$\sum_{l \in \mathbb{Z}_+} \tilde{\mathbf{f}}(w+l) \tilde{\mathbf{f}}(w+l)^* = \mathbf{I}_N, w \in \mathbb{R}_+. \tag{3.16}$$

Proof. Let $\mathbf{f}(t) = \left(\ f_1(t), f_2(t), \ldots, f_N(t) \ \right)^T \in L^2(\mathbb{R}_+, \mathbb{C}^N)$,

$$\delta_{0,k} \mathbf{I}_N = \int_{\mathbb{R}_+} \mathbf{f}(t) \mathbf{f}(t \ominus k)^* dt$$

$$= \begin{pmatrix} \int_{\mathbb{R}_+} f_1(t) \overline{f_1(t \ominus k)} dt, \int_{\mathbb{R}_+} f_1(t) \overline{f_2(t \ominus k)} dt, \ldots, \int_{\mathbb{R}_+} f_1(t) \overline{f_N(t \ominus k)} dt \\ \int_{\mathbb{R}_+} f_2(t) \overline{f_1(t \ominus k)} dt, \int_{\mathbb{R}_+} f_2(t) \overline{f_2(t \ominus k)} dt, \ldots, \int_{\mathbb{R}_+} f_2(t) \overline{f_N(t \ominus k)} dt \\ \cdots \\ \int_{\mathbb{R}_+} f_N(t) \overline{f_1(t \ominus k)} dt, \int_{\mathbb{R}_+} f_N(t) \overline{f_2(t \ominus k)} dt, \ldots, \int_{\mathbb{R}_+} f_N(t) \overline{f_N(t \ominus k)} dt \end{pmatrix}.$$

By identity (2.9), we arrive at

$$\delta_{0,k} \mathbf{I}_N = \int_{\mathbb{R}_+} \tilde{\mathbf{f}}(w) \tilde{\mathbf{f}}(w)^* \overline{\chi(k, w)} dw$$

$$= \sum_{l\in\mathbb{Z}_+} \int_l^{l+1} \tilde{\mathbf{f}}(w)\tilde{\mathbf{f}}(w)^* \overline{\chi(k,w)}dw$$

$$= \int_0^1 \sum_{l\in\mathbb{Z}_+} \tilde{\mathbf{f}}(w+l)\tilde{\mathbf{f}}(w+l)^* \overline{\chi(k,w)}dw$$

So, $\mathbf{f}(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N)$ is orthogonal $\Leftrightarrow \sum_{l\in\mathbb{Z}_+} \tilde{\mathbf{f}}(w+l)\tilde{\mathbf{f}}(w+l)^* = \mathbf{I}_N$. □

Lemma 3.2. *If $\Phi(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N)$, defined by (3.5), is an orthogonal vector-valued scaling function, then $\forall\ k \in \mathbb{Z}_+$, we have*

$$\sum_{u\in\mathbb{Z}_+} R_u(R_{u\oplus pk})^* = p\delta_{0,k}\mathbf{I}_N. \tag{3.17}$$

Proof. Now

$$\begin{aligned}
\delta_{0,k}\mathbf{I}_N &= \langle \Phi(t\ominus k), \Phi(t)\rangle \\
&= \sum_{u\in\mathbb{Z}_+}\sum_{v\in\mathbb{Z}_+}\int_{\mathbb{R}_+} R_u\Phi(pt\ominus pk\ominus u)\Phi(pt\ominus v)R_v^* dt \\
&= \frac{1}{p}\sum_{u,v\in\mathbb{Z}_+} R_u \left\langle \Phi(t\ominus pk\ominus u), \Phi(t\ominus v)\right\rangle R_v^* \\
&= \frac{1}{p}\sum_{u\in\mathbb{Z}_+} R_u(R_{u\oplus pk})^*.
\end{aligned}$$

□

4. The Existence of Orthogonal Vector-Valued Wavelets on $\mathbb{R}_+$

In this section, we consider the existence of compactly supported vector-valued wavelets on $\mathbb{R}_+$. We obtain the necessary and sufficient condition for the existence of vector valued orthogonal wavelets.

Theorem 4.1. *Let $\Phi(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N)$ defined in (3.5) be an orthogonal vector-valued scaling function. Assume that $\Psi_m(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N), m \in \mathbf{\Lambda}$, and $\mathcal{R}(w)$ and $\mathcal{S}^{(m)}(w)$ are defined by (3.7) and (3.12) respectively. Then $\Psi_m(t), m \in \mathbf{\Lambda}$ are orthogonal vector-valued wavelet functions associated with $\Phi(t)$ if and only if*

$$\sum_{l=0}^{p-1} \mathcal{R}\left(\frac{w+l}{p}\right)\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)^* = \mathbf{O}, \quad m\in\mathbf{\Lambda}, w\in\mathbb{R}_+, \tag{4.1}$$

$$\sum_{l=0}^{p-1} \mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)\mathcal{S}^{(n)}\left(\frac{w+l}{p}\right)^* = \mathbf{I}_N, \quad m,n \in \mathbf{\Lambda}, w \in \mathbb{R}_+. \tag{4.2}$$

Proof. Firstly, we prove the necessary part of the theorem.

By Lemma 3.1 and (3.14), we have

$$\mathbf{O} = \sum_{l\in\mathbb{Z}_+} \tilde{\Phi}(w+l)\tilde{\Psi}_m(w+l)^*$$

$$= \sum_{l\in\mathbb{Z}_+} \mathcal{R}\left(\frac{w+l}{p}\right)\tilde{\Phi}\left(\frac{w+l}{p}\right)\tilde{\Phi}\left(\frac{w+l}{p}\right)^*\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)^*$$

$$= \sum_{l=pn} \mathcal{R}\left(\frac{w}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+n\right)^*\mathcal{S}^{(m)}\left(\frac{w}{p}+n\right)^*$$

$$+ \sum_{l=pn+1} \mathcal{R}\left(\frac{w}{p}+\frac{1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{1}{p}+n\right)^*\mathcal{S}^{(m)}\left(\frac{w}{p}+\frac{1}{p}+n\right)^*$$

$$+\ldots+ \sum_{l=pn+(p-1)} \mathcal{R}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)^*$$

$$\mathcal{S}^{(m)}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)^*$$

$$= \mathcal{R}\left(\frac{w}{p}\right)\left(\sum_{l=pn}\tilde{\Phi}\left(\frac{w}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+n\right)^*\right)\mathcal{S}^{(m)}\left(\frac{w}{p}\right)^*$$

$$+ \mathcal{R}\left(\frac{w}{p}+\frac{1}{p}\right)\left(\sum_{l=pn+1}\tilde{\Phi}\left(\frac{w}{p}+\frac{1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{1}{p}+n\right)^*\right)\mathcal{S}^{(m)}\left(\frac{w}{p}+\frac{1}{p}\right)^*$$

$$+\ldots+ \mathcal{R}\left(\frac{w}{p}+\frac{p-1}{p}\right)\left(\sum_{l=pn+p-1}\tilde{\Phi}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)^*\right)$$

$$\mathcal{S}^{(m)}\left(\frac{w}{p}+\frac{p-1}{p}\right)^*$$

$$= \sum_{l=0}^{p-1} \mathcal{R}\left(\frac{w+l}{p}\right)\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)^*.$$

Again from (3.15) and Lemma 3.1, for $m,n \in \mathbf{\Lambda}$, we get

$$\mathbf{I}_N = \sum_{l\in\mathbb{Z}_+} \tilde{\Psi}_m(w+l)\tilde{\Psi}_n(w+l)^*$$

$$= \sum_{l\in\mathbb{Z}_+} \mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)\tilde{\Phi}\left(\frac{w+l}{p}\right)\tilde{\Phi}\left(\frac{w+l}{p}\right)^{*}\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)^{*}$$

$$= \mathcal{S}^{(m)}\left(\frac{w}{p}\right)\left(\sum_{l=pn}\tilde{\Phi}\left(\frac{w}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+n\right)^{*}\right)\mathcal{S}^{(n)}\left(\frac{w}{p}\right)^{*}$$

$$+ \mathcal{S}^{(m)}\left(\frac{w}{p}+\frac{1}{p}\right)\left(\sum_{l=pn+1}\tilde{\Phi}\left(\frac{w}{p}+\frac{1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{1}{p}+n\right)^{*}\right)\mathcal{S}^{(n)}\left(\frac{w}{p}+\frac{1}{p}\right)^{*}$$

$$+ \ldots + \mathcal{S}^{(m)}\left(\frac{w}{p}+\frac{p-1}{p}\right)$$

$$\left(\sum_{l=pn+p-1}\tilde{\Phi}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)\tilde{\Phi}\left(\frac{w}{p}+\frac{p-1}{p}+n\right)^{*}\right)$$

$$\mathcal{S}^{(n)}\left(\frac{w}{p}+\frac{p-1}{p}\right)^{*}$$

$$= \sum_{l=0}^{p-1}\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)\mathcal{S}^{(n)}\left(\frac{w+l}{p}\right)^{*}.$$

Conversely, suppose identities (4.1), (4.2) hold. From above, we have

$$\sum_{l\in\mathbb{Z}_+}\tilde{\Phi}(w+l)\tilde{\Psi}_m(w+l)^{*} = \sum_{l=0}^{p-1}\mathcal{R}\left(\frac{w+l}{p}\right)\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)^{*} = \mathbf{O},$$

$$\sum_{l\in\mathbb{Z}_+}\tilde{\Psi}_m(w+l)\tilde{\Psi}_n(w+l)^{*} = \sum_{l=0}^{p-1}\mathcal{S}^{(m)}\left(\frac{w+l}{p}\right)\mathcal{S}^{(n)}\left(\frac{w+l}{p}\right)^{*} = \mathbf{I}_N.$$

Therefore

$$\begin{aligned}\langle\Phi(t),\Psi_m(t\ominus k)\rangle &= \sum_{l\in\mathbb{Z}_+}\int_l^{l+1}\tilde{\Phi}(w)\tilde{\Psi}_m(w)^{*}\overline{\chi(k,w)}dw\\ &= \int_0^1\sum_{l\in\mathbb{Z}_+}\tilde{\Psi}(w+l)\tilde{\Psi}_m(w+l)^{*}\overline{\chi(k,w)}dw\\ &= \mathbf{O},\quad m\in\mathbf{\Lambda}, k\in\mathbb{Z}_+.\end{aligned}$$

Similarly, we have

$$\langle\Psi_m(t),\Psi_n(t\ominus k)\rangle = \int_0^1\sum_{l\in\mathbb{Z}_+}\tilde{\Psi}_m(w+l)\tilde{\Psi}_n(w+l)^{*}\overline{\chi(k,w)}dw$$

$$= \quad \delta_{0,k}\delta_{m,n}\mathbf{I}_N, \quad m,n\in\mathbf{\Lambda}, k\in\mathbb{Z}_+.$$

Thus, $\Phi(t)$ and $\Psi_m(t), m\in\mathbf{\Lambda}$ are mutually orthogonal, and $\{\Psi_m(t), m\in\mathbf{\Lambda}\}$ are a finite family of orthogonal vector-valued functions. This proves the orthogonality of $\{\Psi_m(t\ominus k), m\in\mathbf{\Lambda}\}_{k\in\mathbb{Z}_+}$.

Finally, we shall prove the completeness of $\{\Psi_m(t\ominus k), m\in\mathbf{\Lambda}\}_{k\in\mathbb{Z}_+}$ in W_0. For any $\mathbf{f}\in W_0\subset V_1$, there exists a finite supported sequence of $N\times N$ constant matrices $\{A_k\}_{k\in\mathbb{Z}_+}$ such that

$$\mathbf{f}=\sum_{k\in\mathbb{Z}_+}A_k\Phi(pt\ominus k) \tag{4.3}$$

By taking Walsh-Fourier Transform, we have

$$\tilde{\mathbf{f}}(w)=\mathcal{A}(w/p)\tilde{\Phi}(w/p) \tag{4.4}$$

where $\mathcal{A}(w)=\frac{1}{p}\sum_{k\in\mathbb{Z}_+}A_k\overline{\chi(k,w)}$. On the other hand, $\mathbf{f}\in W_0$ and $\mathbf{f}\notin V_0$ means

$$\int_{\mathbb{R}_+}\mathbf{f}(t)\Phi(t\ominus k)^*dt=\mathbf{O}, k\in\mathbb{Z}_+, \tag{4.5}$$

This is equivalent to

$$\sum_{l\in\mathbb{Z}_+}\tilde{\mathbf{f}}(w+l)\tilde{\Phi}(w+l)^*=\mathbf{O}. \tag{4.6}$$

According to (3.6), (4.4) and Lemma 3.1, we have

$$\sum_{l=0}^{p-1}\mathcal{A}\left(\frac{w+l}{p}\right)\mathcal{R}\left(\frac{w+l}{p}\right)^*=\mathbf{O},\quad w\in\mathbb{R}_+ \tag{4.7}$$

Let $\mathcal{A}_1(w)=\left(\ \mathcal{A}\left(\frac{w}{p}\right),\mathcal{A}\left(\frac{w+1}{p}\right),\ldots,\mathcal{A}\left(\frac{w+p-1}{p}\right)\ \right)^*$,
$\mathcal{R}_1(w)=\left(\ \mathcal{R}\left(\frac{w}{p}\right),\mathcal{R}\left(\frac{w+1}{p}\right),\ldots,\mathcal{R}\left(\frac{w+p-1}{p}\right)\ \right)^*$ and for $i=1,2,\ldots,p-1$, we set
$\mathcal{S}_i(w)=\left(\ \mathcal{S}^{(i)}\left(\frac{w}{p}\right),\mathcal{S}^{(i)}\left(\frac{w+1}{p}\right),\ldots,\mathcal{S}^{(i)}\left(\frac{w+p-1}{p}\right)\ \right)^*$. Then the identities (4.1) and (4.2) imply that, for any $w\in\mathbb{R}_+$, the column vectors in the $pN\times N$ matrix $\mathcal{R}_1(w)$ and column vectors in the $pN\times N$ $\mathcal{S}_i(w)$ are orthogonal and all these vectors form an orthonormal basis for the pN-dimensional complex Euclidean space $\mathbb{C}^{pN}$. The identity (4.7) implies that the column vectors in the $pN\times N$ matrix $\mathcal{A}_1(w)$ and column vectors in the $pN\times N$ $\mathcal{R}_1(w)$ are orthogonal.

Thus, there exist $p-1$ matrices $\mathcal{L}^{(m)}(w), m \in \mathbf{\Lambda}$ whose all entries are 1-periodic functions of w such that

$$\mathcal{A}(w) = \sum_{m\in\mathbf{\Lambda}} \mathcal{L}^{(m)}(w)\mathcal{S}^{(m)}(w), \quad w \in \mathbb{R}_+. \tag{4.8}$$

Therefore, by (4.4)

$$\tilde{\mathbf{f}}(w) = \sum_{m\in\mathbf{\Lambda}} \mathcal{L}^{(m)}(w/p)\mathcal{S}^{(m)}(w/p)\Phi(w/p) = \sum_{m\in\mathbf{\Lambda}} \mathcal{L}^{(m)}(w/p)\tilde{\Psi}_m(w). \tag{4.9}$$

By the orthonormality of $\{\Psi_m(t \ominus k), m \in \mathbf{\Lambda}\}_{k\in\mathbb{Z}_+}$, we get

$$\begin{aligned}\int_{\mathbb{R}_+} \tilde{\mathbf{f}}(pw)\tilde{\mathbf{f}}(pw)^* dw &= \sum_{l\in\mathbb{Z}_+} \int_l^{l+1} \sum_{m\in\mathbf{\Lambda}} \mathcal{L}^{(m)}(w)\tilde{\Psi}_m(pw)\tilde{\Psi}_m(pw)^*\mathcal{L}^{(m)}(w)^* dw \\ &= \int_0^1 \sum_{m\in\mathbf{\Lambda}} \mathcal{L}^{(m)}(w)\mathcal{L}^{(m)}(w)^* dw.\end{aligned}$$

This proves that $\mathcal{L}(w)$ has Walsh-Fourier series expansion. Let constant $N \times N$ matrices $\{Q_k^{(m)}\}_{k\in\mathbb{Z}_+}, m \in \mathbf{\Lambda}$ be its Walsh-Fourier coefficients. Then

$$\mathbf{f} = \sum_{k\in\mathbb{Z}_+} \sum_{m\in\mathbf{\Lambda}} Q_k^{(m)} \Psi_m(t \ominus k).$$

This proves the completeness of $\{\Psi_m(t \ominus k), m \in \mathbf{\Lambda}\}_{k\in\mathbb{Z}_+}$ in W_0. □ □

By Walsh-Fourier analysis and (3.7), (3.12), identities (4.1) and (4.2) are equivalent to, respectively,

$$\sum_{v\in\mathbb{Z}_+} P_v(S_{v\oplus pk}^{(m)})^* = \mathbf{O}, \;\; m \in \mathbf{\Lambda}, k \in \mathbb{Z}_+ \tag{4.10}$$

$$\sum_{v\in\mathbb{Z}_+} S_v^{(m)}(S_{v\oplus pk}^{(n)})^* = p\delta_{m,n}\delta_{0,k}\mathbf{I}_N, \;\; m, n \in \mathbf{\Lambda}, k \in \mathbb{Z}_+. \tag{4.11}$$

Theorem 4.1 implies that a vector valued multiresolution p-analysis in $L^2(\mathbb{R}_+, \mathbb{C}^N)$ gives us vector-valued scaling function Φ and moreover associated $p-1$ orthogonal vector valued wavelet functions $\Psi_m(t), m \in \mathbf{\Lambda}$ such that whose dilations and translations $\Psi_{j,k,m}(t) = p^{j/2}\Psi_m(p^j t \ominus k), j \in \mathbb{Z}, k \in \mathbb{Z}_+, m \in \mathbf{\Lambda}$ form an orthonormal basis for $L^2(\mathbb{R}_+, \mathbb{C}^N)$. Therefore to construct a vector-valued wavelet functions, we only need to construct a vector-valued scaling function.

Theorem 4.2. *Let*

$$\mathcal{R}(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}_+} R_k \overline{\chi(k, w)}$$

be a matrix valued scaling filter satisfying following conditions:

(a).

$$\sum_{l=0}^{p-1} \mathcal{R}\left(\frac{w+l}{p}\right) \mathcal{R}\left(\frac{w+l}{p}\right)^* = \mathbf{I}_N. \tag{4.12}$$

(b). There exists a constant $C > 0$ and integer $M > 0$ such that for $w \in (0, p^M)$

$$\| \prod_{l=1}^{M} \mathcal{R}\left(\frac{w}{p^l}\right) \| < C \| \prod_{l=1}^{\infty} \mathcal{R}\left(\frac{w}{p^l}\right) \| \tag{4.13}$$

If $\Phi \in L^2(\mathbb{R}_+, \mathbb{C}^N)$ such that $\tilde{\Phi}(0)\tilde{\Phi}(0)^* = I_N$ and its Walsh-Fourier transform can be written as

$$\tilde{\Phi}(w) = \prod_{l=1}^{\infty} \mathcal{R}\left(\frac{w}{p^l}\right) \tilde{\Phi}(0) \tag{4.14}$$

then $\Phi(t)$ is a vector valued scaling function for a vector valued multiresolution p-analysis in $L^2(\mathbb{R}_+, \mathbb{C}^N)$. Thus, the corresponding $\Psi_{j,k,m}(t) = p^{\frac{j}{2}} \Psi_m(p^j t \ominus k), j \in \mathbb{Z}, k \in \mathbb{Z}_+, m \in \mathbf{\Lambda}$ form an orthonormal basis for $L^2(\mathbb{R}_+, \mathbb{C}^N)$.

Proof. To prove Theorem 4.2, we only need to prove the orthonormality of $\Phi(t \ominus k), k \in \mathbb{Z}_+$. The rest is similar to Daubechies [2], Mallat [9] and Farkov [3].

Now

$$\int_{\mathbb{R}_+} \Phi(t)\Phi(t \ominus k)dt = \int_{\mathbb{R}_+} \tilde{\Phi}(w)\tilde{\Phi}(w)^* \overline{\chi(k, w)} dw. \tag{4.15}$$

For an integer$M > 0$, let

$$\mu_M(w) = \prod_{l=1}^{M} \mathcal{R}\left(\frac{w}{p^l}\right) \tilde{\Phi}(0) \mathbf{1}_{(0,p^M)}(w). \tag{4.16}$$

where $\mathbf{1}_{(0,p^M)}(w)$ is the characteristic function of a subset $(0, p^M)$ of $\mathbb{R}_+$.

Now

$$\int_{\mathbb{R}_+} \mu_M(w)\mu_M(w)^* \overline{\chi(k, w)} dw$$

$$
\begin{aligned}
&= \int_0^{p^M} \left(\mathcal{R}\left(\frac{w}{p}\right)\mathcal{R}\left(\frac{w}{p^2}\right)\dots\mathcal{R}\left(\frac{w}{p^M}\right)\tilde{\Phi}(0)\right) \\
&\quad \left(\tilde{\Phi}(0)^*\mathcal{R}\left(\frac{w}{p^M}\right)^*\mathcal{R}\left(\frac{w}{p^{M-1}}\right)^*\dots\mathcal{R}\left(\frac{w}{p}\right)^*\right)\overline{\chi(k,w)}dw \\
&= p^M\int_0^1 \left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)\mathcal{R}(w)\mathcal{R}(w)^*\left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)^*\overline{\chi(k,p^Mw)}dw \\
&= p^M\int_0^{\frac{1}{p}} \left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)\mathcal{R}(w)\mathcal{R}(w)^*\left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)^*\overline{\chi(k,p^Mw)}dw \\
&+ p^M\int_{\frac{1}{p}}^{\frac{2}{p}} \left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)\mathcal{R}(w)\mathcal{R}(w)^*\left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)^*\overline{\chi(k,p^Mw)}dw \\
&+ \dots \\
&+ p^M\int_{\frac{p-1}{p}}^{1} \left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)\mathcal{R}(w)\mathcal{R}(w)^*\left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)^*\overline{\chi(k,p^Mw)}dw \\
&= p^M\int_0^{\frac{1}{p}} \left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)\left(\sum_{l=0}^{p-1}\mathcal{R}\left(w+\frac{l}{p}\right)\mathcal{R}\left(w+\frac{l}{p}\right)^*\right) \\
&\quad \left(\prod_{l=1}^{M-1}\mathcal{R}(p^{M-l}w)\right)^*\overline{\chi(k,p^Mw)}dw.
\end{aligned}
$$

Using (4.12) and $\tilde{\Phi}(0)\tilde{\Phi}(0)^* = I_N$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+} \mu_M(w)\mu_M(w)^*\overline{\chi(k,w)}dw \\
&= \int_0^{p^{M-1}} \left(\prod_{l=1}^{M-1}\mathcal{R}\left(\frac{w}{p^l}\right)\right)\left(\prod_{l=1}^{M-1}\mathcal{R}\left(\frac{w}{p^l}\right)\right)^*\overline{\chi(k,w)}dw \\
&= \int_{\mathbb{R}_+} \mu_{M-1}(w)\mu_{M-1}(w)^*\overline{\chi(k,w)}dw \\
&= \dots = \\
&= \int_{\mathbb{R}_+} \mu_1(w)\mu_1(w)^*\overline{\chi(k,w)}dw \\
&= \int_0^1 \mathbf{I}_N\overline{\chi(k,w)}dw \\
&= \delta_{0,k}\mathbf{I}_N.
\end{aligned}
$$

From (4.16), we get that $\mu_k(w)$ converges to $\Phi(w)$ pointwise. In view of (4.13), we have

$$\|\mu_M(w)\mu_M(w)^* - \tilde{\Phi}(w)\tilde{\Phi}(w)^*\| \leq (C+1)\|\tilde{\Phi}(w)\tilde{\Phi}(w)^*\|, \quad w \in \mathbb{R}_+.$$

Since all matrix norms are equivalent, there exists a constant $C_1 > 0$ such that

$$\|\mu_M\mu_M^* - \tilde{\Phi}\tilde{\Phi}^*\| \leq C_1 \int_{\mathbb{R}_+} \|\mu_M(w)\mu_M(w)^* - \tilde{\Phi}(w)\tilde{\Phi}(w)^*\|^2 dw.$$

By the Dominated convergence theorem, we get $\|\mu_M\mu_M^* - \tilde{\Phi}\tilde{\Phi}^*\| \to 0$ as $M \to \infty$. Therefore

$$\begin{aligned} \int_{\mathbb{R}_+} \tilde{\Phi}(w)\tilde{\Phi}(w)^*\overline{\chi(k,w)}dw &= \lim_{M\longrightarrow\infty} \int_{\mathbb{R}_+} \mu_M(w)\mu_M(w)^*\overline{\chi(k,w)}dw \\ &= \delta_{0,k}\mathbf{I}_N. \end{aligned}$$

This proves the orthonormality of $\Phi(t \ominus k), k \in \mathbb{Z}_+$. □ □

Acknowledgments

The authors are thankful to University Grants Commission and Council of Scientific and Industrial Research, New Delhi for providing the financial assistance for the preparation of the manuscript.

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