ORTHOGONAL VECTOR VALUED WAVELETS ON $\mathbb{R}_+$

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Abstract: Xia and Suter [15] have introduced the notion of vector valued multiresolution analysis on real line $\mathbb{R}$. Chen and Chang [1] have given an algorithm for construction of vector valued wavelets. Farkov [3] has studied the notion of multiresolution analysis on locally abelian groups and constructed the compactly supported orthogonal $p$-wavelets on $L^2(\mathbb{R}_+)$. In this paper, we introduce vector valued multiresolution $p$-analysis on positive half line. We find necessary and sufficient condition for the existence of associated vector valued wavelets. We construct vector valued wavelets on $\mathbb{R}_+$. Our approach is connected with Walsh-Fourier theory.

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1. Introduction

Wavelet theory has been studied extensively in both theory and applications. The main advantage of wavelets is their time-frequency localization property. The wavelet transform is a simple mathematical tool that cuts up data or functions into different frequency components, studies each components with a resolution matched to its scale. Many signals in areas like music, speech, image and video images can be efficiently represented by wavelets that are translations and dilations of a single function called mother wavelet with bandpass property.

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Multiresolution Analysis is the heart of wavelet theory. A multiresolution analysis on the set of real numbers \( \mathbb{R} \), introduced by Mallat [9] is an increasing sequence of closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) of \( L^2(\mathbb{R}) \) such that \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( \{V_j\}_{j \in \mathbb{Z}} \) and which satisfies \( f(t) \in V_j \) if and only if \( f(2t) \in V_{j+1} \). Also there exists an element \( \phi \in V_0 \) called scaling function such that the collection of integer translates of \( \phi \), i.e. \( \{\phi(t-k)\}_{k \in \mathbb{Z}} \) is a complete orthonormal basis for \( V_0 \). In the definition of multiresolution analysis the dilation factor of 2 can be replaced by an integer \( n > 2 \) and one can construct \( n - 1 \) wavelets to generate the whole space \( L^2(\mathbb{R}) \). A similar generalization of multiresolution analysis can be made in higher dimensions by considering matrix dilations.

Walsh analysis or Dyadic harmonic analysis has been extensively studied: both aspects theory as well as applications, see Golubov et al. [5], Schipp et al. [11]. In his papers, Lang [6-8] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group \( C \) by following the rules and procedures of Mallat and Daubechies via scaling filters. These wavelets turn out to be certain lacunary Walsh series on the \( \mathbb{R}_+ \). Later on, Farkov [4] extended the results of Lang [6-8] on the wavelet analysis on the Cantor dyadic group \( C \) to the locally compact Abelian group \( G \) which is defined for an integer \( \geq 2 \) and coincides with \( C \) when \( p = 2 \). Subsequently, Protasov and Farkov [10] constructed dyadic compactly supported wavelets in \( L^2(\mathbb{R}_+) \), whereas Farkov [3] has given the general construction of all compactly supported orthogonal \( p \)-wavelets in \( L^2(\mathbb{R}_+) \) and proved necessary and sufficient conditions for scaling filters with \( n \) many terms \( p, n \geq 2 \) to generate a \( p \)-MRA in \( L^2(\mathbb{R}_+) \). The approach adopted by Farkov is connected with Walsh-Fourier transform and the elements of M-band wavelet theory.

The paper is organized as follows. In Section 2, we explain certain results of Walsh-Fourier analysis. We present brief review of generalized Walsh functions, Walsh-Fourier transforms and its various properties, multiresolution \( p \)-analysis in \( L^2(\mathbb{R}_+) \) introduced by Farkov [3]. We introduce the concept of vector valued multiresolution \( p \)-analysis on \( \mathbb{R}_+ \) in Section 3. In Section 4, necessary and sufficient condition for the existence of associated vector valued wavelets is given. We construct a vector valued multiresolution analysis on the positive half line with a compactly supported vector valued scaling function \( \Phi \).

## 2. Walsh-Fourier Analysis

Let \( p \) be a fixed natural number greater than 1. As usual, let \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{Z}_+ = \{0, 1, \ldots\} \). Denote by \( [x] \) the integer part of \( x \). For \( x \in \mathbb{R}_+ \) and for any
positive integer \( j \),

\[
x_j = [p^j x](\text{mod } p), \quad x_{-j} = [p^{1-j} x](\text{mod } p)
\]  

(2.1)

where \( x_j, x_{-j} \in \{0, 1, \ldots, p-1\} \). It is clear that for each \( x \in \mathbb{R}_+ \), \( \exists k = k(x) \in \mathbb{N} \) such that \( x_{-j} = 0, \forall j > k \).

Consider on \( \mathbb{R}_+ \) the addition defined as follows:

\[
x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j}
\]  

(2.2)

with

\[
\xi_j = x_j + y_j(\text{mod } p), \quad j \in \mathbb{Z} \setminus \{0\},
\]  

(2.3)

where \( \xi_j \in \{0, 1, 2, \ldots, p-1\} \) and \( x_j, y_j \) are calculated by (2.1). As usual, we write \( z = x \ominus y \) if \( z \oplus y = x \), where \( \ominus \) denotes subtraction modulo \( p \) in \( \mathbb{R}_+ \).

For \( x \in [0, 1) \), let \( r_0(x) \) is given by

\[
r_0(x) = \begin{cases} 
1, & x \in [0, 1/p) \\
\varepsilon_p^j, & x \in [jp^{-1}, (j+1)p^{-1}), j = 1, 2, \ldots, p-1
\end{cases}
\]  

(2.4)

where \( \varepsilon = \exp\left(\frac{2\pi i}{p}\right) \).

The extension of the function \( r_0 \) to \( \mathbb{R}_+ \) is defined by the equality \( r_0(x+1) = r_0(x), x \in \mathbb{R}_+ \). Then the generalized Walsh functions \( \{\omega_m(x)\}_{m \in \mathbb{Z}_+} \) are defined by

\[
\omega_0(x) \equiv 1, \quad \omega_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^\mu_j,
\]

where \( m = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, 2, \ldots, p-1\}, \quad \mu_k \neq 0 \).

For \( x, w \in \mathbb{R}_+ \), let

\[
\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j)\right)
\]  

(2.5)

where \( x_j \) and \( w_j \) are given by (2.1). We note that

\[
\chi\left(x, \frac{m}{p^{n-1}}\right) = \chi\left(\frac{m}{p^{n-1}}, x\right) = \omega_m\left(\frac{x}{p^{n-1}}\right), \quad \forall x \in [0, p^{n-1}), m \in \mathbb{Z}_+.
\]

The **Walsh-Fourier transform** of a function \( f \in L^2(\mathbb{R}_+) \) is defined by

\[
\hat{f}(w) = \int_{\mathbb{R}_+} f(x) \overline{\chi(x, w)} \, dw
\]  

(2.6)
where \( \chi(x, w) \) is given by (2.5). The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform [5, 11]. In particular, if \( f \in L^2(\mathbb{R}_+) \), then \( \tilde{f} \in L^2(\mathbb{R}_+) \) and

\[
\|f\|_{L^2(\mathbb{R}_+)} = \|\tilde{f}\|_{L^2(\mathbb{R}_+)}.
\]

(2.7)

If \( x, y, w \in \mathbb{R}_+ \) and \( x \oplus y \) is \( p \)-adic irrational, then

\[
\chi(x \oplus y, w) = \chi(x, w)\chi(y, w), \quad \chi(x \ominus y, w) = \chi(x, w)\chi(y, w),
\]

(2.8)

see Golubov et al. [5], Schipp et al. [11]. Thus for fixed \( x \) and \( w \), the equality (2.8) holds for all \( y \in \mathbb{R}_+ \) except for countably many. It was shown by Golubov et al. [5] that systems \( \{\chi(\alpha, \cdot)\}_{\alpha=0}^{\infty} \) and \( \{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty} \) are orthonormal basis in \( L^2(0, 1) \).

According to Schipp et al. [11] for any \( \phi \in L^2(\mathbb{R}_+) \), we have

\[
\int_{\mathbb{R}_+} \phi(t)\overline{\phi(t \ominus k)}dt = \int_{\mathbb{R}_+} \tilde{\phi}(w)\overline{\phi(w)}\chi(k, w)dw
\]

(2.9)

Let \( \{w\} \) be the fractional part of \( w \). For any \( \phi \in L^2(\mathbb{R}_+) \) and \( k \in \mathbb{Z}_+ \), we have \( \chi(k, w) = \chi(k, \{w\}) \). Therefore \( \chi(k, w + l) = \chi(k, w), l \in \mathbb{Z}_+ \). It follows from (2.9) that

\[
\int_{\mathbb{R}_+} \phi(t)\overline{\phi(t \ominus k)}dt = \sum_{l \in \mathbb{Z}_+} \int_{0}^{1} |\tilde{\phi}(w)|^2 \chi(k, w)dw
\]

\[
= \int_{0}^{1} (\sum_{l \in \mathbb{R}_+} |\tilde{\phi}(w + 1)|^2)\chi(k, w)dw
\]

Therefore, a necessary and sufficient condition for a system \( \{\phi(t \ominus k) | k \in \mathbb{Z}_+ \} \) to be orthonormal in \( L^2(\mathbb{R}_+) \) is

\[
\sum_{l \in \mathbb{R}_+} |\tilde{\phi}(w + 1)|^2 = 1 \quad a.e.
\]

(2.10)

Multiresolution \( p \)-analysis in \( L^2(\mathbb{R}_+) \) defined by Farkov [3] is as follows:

**Definition 2.1.** A multiresolution \( p \)-analysis on \( L^2(\mathbb{R}_+) \) is a nested sequence of closed subspaces \( V_j, j \in \mathbb{Z} \) of \( L^2(\mathbb{R}_+) \) such that following hold:

(a). \( V_j \subset V_{j+1}, j \in \mathbb{Z} \).

(b). \( \bigcup_j V_j \) is dense in \( L^2(\mathbb{R}_+) \) and \( \bigcap_j V_j = 0 \).
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(c). $f(t) \in V_j$ if and only if $f(pt) \in V_{j+1}$.

(d). $f(t) \in V_0 \Rightarrow f(t \oplus k) \in V_0$ for all $k \in \mathbb{Z}_+$.

(e). there exists a function called scaling function $\Phi \in V_0$ such that its translations $\Phi_k(t) = \Phi(t \ominus k), k \in \mathbb{Z}_+$, form an orthonormal basis for $V_0$.

The function $\phi$ is called the scaling function in $L^2(\mathbb{R}_+)$.  

Farkov has given a general construction of compactly supported orthogonal $p$-wavelets in $L^2(\mathbb{R}_+)$ arising from scaling filters with $p^n$ many terms. For all integer $p \geq 2$ these wavelets are identified with certain lacunary Walsh series on $\mathbb{R}_+$. In this new setting Farkov [3] has proved the extension of classical results concerning necessary and sufficient condition of wavelets associated with the classical multiresolution analysis.

The following theorem by Farkov [3] generalizes A. Cohen’s result, see Daubechies [2]:

**Theorem 2.1.** Let

$$m_0(w) = \sum_{\alpha=0}^{p^n-1} a_\alpha \chi(k, w)$$

be a polynomial satisfying the following conditions:

(i). $m_0(0) = 1$.

(ii). $\sum_{j=0}^{p^n-1} |m_0(sp^{-n} \oplus jp^{-1})|^2 = 1$ for $s = 0, 1, \ldots p^{n-1} - 1$.

(iii). There exists a $W$-compact set $E$ such that $0 \in \text{int}(E)$, $\mu(E) = 1$, $E \equiv [0, 1)(mod \mathbb{Z}_+)$ and

$$\inf_{j \in \mathbb{N}} \inf_{w \in E} |m_0(p^{-j}w)| > 0$$

If the Walsh-Fourier transform if $\phi \in L^2(\mathbb{R}_+)$ can be written as

$$\tilde{\phi}(w) = \prod_{j=1}^{\infty} m_0(p^{-j}w),$$

then $\phi$ is scaling function in $L^2(\mathbb{R}_+)$.  

3. Vector-Valued Multiresolution $p$-Analysis on $\mathbb{R}_+$

We use the following notations. Let $\mathbb{C}$ be the set of all complex numbers, $\mathbf{I}_N$ and $\mathbf{O}$ represent $N \times N$ identity matrix and the zero matrix respectively.

$L^2(\mathbb{R}_+, \mathbb{C}^N)$ represents the set of square integrable vector-valued functions $f(t)$ on positive half line, $\mathbb{R}_+$ i.e.,

$$L^2(\mathbb{R}_+, \mathbb{C}^N) = \{ f(t) = ( f_1(t), f_2(t), \ldots, f_N(t) )^T : t \in \mathbb{R}_+, \quad f_v(t) \in L^2(\mathbb{R}_+), v = 1, 2, \ldots, N \}$$

where $T$ denotes Transpose.

For $f \in L^2(\mathbb{R}_+, \mathbb{C}^N)$, $\|f\|_{L^2(\mathbb{R}_+, \mathbb{C}^N)}$ is the norm of the function $f$, i.e.,

$$\|f\|_{L^2(\mathbb{R}_+, \mathbb{C}^N)} = \sqrt{\sum_{v=1}^{N} \int_{\mathbb{R}_+} |f_v(t)|^2 dt}$$

and integration of $f$ is given by

$$\int_{\mathbb{R}_+} f(t) dt = \left( \int_{\mathbb{R}_+} f_1(t) dt, \int_{\mathbb{R}_+} f_2(t) dt, \ldots, \int_{\mathbb{R}_+} f_N(t) dt \right)^T.$$

The Walsh-Fourier transform of $f(t)$ is defined by

$$\tilde{f}(w) = \int_{\mathbb{R}_+} f(t) \chi(k, w) dt = \left( \tilde{f}_1(w), \tilde{f}_2(w), \ldots, \tilde{f}_N(w) \right)^T. \quad (3.1)$$

For two vector valued functions $f, h \in L^2(\mathbb{R}_+, \mathbb{C}^N)$, their symbol inner product is defined by

$$\langle f, h \rangle_{L^2(\mathbb{R}_+, \mathbb{C}^N)} = \int_{\mathbb{R}_+} f(t) h(t)^* dt,$$

where $^*$ means complex conjugate and transpose. The inner product defined above is matrix valued (usually it is a scaler valued).

A sequence $\{ f_k(t) \}_{k \in \mathbb{Z}_+} \subset U \subseteq L^2(\mathbb{R}_+, \mathbb{C}^N)$ is called orthonormal set of $U$, if it satisfies

$$\langle f_k(t), f_n(t) \rangle = \delta_{k,n} \mathbf{I}_N, \quad (3.2)$$

where $\delta_{k,n}$ is the Kronecker delta such that $\delta_{k,n} = 1$ when $k = n$ and $\delta_{k,n} = 0$ when $k \neq n$. 
Definition 3.1. We say that \( f(t) \in U \subseteq L^2(\mathbb{R}^+, \mathbb{C}^N) \) is an orthogonal vector-valued function in \( U \) if its translations \( \{ f(t \ominus k) \}_{k \in \mathbb{Z}^+} \) is an orthonormal set in \( U \), i.e.,
\[
\langle f(t \ominus k), f(t \ominus n) \rangle = \delta_{k,n} I_N, \quad k, n \in \mathbb{Z}^+.
\] (3.3)

Definition 3.2. A sequence \( \{ f_k(t) \}_{k \in \mathbb{Z}^+} \subset U \subseteq L^2(\mathbb{R}^+, \mathbb{C}^N) \) is called an orthonormal basis of \( U \) if it satisfies (3.2), and for any \( h(t) \in U \), there exists a unique sequence of \( N \times N \) constant matrices \( \{ A_k \}_{k \in \mathbb{Z}^+} \) such that
\[
h(t) = \sum_{k \in \mathbb{Z}^+} A_k f_k(t).
\]

The multiresolution analysis approach is one of the main approaches in the construction of wavelets. We introduce vector-valued multiresolution \( p \)-analysis on positive half line and give the definition for associated orthogonal vector valued wavelets.

Definition 3.3. A vector valued multiresolution \( p \)-analysis on \( L^2(\mathbb{R}^+, \mathbb{C}^N) \) is a nested sequence of closed subspaces \( V_j, \in \mathbb{Z} \) of \( L^2(\mathbb{R}^+, \mathbb{C}^N) \) such that following hold:

(a). \( V_j \subset V_{j+1}, j \in \mathbb{Z} \).

(b). \( \bigcup_j V_j \) is dense in \( L^2(\mathbb{R}^+, \mathbb{C}^N) \) and \( \bigcap_j V_j = \{ 0 \} \), where \( 0 \) is the zero vector of \( L^2(\mathbb{R}^+, \mathbb{C}^N) \).

(c). \( f(t) \in V_j \) if and only if \( f(pt) \in V_{j+1} \).

(d). \( f(t) \in V_0 \Rightarrow f(t \oplus k) \in V_0 \) for all \( k \in \mathbb{Z}^+ \).

(e). there exists a function called scaling function \( \Phi \in V_0 \) such that its translations \( \Phi_k(t) = \Phi(t \ominus k), k \in \mathbb{Z}^+ \), form an orthonormal basis for \( V_0 \).

Now \( \Phi(t) \in V_0 \Rightarrow \Phi(pt) \in V_1 \), by (e)
\[
\delta_{0,k} I_N = \int_{\mathbb{R}^+} \Phi(t)\Phi(t \ominus k)dt
= \int_{\mathbb{R}^+} \sqrt{p} \Phi(pt)\sqrt{p} \Phi(pt \ominus k)dt.
\]
So \( \Phi_{1,k}(t) = \{ \sqrt{p} \Phi(pt \ominus k) \}_{k \in \mathbb{Z}^+} \) form an orthonormal basis for \( V_1 \).
Therefore, the space $V_j$ is defined by
\[
V_j = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{p^j \Phi(p^j t \ominus k)\}, k \in \mathbb{Z}_+), j \in \mathbb{Z}. \tag{3.4}
\]

Now $\Phi \in V_1$, there exists a sequence of $N \times N$ constant matrices $\{R_k\}_{k \in \mathbb{Z}^+}$ such that
\[
\Phi(t) = \sum_{k \in \mathbb{Z}^+} R_k \Phi(pt \ominus k), t \in \mathbb{R}_+. \tag{3.5}
\]
If the sequence $\{R_k\}_{k \in \mathbb{Z}^+}$ is finite, we say that $\Phi(t)$ is a compactly supported vector-valued function. By Walsh Fourier Transform, we have
\[
\tilde{\Phi}(w) = R(w/p)\tilde{\Phi}(w/p), \; w \in \mathbb{R}_+. \tag{3.6}
\]
where
\[
R(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}^+} R_k \chi(k, w). \tag{3.7}
\]
Noting that $\chi(k, w + l) = \chi(k, w), k, l \in \mathbb{R}_+$, so $R(w)$ is $1$-periodic function of $w$. By (3.6), we have
\[
\tilde{\Phi}(w) = R(w/p)R(w/p^2)\ldots \tilde{\Phi}(0) = \prod_{l=1}^{\infty} R(w/p^l)\tilde{\Phi}(0). \tag{3.8}
\]
Let $W_j, j \in \mathbb{Z}$ denote the orthocomplement subspace of $V_j$ in $V_{j+1}$ and there exists $p - 1$ vector valued functions $\Psi_m(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N), m \in \Lambda$, where $\Lambda = \{1, 2, \ldots, p - 1\}$, such that their translations and dilations form Riesz basis of $W_j$, i.e.,
\[
W_j = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{p^j \Psi_m(p^j t \ominus k)\}, m \in \Lambda, k \in \mathbb{Z}_+), j \in \mathbb{Z}. \tag{3.9}
\]
We say that $\{\Phi(t), \Psi_1(t), \Psi_2(t), \ldots, \Psi_{p-1}(t)\}$ is a vector-valued wavelet system. For each $m \in \Lambda$, $\Psi_m(t) \in W_0 \subset V_1$, there exist $p - 1$ sequences of $N \times N$ constant matrices $\{S_k^{(m)}\}_{k \in \mathbb{Z}^+}$ such that
\[
\Psi_m(t) = \sum_{k \in \mathbb{Z}^+} S_k^{(m)} \Phi(pt \ominus k), \; m \in \Lambda, t \in \mathbb{R}_+. \tag{3.10}
\]
By taking Walsh-Fourier Transform, the refinement equation (3.10) becomes
\[
\tilde{\Psi}_m(w) = S^{(m)}(w/p)\tilde{\Phi}(w/p), \; w \in \mathbb{R}_+, m \in \Lambda \tag{3.11}
\]
where

\[ S^{(m)}(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}^+} S^{(m)}_k \chi(k, w). \] (3.12)

If \( \Phi(t) \in L^2(\mathbb{R}^+, \mathbb{C}^N) \) is an orthogonal vector-valued scaling function, then it follows from (3.3) that

\[ \langle \Phi(t), \Phi(t \oplus k) \rangle = \delta_{0,k} \mathbf{I}_N, \quad k \in \mathbb{Z}^+. \] (3.13)

We say \( p - 1 \) vector-valued functions \( \Psi_1(t), \Psi_2(t), \ldots, \Psi_{p-1}(t) \) are orthogonal vector-valued wavelet functions associated with the orthogonal vector-valued scaling function \( \Phi(t) \), if they satisfy

\[ \langle \Psi_m(t), \Phi(t \oplus k) \rangle = 0, \quad m \in \Lambda, k \in \mathbb{Z}^+. \] (3.14)

and the family \( \{ \Psi_m(t \oplus k), m \in \Lambda \}_{k \in \mathbb{Z}^+} \) is an orthonormal basis of the subspace \( W_0 \). So we have

\[ \langle \Psi_m(t), \Psi_n(t \oplus k) \rangle = \delta_{0,k} \delta_{m,n} \mathbf{I}_N, \quad m, n \in \Lambda, k \in \mathbb{Z}^+. \] (3.15)

The following lemma, which will be used in next section, gives a characterization in the frequency domain of an orthogonal vector-valued function \( f(t) \).

**Lemma 3.1.** Let \( f(t) \in L^2(\mathbb{R}^+, \mathbb{C}^N) \). Then \( f(t) \) is an orthogonal vector-valued function if and only if

\[ \sum_{l \in \mathbb{Z}^+} \tilde{f}(w + l) \tilde{f}(w + l)^* = \mathbf{I}_N, w \in \mathbb{R}^+. \] (3.16)

**Proof.** Let \( f(t) = (f_1(t), f_2(t), \ldots, f_N(t))^T \in L^2(\mathbb{R}^+, \mathbb{C}^N) \),

\[ \delta_{0,k} \mathbf{I}_N = \int_{\mathbb{R}^+} f(t) \overline{f(t \oplus k)^*} dt \]

\[ = \begin{pmatrix} \int_{\mathbb{R}^+} f_1(t) f_1(t \oplus k) dt, & \int_{\mathbb{R}^+} f_1(t) f_2(t \oplus k) dt, & \cdots, & \int_{\mathbb{R}^+} f_1(t) f_N(t \oplus k) dt \\ \int_{\mathbb{R}^+} f_2(t) f_1(t \oplus k) dt, & \int_{\mathbb{R}^+} f_2(t) f_2(t \oplus k) dt, & \cdots, & \int_{\mathbb{R}^+} f_2(t) f_N(t \oplus k) dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}^+} f_N(t) f_1(t \oplus k) dt, & \int_{\mathbb{R}^+} f_N(t) f_2(t \oplus k) dt, & \cdots, & \int_{\mathbb{R}^+} f_N(t) f_N(t \oplus k) dt \end{pmatrix}. \]

By identity (2.9), we arrive at

\[ \delta_{0,k} \mathbf{I}_N = \int_{\mathbb{R}^+} \tilde{f}(w) \overline{\tilde{f}(w)^*} \chi(k, w) dw \]
\[
\begin{align*}
&= \sum_{l \in \mathbb{Z}_+} \int_{l}^{l+1} \tilde{f}(w)\tilde{f}(w)^* \chi(k, w) dw \\
&= \int_{0}^{1} \sum_{l \in \mathbb{Z}_+} \tilde{f}(w + l)\tilde{f}(w + l)^* \chi(k, w) dw
\end{align*}
\]

So, \( f(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N) \) is orthogonal \( \iff \sum_{l \in \mathbb{Z}_+} \tilde{f}(w + l)\tilde{f}(w + l)^* = \mathbb{I}_N \).

**Lemma 3.2.** If \( \Phi(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N) \), defined by (3.5), is an orthogonal vector-valued scaling function, then \( \forall \ k \in \mathbb{Z}_+ \), we have

\[
\sum_{u \in \mathbb{Z}_+} R_u(R_{u\oplus pk})^* = p\delta_{0,k}\mathbb{I}_N.
\]

(3.17)

**Proof.** Now

\[
\delta_{0,k}\mathbb{I}_N = \langle \Phi(t \ominus k), \Phi(t) \rangle
\]

\[
= \sum_{u \in \mathbb{Z}_+} \sum_{v \in \mathbb{Z}_+} \int_{\mathbb{R}_+} R_u \Phi(pt \ominus pk \ominus u) \Phi(pt \ominus v) R_v^* dt
\]

\[
= p \sum_{u,v \in \mathbb{Z}_+} R_u \langle \Phi(t \ominus pk \ominus u), \Phi(t \ominus v) \rangle R_v^*
\]

\[
= p \sum_{u \in \mathbb{Z}_+} R_u(R_{u\oplus pk})^*.
\]

4. The Existence of Orthogonal Vector-Valued Wavelets on \( \mathbb{R}_+ \)

In this section, we consider the existence of compactly supported vector-valued wavelets on \( \mathbb{R}_+ \). We obtain the necessary and sufficient condition for the existence of vector valued orthogonal wavelets.

**Theorem 4.1.** Let \( \Phi(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N) \) defined in (3.5) be an orthogonal vector-valued scaling function. Assume that \( \Psi_m(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N), m \in \Lambda, \) and \( \mathcal{R}(w) \) and \( S^{(m)}(w) \) are defined by (3.7) and (3.12) respectively. Then \( \Psi_m(t), m \in \Lambda \) are orthogonal vector-valued wavelet functions associated with \( \Phi(t) \) if and only if

\[
\sum_{l=0}^{p-1} \mathcal{R} \left( \frac{w + l}{p} \right) S^{(m)} \left( \frac{w + l}{p} \right)^* = \mathbb{O}, \quad m \in \Lambda, w \in \mathbb{R}_+.
\]

(4.1)
\[
\sum_{l=0}^{p-1} \mathcal{S}^{(m)} \left( \frac{w+l}{p} \right) \mathcal{S}^{(n)} \left( \frac{w+l}{p} \right)^* = \mathbf{I}_N, \quad m, n \in \Lambda, w \in \mathbb{R}_+.
\] 

**Proof.** Firstly, we prove the necessary part of the theorem.

By Lemma 3.1 and (3.14), we have

\[
\mathcal{O} = \sum_{l \in \mathbb{Z}_+} \tilde{\Phi}(w+l) \tilde{\Psi}_m(w+l)^*
\]

\[
= \sum_{l \in \mathbb{Z}_+} \mathcal{R} \left( \frac{w+l}{p} \right) \tilde{\Phi} \left( \frac{w+l}{p} \right) \tilde{\Phi} \left( \frac{w+l}{p} \right)^* \mathcal{S}^{(m)} \left( \frac{w+l}{p} \right)^*
\]

\[
= \sum_{l=pn} \mathcal{R} \left( \frac{w}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + n \right)^* \mathcal{S}^{(m)} \left( \frac{w}{p} + n \right)^*
\]

\[
+ \sum_{l=pn+1} \mathcal{R} \left( \frac{w}{p} + \frac{1}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + \frac{1}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + \frac{1}{p} + n \right)^* \mathcal{S}^{(m)} \left( \frac{w}{p} + \frac{1}{p} + n \right)^*
\]

\[
+ \ldots + \sum_{l=pn+(p-1)} \mathcal{R} \left( \frac{w}{p} + \frac{p-1}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + \frac{p-1}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + \frac{p-1}{p} + n \right)^* \mathcal{S}^{(m)} \left( \frac{w}{p} + \frac{p-1}{p} + n \right)^*
\]

\[
= \mathcal{R} \left( \frac{w}{p} \right) \left( \sum_{l=pn} \tilde{\Phi} \left( \frac{w}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + n \right)^* \mathcal{S}^{(m)} \left( \frac{w}{p} \right)^* \right)
\]

\[
+ \mathcal{R} \left( \frac{w}{p} + \frac{1}{p} \right) \left( \sum_{l=pn+1} \tilde{\Phi} \left( \frac{w}{p} + \frac{1}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + \frac{1}{p} + n \right)^* \mathcal{S}^{(m)} \left( \frac{w}{p} + \frac{1}{p} \right)^* \right)
\]

\[
+ \ldots + \mathcal{R} \left( \frac{w}{p} + \frac{p-1}{p} \right) \left( \sum_{l=pn+p-1} \tilde{\Phi} \left( \frac{w}{p} + \frac{p-1}{p} + n \right) \tilde{\Phi} \left( \frac{w}{p} + \frac{p-1}{p} + n \right)^* \mathcal{S}^{(m)} \left( \frac{w}{p} + \frac{p-1}{p} \right)^* \right)
\]

\[
= \sum_{l=0}^{p-1} \mathcal{R} \left( \frac{w+l}{p} \right) \mathcal{S}^{(m)} \left( \frac{w+l}{p} \right)^*.
\]

Again from (3.15) and Lemma 3.1, for \( m, n \in \Lambda \), we get

\[
\mathbf{I}_N = \sum_{l \in \mathbb{Z}_+} \tilde{\Psi}_m(w+l) \tilde{\Psi}_n(w+l)^*
\]
\[
\sum_{l \in \mathbb{Z}_+} S^{(m)} \left( \frac{w + l}{p} \right) \tilde{\Phi} \left( \frac{w + l}{p} \right) \tilde{\Phi} \left( \frac{w + l}{p} \right)^* S^{(m)} \left( \frac{w + l}{p} \right)^* = S^{(m)} \left( \frac{w}{p} \right) \left( \sum_{l=0}^{p-1} \text{R} \left( \frac{w + l}{p} \right) S^{(m)} \left( \frac{w + l}{p} \right)^* \right) = O,
\]
\[
\sum_{l \in \mathbb{Z}_+} \tilde{\Psi}_m(w + l) \tilde{\Psi}_n(w + l)^* = \sum_{l=0}^{p-1} S^{(m)} \left( \frac{w + l}{p} \right) S^{(n)} \left( \frac{w + l}{p} \right)^* = \text{I}_N.
\]

Conversely, suppose identities (4.1), (4.2) hold. From above, we have
\[
\sum_{l \in \mathbb{Z}_+} \tilde{\Phi}(w + l) \tilde{\Psi}_m(w + l)^* = \sum_{l=0}^{p-1} \text{R} \left( \frac{w + l}{p} \right) S^{(m)} \left( \frac{w + l}{p} \right)^* = O,
\]
\[
\sum_{l \in \mathbb{Z}_+} \tilde{\Psi}_m(w + l) \tilde{\Psi}_n(w + l)^* = \sum_{l=0}^{p-1} S^{(m)} \left( \frac{w + l}{p} \right) S^{(n)} \left( \frac{w + l}{p} \right)^* = \text{I}_N.
\]
Therefore
\[
\langle \Phi(t), \Psi_m(t \ominus k) \rangle = \sum_{l \in \mathbb{Z}_+} \int_{l}^{l+1} \tilde{\Phi}(w) \tilde{\Psi}_m(w)^* \chi(k, w)dw = \int_{0}^{1} \sum_{l \in \mathbb{Z}_+} \tilde{\Psi}(w + l) \tilde{\Psi}_m(w + l)^* \chi(k, w)dw = O, \quad m \in \Lambda, k \in \mathbb{Z}_+.
\]
Similarly, we have
\[
\langle \Psi_m(t), \Psi_n(t \ominus k) \rangle = \int_{0}^{1} \sum_{l \in \mathbb{Z}_+} \tilde{\Psi}_m(w + l) \tilde{\Psi}_n(w + l)^* \chi(k, w)dw
\]
Thus, $\Phi(t)$ and $\Psi_m(t), m \in \Lambda$ are mutually orthogonal, and \{\Psi_m(t), m \in \Lambda\} are a finite family of orthogonal vector-valued functions. This proves the orthogonality of \{\Psi_m(t \ominus k), m \in \Lambda\}_{k \in \mathbb{Z}^+}.

Finally, we shall prove the completeness of \{\Psi_m(t \ominus k), m \in \Lambda\}_{k \in \mathbb{Z}^+} in $W_0$. For any $f \in W_0 \subset V_1$, there exists a finite supported sequence of $N \times N$ constant matrices \{\mathcal{A}_k\}_{k \in \mathbb{Z}^+}$ such that

$$f = \sum_{k \in \mathbb{Z}^+} \mathcal{A}_k \Phi(pt \ominus k)$$

(4.3)

By taking Walsh-Fourier Transform, we have

$$\tilde{f}(w) = \mathcal{A}(w/p) \tilde{\Phi}(w/p)$$

(4.4)

where $\mathcal{A}(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}^+} \mathcal{A}_k \chi(k, w)$. On the other hand, $f \in W_0$ and $f \notin V_0$ means

$$\int_{\mathbb{R}_+} f(t) \Phi(t \ominus k)^* dt = O, k \in \mathbb{Z}^+, \quad (4.5)$$

This is equivalent to

$$\sum_{l \in \mathbb{Z}^+} \tilde{f}(w + l) \tilde{\Phi}(w + l)^* = O.$$  

(4.6)

According to (3.6), (4.4) and Lemma 3.1, we have

$$\sum_{l=0}^{p-1} \mathcal{A} \left( \frac{w + l}{p} \right) \mathcal{R} \left( \frac{w + l}{p} \right)^* = O, \quad w \in \mathbb{R}_+$$

(4.7)

Let $\mathcal{A}_1(w) = \left( \mathcal{A} \left( \frac{w}{p} \right), \mathcal{A} \left( \frac{w+1}{p} \right), \ldots, \mathcal{A} \left( \frac{w+p-1}{p} \right) \right)^*$, $\mathcal{R}_1(w) = \left( \mathcal{R} \left( \frac{w}{p} \right), \mathcal{R} \left( \frac{w+1}{p} \right), \ldots, \mathcal{R} \left( \frac{w+p-1}{p} \right) \right)^*$ and for $i = 1, 2, \ldots, p - 1$, we set $S_i(w) = \left( S^{(i)} \left( \frac{w}{p} \right), S^{(i)} \left( \frac{w+1}{p} \right), \ldots, S^{(i)} \left( \frac{w+p-1}{p} \right) \right)^*$. Then the identities (4.1) and (4.2) imply that, for any $w \in \mathbb{R}_+$, the column vectors in the $pN \times N$ matrix $\mathcal{R}_1(w)$ and column vectors in the $pN \times N$ $S_i(w)$ are orthogonal and all these vectors form an orthonormal basis for the $pN$-dimensional complex Euclidean space $\mathbb{C}^{pN}$. The identity (4.7) implies that the column vectors in the $pN \times N$ matrix $\mathcal{A}_1(w)$ and column vectors in the $pN \times N$ $\mathcal{R}_1(w)$ are orthogonal.
Thus, there exist \( p - 1 \) matrices \( \mathcal{L}^{(m)}(w), m \in \Lambda \) whose all entries are 1-periodic functions of \( w \) such that
\[
\mathcal{A}(w) = \sum_{m \in \Lambda} \mathcal{L}^{(m)}(w)S^{(m)}(w), \quad w \in \mathbb{R}_+.
\] (4.8)

Therefore, by (4.4)
\[
\tilde{f}(w) = \sum_{m \in \Lambda} \mathcal{L}^{(m)}(w/p)S^{(m)}(w/p)\Phi(w/p) = \sum_{m \in \Lambda} \mathcal{L}^{(m)}(w/p)\tilde{\Psi}_m(w). \quad (4.9)
\]

By the orthonormality of \( \{\Psi_m(t \ominus k), m \in \Lambda\}_{k \in \mathbb{Z}_+} \), we get
\[
\int_{\mathbb{R}_+} \tilde{f}(pw)\tilde{f}(pw)^*dw = \sum_{l \in \mathbb{Z}_+} \sum_{m \in \Lambda} \mathcal{L}^{(m)}(w)\tilde{\Psi}_m(pw)\tilde{\Psi}_m(pw)^*\mathcal{L}^{(m)}(w)^*dw
\]
\[
= \int_{0}^{1} \sum_{m \in \Lambda} \mathcal{L}^{(m)}(w)\mathcal{L}^{(m)}(w)^*dw.
\]

This proves that \( \mathcal{L}(w) \) has Walsh-Fourier series expansion. Let constant \( N \times N \) matrices \( \{Q_k^{(m)}\}_{k \in \mathbb{Z}_+}, m \in \Lambda \) be its Walsh-Fourier coefficients. Then
\[
f = \sum_{k \in \mathbb{Z}_+} \sum_{m \in \Lambda} Q_k^{(m)}\Psi_m(t \ominus k).
\]

This proves the completeness of \( \{\Psi_m(t \ominus k), m \in \Lambda\}_{k \in \mathbb{Z}_+} \) in \( W_0 \). \( \Box \) \( \Box \)

By Walsh-Fourier analysis and (3.7), (3.12), identities (4.1) and (4.2) are equivalent to, respectively,
\[
\sum_{v \in \mathbb{Z}_+} P_v(S_v^{(m)}S_v^{(n)}^{*}) = \mathbf{0}, \quad m \in \Lambda, k \in \mathbb{Z}_+ \quad (4.10)
\]
\[
\sum_{v \in \mathbb{Z}_+} S_v^{(m)}(S_v^{(n)}S_v^{(n)}^{*}) = p\delta_m,n\delta_0,kI_N, \quad m, n \in \Lambda, k \in \mathbb{Z}_+. \quad (4.11)
\]

Theorem 4.1 implies that a vector valued multiresolution \( p \)-analysis in \( L^2(\mathbb{R}_+, \mathbb{C}^N) \) gives us vector-valued scaling function \( \Phi \) and moreover associated \( p - 1 \) orthogonal vector valued wavelet functions \( \Psi_m(t), m \in \Lambda \) such that whose dilations and translations \( \Psi_{j,k,m}(t) = p^{j/2}\Psi_m(p^j t \ominus k), j \in \mathbb{Z}, k \in \mathbb{Z}_+, m \in \Lambda \) form an orthonormal basis for \( L^2(\mathbb{R}_+, \mathbb{C}^N) \). Therefore to construct a vector-valued wavelet functions, we only need to construct a vector-valued scaling function.
Theorem 4.2. Let
\[ R(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}_+} R(T_k(\chi(k, w))) \]
be a matrix valued scaling filter satisfying following conditions:
(a).
\[ \sum_{l=0}^{p-1} R\left(\frac{w + l}{p}\right) R\left(\frac{w + l}{p}\right)^* = I_N. \] (4.12)
(b). There exists a constant \( C > 0 \) and integer \( M > 0 \) such that for \( w \in (0, p^M) \)
\[ \| \prod_{l=1}^{M} R\left(\frac{w}{p^l}\right) \| < C \| \prod_{l=1}^{\infty} R\left(\frac{w}{p^l}\right) \| \] (4.13)
If \( \Phi \in L^2(\mathbb{R}_+, \mathbb{C}^N) \) such that \( \tilde{\Phi}(0)\tilde{\Phi}(0)^* = I_N \) and its Walsh-Fourier transform can be written as
\[ \tilde{\Phi}(w) = \prod_{l=1}^{\infty} R\left(\frac{w}{p^l}\right) \tilde{\Phi}(0) \] (4.14)
then \( \Phi(t) \) is a vector valued scaling function for a vector valued multiresolution \( p \)-analysis in \( L^2(\mathbb{R}_+, \mathbb{C}^N) \). Thus, the corresponding \( \Psi_{j,k,m}(t) = p^j \tilde{\Psi}(p^j t \ominus k), j \in \mathbb{Z}, k \in \mathbb{Z}_+, m \in \Lambda \) form an orthonormal basis for \( L^2(\mathbb{R}_+, \mathbb{C}^N) \).

Proof. To prove Theorem 4.2, we only need to prove the orthonormality of \( \Phi(t \ominus k), k \in \mathbb{Z}_+ \). The rest is similar to Daubechies [2], Mallat [9] and Farkov [3].

Now
\[ \int_{\mathbb{R}_+} \Phi(t)\Phi(t \ominus k) dt = \int_{\mathbb{R}_+} \tilde{\Phi}(w)\tilde{\Phi}(w)^* \chi(k, w) dw. \] (4.15)
For an integer \( M > 0 \), let
\[ \mu_M(w) = \prod_{l=1}^{M} R\left(\frac{w}{p^l}\right) \tilde{\Phi}(0) \mathbf{1}_{(0,p^M)}(w). \] (4.16)
where \( \mathbf{1}_{(0,p^M)}(w) \) is the characteristic function of a subset \( (0, p^M) \) of \( \mathbb{R}_+ \).

Now
\[ \int_{\mathbb{R}_+} \mu_M(w)\mu_M(w)^* \chi(k, w) dw \]
\[ \int_{0}^{p^{M}} \left( \mathcal{R} \left( \frac{w}{p} \right) \mathcal{R} \left( \frac{w}{p^{2}} \right) \ldots \mathcal{R} \left( \frac{w}{p^{M}} \right) \tilde{\Phi}(0) \right) \left( \tilde{\Phi}(0)^{*} \mathcal{R} \left( \frac{w}{p^{M}} \right)^{*} \mathcal{R} \left( \frac{w}{p^{M-1}} \right)^{*} \ldots \mathcal{R} \left( \frac{w}{p} \right)^{*} \right) \chi(k, w) \, dw \]

\[ = p^{M} \int_{0}^{1} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right) \mathcal{R}(w) \mathcal{R}(w)^{*} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right)^{*} \chi(k, p^{M}w) \, dw \]

\[ = p^{M} \int_{0}^{1} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right) \mathcal{R}(w) \mathcal{R}(w)^{*} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right)^{*} \chi(k, p^{M}w) \, dw \]

\[ + p^{M} \int_{0}^{\frac{1}{p}} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right) \mathcal{R}(w) \mathcal{R}(w)^{*} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right)^{*} \chi(k, p^{M}w) \, dw \]

\[ + \ldots \]

\[ + p^{M} \int_{\frac{1}{p}}^{1} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right) \mathcal{R}(w) \mathcal{R}(w)^{*} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right)^{*} \chi(k, p^{M}w) \, dw \]

\[ = p^{M} \int_{0}^{1} \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right) \left( \sum_{l=0}^{p-1} \mathcal{R} \left( \frac{w}{p} + \frac{l}{p} \right) \mathcal{R} \left( \frac{w}{p} + \frac{l}{p} \right)^{*} \right) \left( \prod_{l=1}^{M-1} \mathcal{R}(p^{M-l}w) \right)^{*} \chi(k, p^{M}w) \, dw \]

Using (4.12) and \( \tilde{\Phi}(0)\tilde{\Phi}(0)^{*} = I_{N} \), we have

\[ \int_{\mathbb{R}^{+}} \mu_{M}(w) \mu_{M}(w)^{*} \chi(k, w) \, dw \]

\[ = \int_{0}^{p^{M-1}} \left( \prod_{l=1}^{M-1} \mathcal{R} \left( \frac{w}{p^{l}} \right) \right) \left( \prod_{l=1}^{M-1} \mathcal{R} \left( \frac{w}{p^{l}} \right) \right)^{*} \chi(k, w) \, dw \]

\[ = \int_{\mathbb{R}^{+}} \mu_{M-1}(w) \mu_{M-1}(w)^{*} \chi(k, w) \, dw \]

\[ = \ldots \]

\[ = \int_{\mathbb{R}^{+}} \mu_{1}(w) \mu_{1}(w)^{*} \chi(k, w) \, dw \]

\[ = \int_{0}^{1} I_{N} \chi(k, w) \, dw \]

\[ = \delta_{0,k} I_{N}. \]

From (4.16), we get that \( \mu_{k}(w) \) converges to \( \Phi(w) \) pointwise. In view of (4.13),
we have
\[ \| \mu_M(w)\mu_M(w)^* - \Phi(w)\Phi(w)^* \| \leq (C + 1)\| \Phi(w)\Phi(w)^* \|, \quad w \in \mathbb{R}_+. \]

Since all matrix norms are equivalent, there exists a constant \( C_1 > 0 \) such that
\[ \| \mu_M\mu_M^* - \Phi\Phi^* \| \leq C_1 \int_{\mathbb{R}^+} \| \mu_M(w)\mu_M(w)^* - \Phi(w)\Phi(w)^* \|^2 dw. \]

By the Dominated convergence theorem, we get \( \| \mu_M\mu_M^* - \Phi\Phi^* \| \to 0 \) as \( M \to \infty \). Therefore
\[
\int_{\mathbb{R}^+} \bar{\Phi}(w)\Phi(w)^*\chi(k,w)dw = \lim_{M \to \infty} \int_{\mathbb{R}^+} \mu_M(w)\mu_M(w)^*\chi(k,w)dw = \delta_{0,k}I_N.
\]

This proves the orthonormality of \( \Phi(t \ominus k), k \in \mathbb{Z}_+ \). □ □

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References


