OSCILLATION OF THIRD-ORDER NEUTRAL RETARDED DIFFERENTIAL EQUATIONS

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Abstract: This paper is concerned with the oscillatory and asymptotic behavior of the solutions of the third-order neutral retarded differential equation

\[ a(t) \left( b(t) (x(t) + p(t)x(t - \tau))' \right)' + q(t)x(t - \sigma) = 0, \]

where \( a(t) > 0, b(t) > 0, q(t) \geq 0, 0 \leq p(t) < 1, \) and \( \sigma > \tau \geq 0. \) Examples illustrating the main result are given.

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1. Introduction

This paper is concerned with the oscillatory and asymptotic behavior of solutions of the third-order neutral differential equation

\[
\left[ a(t) \left( b(t) (x(t) + p(t)x(t - \tau))' \right)' \right]' + q(t)x(t - \sigma) = 0. \quad (E)
\]

Throughout this paper, and without further mention, we always assume that

(C1) \( a, b, p, q \in C([t_0, \infty), \mathbb{R}), a(t) > 0, b(t) > 0, 0 \leq p(t) \leq p < 1, q(t) \geq 0, \) and \( q(t) \not\equiv 0 \) on any ray of the form \([t^*, \infty)\) for any \( t^* \geq t_0; \)

(C2) \( \tau \) and \( \sigma \) are nonnegative constants with \( \sigma > \tau \geq 0. \)

By a solution of Eq. \((E)\) we mean a nontrivial function \( x(t) \in C([T_x, \infty)), T_x \geq t_0, \) which satisfies \((E)\) on \([T_x, \infty).\) We only consider those solutions \( x(t) \) of \((E)\) satisfying \( \sup\{|x(t)| : t \geq T\} > 0 \) for all \( T \geq T_x, \) and we assume that \((E)\) possesses such solutions. A solution of \((E)\) is called oscillatory if it has arbitrarily large zeros on \([T_x, \infty),\) and is called nonoscillatory otherwise. Equation \((E)\) is said to be oscillatory if all its solutions are oscillatory.

The oscillation of solutions of differential equations has been a topic of interest for many years and we cite as recent works the results in [1]–[18]. A number of authors including Baculíková and Džurina [2, 3], Candan and Dahiya [5], Graef et al. [11], and Thandapani and Li [14] have studied the oscillatory behavior of solutions of third order neutral retarded differential equations in the form of equation \((E)\) under the assumption that

\[
\int_{t_0}^{\infty} \frac{1}{a(t)} \, dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(t)} \, dt = \infty.
\]

In this note, we continue the study of the behavior of solutions \((E)\) but under the assumption that

\[
\int_{t_0}^{\infty} \frac{1}{a(t)} \, dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(t)} \, dt = \infty. \quad (1)
\]

In what follows, all functional inequalities considered are assumed to hold eventually, that is, they are satisfied for all sufficiently large \( t.\)
2. Oscillation and Asymptotic Behavior

In this section, we present our main result in the paper. For convenience, we define

\[ z(t) := x(t) + p(t)x(t - \tau). \]

**Theorem 2.1.** Assume that (1) holds and there exists a function \( \rho \in C^1([t_0, \infty), (0, \infty)) \) such that for all sufficiently large \( t_3 > t_2 > t_1 \geq t_0 \), we have

\[
\limsup_{t \to \infty} \int_{t_3}^{t} \left( \rho(s)q(s)(1 - p(s - \sigma)) \int_{t_2}^{s-\sigma} \frac{1}{b(v)} \int_{t_1}^{v} \frac{1}{a(u)} du dv 
- \frac{a(s)(\rho'(s))^2}{4\rho(s)} \right) ds = \infty, \quad (2)
\]

\[
\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{t}^{\infty} \frac{1}{a(u)} \int_{t}^{\infty} q(s) ds dv du = \infty, \quad (3)
\]

and

\[
\limsup_{t \to \infty} \int_{t_2}^{t} \left( \delta(s)q(s)(1 - p(s - \sigma)) \int_{t_1}^{s-\sigma} \frac{1}{b(v)} \int_{a(s)}^{v} 4\delta(s)a(s) \right) ds = \infty, \quad (4)
\]

where

\[ \delta(t) := \int_{t}^{\infty} \frac{1}{a(s)} ds. \]

Then any solution \( x(t) \) of (E) is either oscillatory or satisfies \( x(t) \to 0 \) as \( t \to \infty \).

**Proof.** Let \( x \) be a nonoscillatory solution of (E). Without loss of generality, we may suppose that \( x(t) > 0, x(t - \tau) > 0, \) and \( x(t - \sigma) > 0 \) for \( t \geq t_1 \) for some \( t_1 \geq t_0 \). From (1), there exist three possible cases:

(1) \( z(t) > 0, z'(t) > 0, (b(t)z'(t))' > 0, \) and \( [a(t)(b(t)z'(t))]' \leq 0; \)

(2) \( z(t) > 0, z'(t) < 0, (b(t)z'(t))' > 0, \) and \( [a(t)(b(t)z'(t))]' \leq 0; \) or

(3) \( z(t) > 0, z'(t) > 0, (b(t)z'(t))' < 0, \) and \( [a(t)(b(t)z'(t))]' \leq 0; \) for \( t \geq t_2 \)

for a sufficiently large \( t_2 \geq t_1 \).

Assume that case (1) holds. We define the function \( \omega \) by

\[
\omega(t) = \rho(t) \frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, \quad t \geq t_2, \quad (5)
\]
and note that \( \omega(t) > 0 \) for \( t \geq t_2 \). Using the fact that \( z'(t) > 0 \), we have

\[
x(t) \geq (1 - p(t))z(t).
\]

(6)

Since

\[
b(t)z'(t) \geq \int_{t_2}^{t} \frac{a(s)(b(s)z'(s))'}{a(s)} ds \geq a(t)(b(t)z'(t))' \int_{t_2}^{t} \frac{1}{a(s)} ds,
\]

we have that

\[
\left( \frac{b(t)z'(t)}{\int_{t_2}^{t} \frac{1}{a(s)} ds} \right)' \leq 0.
\]

(7)

Thus,

\[
z(t) = z(t_2) + \int_{t_2}^{t} \frac{b(s)z'(s)}{b(s)} \int_{t_2}^{s} \frac{1}{a(u)} du \, ds
\]

\[
\geq \int_{t_2}^{t} \frac{b(t)z'(t)}{\int_{t_2}^{t} \frac{1}{a(u)} du} \int_{t_2}^{s} \frac{1}{b(s)} du \, ds
\]

(8)

for \( t \geq t_2 \). Differentiating (5), we obtain

\[
\omega'(t) = \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\rho(t)}{\rho(t)a(t)} \omega(t) - \frac{\rho(t)}{\rho(t)} \omega^2(t)
\]

\[
\left( \frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} \right) + \left( \frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} \right) \left( \frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} \right)
\]

\[
- \frac{\rho(t)}{\rho(t)} \omega^2(t)
\]

It follows from (E), (5), and (6) that

\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t)q(t)(1 - p(t - \sigma)) \frac{z(t - \sigma)}{b(t)z'(t)} - \frac{\omega^2(t)}{\rho(t)a(t)},
\]

i.e.,

\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\omega^2(t)}{\rho(t)a(t)}
\]

\[
- \rho(t)q(t)(1 - p(t - \sigma)) \frac{z(t - \sigma)}{b(t - \sigma)z'(t - \sigma)} \frac{b(t - \sigma)z'(t - \sigma)}{b(t)z'(t)}
\]

Now (7) and (8) imply

\[
\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\omega^2(t)}{\rho(t)a(t)}
\]
and decreasing, we obtain
\[ -\rho(t)q(t)(1-p(t-\sigma)) \frac{\int_{t_2}^{t_2} \frac{1}{b(u)} du}{\int_{t_1}^{t_1} \frac{1}{a(u)} du} \frac{\int_{t_2}^{t_2} \frac{1}{b(s)} ds}{\int_{t_1}^{t_1} \frac{1}{a(u)} du} = \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\omega^2(t)}{\rho(t)a(t)} - \rho(t)q(t)(1-p(t-\sigma)) \frac{\int_{t_2}^{t_2} \frac{1}{b(s)} ds}{\int_{t_1}^{t_1} \frac{1}{a(u)} du}.
\]

Hence, we have
\[ \omega'(t) \leq -\rho(t)q(t)(1-p(t-\sigma)) \frac{\int_{t_2}^{t_2} \frac{1}{b(s)} ds}{\int_{t_1}^{t_1} \frac{1}{a(u)} du} + \frac{a(t)(\rho'(t))^2}{4\rho(t)}. \]

Integrating the last inequality from \( t_3 > t_2 \) to \( t \) gives
\[ \int_{t_3}^{t} \left( \rho(s)q(s)(1-p(s-\sigma)) \frac{\int_{t_2}^{t_2} \frac{1}{b(v)} dv}{\int_{t_1}^{t_1} \frac{1}{a(u)} du} - \frac{a(s)(\rho'(s))^2}{4\rho(s)} \right) ds \leq \omega(t_3), \]
which contradicts (2).

Next, assume that case (2) holds. Since \( z(t) > 0 \) and \( z'(t) < 0 \), we have \( z(t) \to L \geq 0 \). If \( L > 0 \), then for \( \epsilon = \frac{L(1-p)}{2p} > 0 \), there exists \( t_4 \geq t_1 \) such that \( L < z(t) < L + \epsilon \) for \( t \geq t_4 \). Then for \( t \geq t_4 \), we have
\[ x(t) = z(t) - p(t)x(t-\tau) > L - pz(t) > L - p(L + \epsilon) = \frac{L(1-p)}{2} = L_1. \]

From (E), we have
\[ \left[ a(t) \left( b(t)(x(t) + p(t)x(t-\tau))' \right)' \right]' \leq -q(t)x(t-\sigma). \quad (9) \]

Integrating from \( t \geq t_4 \) to \( \infty \) and using the fact that \( a(t)(b(t)z'(t))' \) is positive and decreasing, we obtain
\[ a(t)(b(t)z'(t))' \geq \int_{t}^{\infty} q(s)x(s-\sigma)ds \geq L_1 \int_{t}^{\infty} q(s)ds. \quad (10) \]

Integrating again gives
\[ b(t)z'(t) \leq -L_1 \int_{t}^{\infty} \frac{1}{a(u)} \int_{t}^{\infty} q(s)ds du, \quad (11) \]
and a final integration yields
\[
z(t_4) \geq L_1 \int_{t_4}^{\infty} \frac{1}{b(v)} \int_{t}^{\infty} \frac{1}{a(u)} \int_{t}^{\infty} q(s) ds du dv. \tag{12}
\]
This contradicts (3) and shows that \(L = 0\), i.e., \(z(t) \to 0\). Since \(z(t) > x(t) > 0\), we have \(x(t) \to 0\) as \(t \to \infty\).

Finally, assume that case (3) holds. Now
\[
\left[ a(t) (b(t)z'(t))' \right]' \leq 0,
\]
so \(a(t) (b(t)z'(t))'\) is nonincreasing. Thus,
\[
a(s)(b(s)z'(s))' \leq a(t)(b(t)z'(t))', \quad s \geq t \geq t_5,
\]
for some \(t_5 \geq t_0\). Dividing the above inequality by \(a(s)\) and integrating from \(t\) to \(l\), we obtain
\[
b(l)z'(l) \leq b(t)z'(t) + a(t)(b(t)z'(t))' \int_{t}^{l} \frac{ds}{a(s)}.
\]
Letting \(l \to \infty\), we have
\[
0 \leq b(t)z'(t) + a(t)(b(t)z'(t))' \int_{t}^{\infty} \frac{ds}{a(s)}.
\]
That is,
\[
-\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} \int_{t}^{\infty} \frac{ds}{a(s)} \leq 1. \tag{13}
\]
Define function \(\phi\) by
\[
\phi(t) := \frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, \quad t \geq t_5. \tag{14}
\]
Then \(\phi(t) < 0\) for \(t \geq t_5\). Hence, from (13) and (14), we obtain
\[
-\delta(t)\phi(t) \leq 1. \tag{15}
\]
Differentiating (14) gives
\[
\phi'(t) = \frac{(a(t)(b(t)z'(t))')'}{b(t)z'(t)} - \frac{a(t)(b(t)z'(t))' (b(t)z'(t))'}{(b(t)z'(t))^2}.
\]
Now \( z'(t) > 0 \), so from (E) and (6), we have

\[
\phi'(t) \leq -q(t)(1 - p(t - \sigma)) \frac{z(t - \sigma)}{b(t)z'(t)} - \frac{a(t)(b(t)z'(t))'(b(t)z'(t))'}{(b(t)z'(t))^2}. \quad (16)
\]

From the third inequality in case (3), we see that

\[
z(t) \geq b(t) \int_{t_5}^{t} \frac{ds}{b(s)} z'(t). \quad (17)
\]

Hence,

\[
\left( \frac{z(t)}{\int_{t_5}^{t} \frac{ds}{b(s)}} \right)' \leq 0,
\]

which implies that

\[
\frac{z(t - \sigma)}{z(t)} \geq \frac{\int_{t_5}^{t - \sigma} \frac{ds}{b(s)}}{\int_{t_5}^{t} \frac{ds}{b(s)}}. \quad (18)
\]

From (14) and (16)–(18), we obtain

\[
\phi'(t) \leq -q(t)(1 - p(t - \sigma)) \int_{t_5}^{t - \sigma} \frac{ds}{b(s)} - \frac{\phi^2(t)}{a(t)}.
\]

Multiplying the last inequality by \( \delta(t) \) and integrating from \( t_6 > t_5 \) to \( t \), we have

\[
\phi(t)\delta(t) = \phi(t_6)\delta(t_6) + \int_{t_6}^{t} \delta(s)q(s)(1 - p(s - \sigma)) \int_{t_5}^{s - \sigma} \frac{dv}{b(v)} \, ds \\
+ \int_{t_6}^{t} \frac{\phi^2(s)\delta(s)}{a(s)} \, ds + \int_{t_6}^{t} \frac{\phi(s)}{a(s)} \, ds \leq 0,
\]

from which it follows that

\[
\int_{t_6}^{t} \left( \delta(s)q(s)(1 - p(s - \sigma)) \int_{t_5}^{s - \sigma} \frac{dv}{b(v)} - \frac{1}{4\delta(s)a(s)} \right) \, ds \leq 1 + \phi(t_2)\delta(t_2)
\]

due to (15). This contradicts (4) and completes the proof of the theorem. \( \blacksquare \)

We conclude this paper with some examples.
Example 2.2. Consider the third-order neutral retarded differential equation
\[
\left( e^t \left( x(t) + \frac{1}{2} x(t - 2\pi) \right)'' \right)' + 3\sqrt{2} e^t x \left( t - \frac{89\pi}{4} \right) = 0, \ t \geq 23\pi. \tag{19}
\]
All conditions of Theorem 2.1 are satisfied for the choice of \( \rho(t) = 1 \), and so any solution of equation (19) is either oscillatory or converges to zero. One such solution is \( x(t) = \cos t \).

Example 2.3. Consider the third-order retarded differential equation
\[
\left( e^t x''(t) \right)' + \sqrt{2} e^t x \left( t - \frac{89\pi}{4} \right) = 0, \ t \geq 23\pi. \tag{20}
\]
The hypotheses of Theorem 2.1 are satisfied for \( \rho(t) = 1 \), and so any solution of equation (20) is either oscillatory or converges to zero. One such solution is \( x(t) = \cos t \). Note that the results of [10] cannot be applied to equation (20).

Example 2.4. Consider the third-order retarded differential equation
\[
\left[ e^t \left( e^{-t} \left( x(t) + e^{-1} x(t - 1) \right)' \right)' \right]' + 4e^{-2} x(t - 2) = 0, \ t \geq 3. \tag{21}
\]
The conditions of Theorem 2.1 are satisfied with \( \rho(t) = 1 \), so any solution of equation (20) is either oscillatory or converges to zero. One such solution is \( x(t) = e^{-t} \).

References


