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SOME COMMON FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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Abstract: In [2], Binayak S. Choudhury has proved that if (X, d) is a complete metric space, then every weak C-contraction on X has a unique fixed point. The main result of this paper is a generalization of this result in complete metric spaces. An example is given to show that our results are proper generalizations of the existing ones.

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1. Introduction

The study of Existence and uniqueness of coincidence points and common fixed points of mappings satisfying certain contractive conditions has been an interesting field of mathematics from 1922, when Banach stated and proved his famous result (Banach contraction principle). The field of fixed point theory that is involving four single valued maps, began with the assumption that all of the maps are commuted. In 1998, the concept of weakly compatible pairs of mappings has been introduced by Jungck [4], that is the class of mappings such that they commute at their coincidence points. In recent years, several authors have obtained coincidence point and common fixed point results for different classes of mappings on various metric spaces such as complete metric spaces,

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partially ordered metric spaces, cone metric spaces and $\cdots$. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [3] and references contained in it.

In this paper, we prove a unique common fixed point theorem for weakly C-contractive mappings satisfying the notion of weakly compatibility without using the notion of continuity which generalizes and extends the result of S. Choudhury [3].

2. Preliminaries

Definition 1. A mapping $T : X \to X$ where (X, d) is a metric space is said to be a C-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)).$$

The concept of C-contraction was defined by S.K. Chatterjea [1] in 1972 and he has proved that if (X, d) is a complete metric space, then every C-contraction on X has a unique fixed point.

In 2009, Choudhury has introduced weak C-contraction given by the following definition.

Definition 2. (see [2]) A mapping $T : X \to X$, where (X, d) is a metric space is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi((d(x, Ty), d(y, Tx))),$$

where $\varphi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

In [2], Binayak S. Choudhury has proved that if (X, d) is a complete metric space, then every weak C-contraction on X has a unique fixed point.

3. Main Results

Definition 3. (see [4]) Let T and S be two self mappings of a metric space (X,d). T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.

Our first result is the following.

Theorem 4. *Let (X,d) be a complete metric space and let E be a nonempty closed subset of X. Let $T, S : E \to E$ be such that,*

$$d(Tx, Sy) \leq \frac{1}{2}(d(Rx, Sy) + d(Ry, Tx)) - \varphi((d(Rx, Sy), d(Ry, Tx))), \qquad (1)$$

for every pair $(x,y) \in X \times X$, where $\varphi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 0$ and $R : E \to X$ satisfying the following hypotheses:

(i) $TE \subseteq RE$ and $SE \subseteq RE$.

(ii) The pairs (T,R) and (S,R) are weakly compatible. xms
In addition, assume that $R(E)$ is a closed subset of X.
Then, T and R and S have a unique common fixed point.

Proof. Let $x_0 \in E$ be arbitrary. Using (i) there exist tow sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = Tx_0 = Rx_1$, $y_1 = Sx_1 = Rx_2$, $y_2 = Tx_2 = Rx_3$, ...,$y_{2n} = Tx_{2n} = Rx_{2n+1}$, $y_{2n+1} = Sx_{2n+1} = Rx_{2n+2}$, ...

We complete the proof in three steps.

Step I. We will prove that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Let $n = 2k$. Using condition 1, we obtain that

$$\begin{aligned} d(y_{2k+1}, y_{2k}) &= d(Tx_{2k}, Sx_{2k+1})) \\ &\leq \frac{1}{2}(d(Rx_{2k}, Sx_{2k+1}) + d(Rx_{2k+1}, Tx_{2k})) \\ &\quad -\varphi(d(Rx_{2k}, Sx_{2k+1}), d(Rx_{2k+1}, Tx_{2k})) \\ &= \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) \\ &\quad -\varphi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})) \\ &\leq \frac{1}{2} d(y_{2k-1}, y_{2k+1}) \\ &\leq \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1})). \end{aligned} \qquad (2)$$

Hence, $d(y_{2k+1}, y_{2k}) \leq d(y_{2k}, y_{2k-1})$.

If $n = 2k+1$, similarly we can prove that

$$d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k}).$$

Thus $d(y_{n+1}, y_n)$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that, $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$.

from the above argument we have

$$\begin{aligned} d(y_{n+1}, y_n) &\leq \frac{1}{2} d(y_{n-1}, y_{n+1}) \\ &\leq \frac{1}{2}(d(y_{n-1}, y_n) + d(y_n, y_{n+1})). \end{aligned} \tag{3}$$

if $n \to \infty$, we have

$$r \leq \lim_{n\to\infty} \frac{1}{2} d(y_{n-1}, y_{n+1}) \leq r.$$

Therefore, $\lim_{n\to\infty} d(y_{n-1}, y_{n+1}) = 2r$.

We have proved in 2

$$\begin{aligned} d(y_{2k+1}, y_{2k}) &= d(Tx_{2k}, Sx_{2k+1})) \\ &\leq \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) \\ &-\varphi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})). \end{aligned} \tag{4}$$

Now, if $k \to \infty$ and using the continuity of φ we obtain

$$r \leq \frac{1}{2} 2r - \varphi(2r, 0),$$

and consequently, $\varphi(2r, 0) = 0$. This gives us that

$$r = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \tag{5}$$

by our assumption about φ.

Step II. $\{y_n\}$ is Cauchy.

Since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, it is sufficient to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the least index for which $n(k) > m(k) > k$ and $d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon$.

This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon. \tag{6}$$

From triangle inequality

$$\begin{aligned} \varepsilon \leq d(y_{2m(k)}, y_{2n(k)}) &\leq d(y_{2m(k)}, y_{2n(k)-2}) \\ &+d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &\leq \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}). \end{aligned} \tag{7}$$

Letting $k \to \infty$ and using 5 we can conclude that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon. \tag{8}$$

Moreover, we have

$$|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1}) \tag{9}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) \tag{10}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-2}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2m(k)-2}, y_{2m(k)-1}). \tag{11}$$

Using 5, 8, 9, 10 and 11 we get

$$\begin{aligned} \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) &= \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) \\ &= \lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) = \varepsilon. \end{aligned} \tag{12}$$

Now, from 1 we have

$$\begin{aligned} d(y_{2m(k)-1}, y_{2n(k)}) &= d(Tx_{2n(k)}, Sx_{2m(k)-1}) \\ &\leq \tfrac{1}{2}(d(Rx_{2n(k)}, Sx_{2m(k)-1}) + d(Rx_{2m(k)-1}, Tx_{2n(k)})) \\ &-\varphi(d(Rx_{2n(k)}, Sx_{2m(k)-1}), d(Rx_{2m(k)-1}, Tx_{2n(k)})) \\ &= \tfrac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)})) \\ &-\varphi(d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2m(k)-2}, y_{2n(k)})) \\ &\leq \tfrac{1}{2}(d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)+1})). \end{aligned} \tag{13}$$

If $k \to \infty$ in the above inequality, from 12 and the continuity of φ, we have

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \varphi(\varepsilon, \varepsilon)$$

and from the last inequality $\varphi(\varepsilon, \varepsilon) = 0$. By our assumption about φ, we have $\varepsilon = 0$ which is a contradiction.

Step III. T, S and R have a common fixed point.

Since (X,d) is complete and $\{y_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Since E is closed and $\{y_n\} \subseteq E$, we have $z \in E$. By assumption $R(E)$ is closed, so there exists $u \in E$ such that $z = Ru$.

For all $n \in N$,

$$\begin{aligned} d(Tu, y_{2n+1}) &= d(Tu, Sx_{2n+1}) \\ &\leq \tfrac{1}{2}(d(Ru, Sx_{2n+1}) + d(Rx_{2n+1}, Tu)) \\ &-\varphi(d(Ru, Sx_{2n+1}), d(Rx_{2n+1}, Tu)) \\ &= \tfrac{1}{2}(d(z, y_{2n+1}) + d(y_{2n}, Tu)) \\ &-\varphi(d(Ru, Sx_{2n+1}), d(Rx_{2n+1}, Tu)). \end{aligned} \tag{14}$$

If $n \to \infty$,

$$d(Tu, z) \leq \frac{1}{2}(d(z,z) + d(z,Tu)) - \varphi(d(Ru,z), d(z,Tu))$$

and hence

$$\varphi(0, d(z,Tu)) \leq -\frac{1}{2}(d(Tu,z)) \leq 0,$$

therefore $d(z,Tu) = 0$. Therefore $Tu = z$.

Similarly $Su = z$. So $Tu = Su = Ru = z$. Since the pairs (R,T) and (R,S) are weakly compatible, we have $Tz = Sz = Rz$.

Now we can have

$$\begin{aligned} d(Tz, y_{2n+1}) = d(Tz, Sx_{2n+1}) &\leq \tfrac{1}{2}(d(Rz, Sx_{2n+1}) + d(Rx_{2n+1}, Tz)) \\ &-\varphi(d(Rz, Sx_{2n+1}), d(Rx_{2n+1}, Tz)) \\ &= \tfrac{1}{2}(d(Rz, y_{2n+1}) + d(y_{2n}, Tz)) \\ &-\varphi(d(Rz, y_{2n+1}), d(y_{2n}, Tz)). \end{aligned} \tag{15}$$

If $n \to \infty$, since $Tz = Sz = Rz$, we obtain

$$\begin{aligned} d(Tz, z) &= \tfrac{1}{2}(d(Tz,z) + d(z,Tz)) \\ &-\varphi(d(Tz,z), d(z,Tz)). \end{aligned} \tag{16}$$

Hence, $\varphi(d(Tz,z), d(z,Tz)) = 0$ and so $d(Tz,z) = 0$. Therefore $Tz = z$ and from $Tz = Sz = Rz$ we conclude that $Tz = Sz = Rz = z$.

Uniqueness of the common fixed point follows from 1. □

Remark 5. If we take R as identity map on X, $T = S$ and $E = X$, then from Theorem 4 we obtain Theorem 2.1 of [2].

Theorem 6. *Let* (X,d) *be a complete metric space and* T,S,φ *and* R *verifying the conditions of Theorem 4. Assume that* R *is a continuous function on* X*. In addition for all* $x\in X$*:*

$$d(RTx,TRx)\leq d(Rx,Tx) \text{ and } d(RSx,SRx)\leq d(Rx,Sx).$$

Then, T *and* R *and* S *have a unique common fixed point.*

Proof. If we review the proof of Theorem 4, we obtain that $\{y_n\}$ is a Cauchy sequence converging to some $z\in X$.

We know that

$$\begin{aligned} z=\lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Rx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Rx_{2n+2}. \end{aligned} \tag{17}$$

Since R is continuous, Ry_n converges to Rz.

On the other hand,

$$\begin{aligned} d(Ty_{2n+1},Rz) &\leq d(Ty_{2n+1},Ry_{2n+2})+d(Ry_{2n+2},Rz) \\ &= d(TRx_{2n+2},RTx_{2n+2})+d(Ry_{2n+2},Rz) \\ &\leq d(Tx_{2n+2},Rx_{2n+2})+d(Ry_{2n+2},Rz) \\ &= d(y_{2n+2},y_{2n+1})+d(Ry_{2n+2},Rz). \end{aligned} \tag{18}$$

Therefore

$$\lim_{n\to\infty} d(Ty_{2n+1},Rz)=0,$$

and we can have

$$\begin{aligned} d(Ty_{2n+1},Sz) &\leq \tfrac{1}{2}(d(Ry_{2n+1},Sz)+d(Rz,Ty_{2n+1})) \\ &-\varphi((d(Ry_{2n+1},Sz),d(Rz,Ty_{2n+1}))). \end{aligned} \tag{19}$$

If $n\to\infty$, we have

$$\begin{aligned} d(Rz,Sz) &\leq \tfrac{1}{2}(d(Rz,Sz)+d(Rz,Rz)) \\ &-\varphi((d(Rz,Sz),d(Rz,Rz))). \end{aligned} \tag{20}$$

So,

$$\frac{1}{2}(d(Rz,Sz))\leq -\varphi((d(Rz,Sz),0)),$$

and hence $Sz=Rz$. We can analogously prove that $Tz=Rz$. That is, $Tz=Sz=Rz=t$.

Using weak compatibility of the pairs (T,R) and (S,R) we have $Rt = Tt = St$. So

$$\begin{aligned} d(Tt,t) = d(Tt,Sz) &\leq \tfrac{1}{2}(d(Rt,Sz) + d(Rz,Tt)) \\ &-\varphi(d(Rt,Sz), d(Rz,Tt)) \\ &\leq \tfrac{1}{2}(d(Tt,t) + d(t,Tt)) \\ &-\varphi(d(Tt,t), d(t,Tt)). \end{aligned} \tag{21}$$

That is $\varphi(d(Tt,t), d(t,Tt)) = 0$ and this implies that $Tt = t$. Therefore $Rt = Tt = St = t$. □

Example 7. Let $X = R$ be endowed with the Euclidean metric and let $E = \{0, \frac{1}{2}, 1\}$. Let $T, S : E \to E$ be defined by $T0 = T\frac{1}{2} = 0$, $T1 = \frac{1}{2}$ and $Sx = 0$, for all $x \in E$.

Let $R : E \to X$ and $\varphi : [0,\infty)^2 \to [0,\infty)$ be defined by $R0 = 0$, $R\frac{1}{2} = \frac{1}{2}$, $R1 = 1$ and $\varphi(t,s) = \frac{t+s}{6}$. By a careful calculous we see that all conditions of Theorem 4 are holds. Hence T, S and R have a unique common fixed point $(x = 0)$ by Theorem 4. But, this example cannot be studied by the main result of [2] (see Theorem 2.1 of [2]).

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